Patty Paper Parabolas

For GSP5

Here's the geometric definition of a *parabola*.

> A parabola is the set of points equidistant from a fixed point (the *focus*) and a fixed line (the *directrix*).

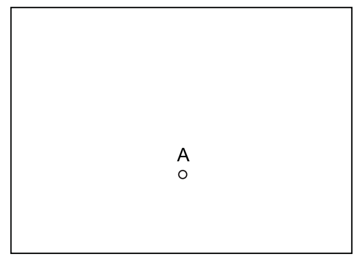
With nothing more than a sheet of paper and a single point on the page, you can create a parabola. No rulers and no measuring required!

CONSTRUCT A PHYSICAL MODEL

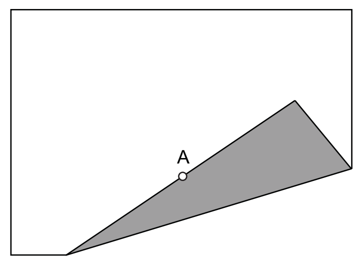
You'll need a rectangular or square piece of wax paper or patty paper. If you don't have these materials, use a plain sheet of paper.

If you're doing this in class with other students, different students should place *A* at different distances from the edge. If you're working alone, do steps 1–5 twice—once with *A* close to the edge, once with *A* farther from the edge.

1. Mark point *A* approximately one inch from the bottom of the paper and centered between the left and right edges.



2. Fold the paper as shown, so that the bottom edge lands directly on point *A*. Make a sharp crease to keep a record of this fold. Unfold the crease.
3. Fold the paper along a new crease so that a different point on the bottom edge lands on point *A*. Unfold the crease and repeat the process.



4. After you've made about a dozen creases, examine them to see if you spot any emerging patterns.

Mathematicians would call your set of creases an *envelope* of creases.

5. Resume creasing the paper. Gradually, you should see a well-outlined curve appear. Be patient—it may take a little while.
6. Discuss what you see with your classmates and compare their folded curves to yours. If you're doing this activity alone, fold a second sheet of paper with point *A* farther from the bottom edge.

Q1 The creases on your paper seem to form the outline of a parabola. Where do its focus and directrix appear to be?

Q2 If you move point *A* closer to the bottom edge of the paper and fold another curve, how do you think its shape will compare to the first curve?

Q3 Each time you crease the paper, some point on the bottom edge of the paper lands on top of point *A*. Call this point *B*. What's the geometric relationship of the crease to points *A* and *B*?

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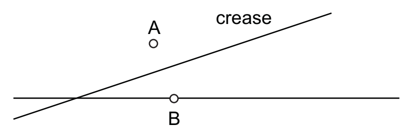
CONSTRUCT A SKETCHPAD MODEL

Fold and unfold. Fold and unfold. Creasing your paper takes work. Folding one or two sheets is fine, but what if you want to test many different locations for point *A*? You'll need to keep starting over with fresh paper, folding new sets of creases.

Sketchpad can streamline your work. After constructing only one set of creases, you can drag point *A* to new locations and watch the creases adjust instantaneously.

Holding the Shift key while drawing the line will help you make it horizontal.

7. Open a new sketch. Use the **Line** tool to draw a horizontal line near the bottom of the screen. This line represents the bottom edge of the paper.
8. Draw point *A* above the line, roughly centered between the left and right edges of the screen.
9. Construct point *B* on the horizontal line.
10. Construct the "crease" formed when point *B* is folded onto point *A*. Make the crease thick.



Q4 What construction did you use to make your crease?

11. Drag point *B* along its line. If you constructed the crease correctly, it should adjust to the new locations of point *B*.

If you don't want your traces to fade, go to **Edit | Preferences | Color** and remove the checkmark from the **Fade traces over time** checkbox.

12. Select the crease line and choose **Display | Trace Perpendicular Line.**
13. Drag point *B* along the horizontal line to create a collection of creases.
14. Drag point *A* to a different location. If necessary, choose **Display | Erase Traces.**
15. Drag point *B* again to create another collection of creases.

Retracing creases for each location of point *A* is certainly faster than folding paper. But you can do even better; you can make the creases relocate automatically as you drag point *A*. The **Construct | Locus** command makes this possible.

16. Turn tracing off for your original crease by selecting it and again choosing **Display | Trace Line.** If you have traces still showing, erase them.
17. Select your crease line and point *B*. Choose **Construct | Locus.** An entire set of creases appears: the locus of crease locations as point *B* moves along its path. If you drag point *A*, the creases will readjust automatically.

Q5 The creases mark out a curve. Describe the shape of the curve.

Q6 How does the curve change as you move point *A* closer to the horizontal line?

Q7 How does the curve change as you move point *A* away from the horizontal line?

Q8 How does the curve change when you move point *A* below the horizontal line?

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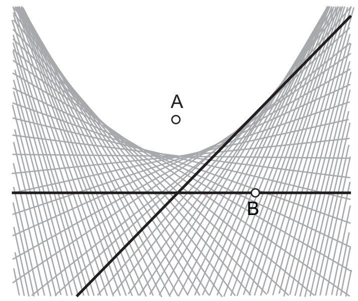
PLAY DETECTIVE

Each crease on your paper touches the parabola at exactly one point. Another way to describe this is to say that each crease is *tangent* to the parabola. By doing some detective work, you can locate the point of tangency and use it to construct only the parabola without its creases.



18. Position point *A* so that your parabola looks similar to the one shown.
19. Drag point *B* and notice that the crease remains tangent to the parabola. If it's hard to see the crease against the locus, set the locus to be thin or dashed. Observe the position of the point of tangency closely as you drag point *B*.

Q9 How does the position of the point of tangency seem to be related to the position of point *B*?

20. The exact point of tangency lies at the intersection of two lines—the crease line and another line not yet constructed. Construct this line in your sketch. Then construct the point of tangency and label it *D*.

Q10 What construction did you use to construct the new line in step 20?

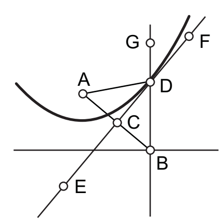
Select this new locus and make it thick so that it's easier to see.

21. Select point *D* and point *B* and choose **Construct | Locus.** If you've constructed point *D* correctly, you should see a curve appear: the parabola itself. Drag point *A* to see how the parabola changes.

PROVE IT

Your construction seems to generate parabolas. Can you prove this? Try developing a proof on your own or work through the following steps and questions.

This picture should resemble your Sketchpad construction. Line *EF* (the perpendicular bisector of segment *AB*) represents the crease formed when point *B* is folded onto point *A*. Point *D* is the point of tangency that defines the curve.



A parabola is the set of points equidistant from a fixed point (the *focus*) and a fixed line (the *directrix*).

Q11 Recall the locus definition of a parabola. To prove that point *D* traces a parabola, which two segments must you prove equal in length?

Q12 Use a triangle congruence theorem to prove that $\triangle ACD \cong \triangle BCD$.

Q13 Use the distance definition of a parabola and the result from Q12 to prove that point *D* traces a parabola.

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EXPLORE MORE

22. Go to page "Tangent Circle" of **Patty Paper Parabolas.gsp.** You'll see a circle, with a center at point *C*, that passes through point *A* and is tangent to a line at point *B*. Drag point *B*. Why does point *C* trace a parabola?

23. A parabola can also be described as an ellipse with one focus at infinity.

    Go to page "Ellipse Connection" of **Patty Paper Parabolas.gsp.** You'll see an ellipse construction based on a circle and two focal points (points *A* and *B*), where *A* is the center of the circle. Press *Send Focus Far Far Away.* Point *A* will travel far off the screen.

    When the movement stops, examine the result. In what ways does it resemble the Folded Rectangle construction?

24. Use the illustration from your parabola proof to show that $\angle GDF \cong \angle ADC$. The page "Headlights" of **Patty Paper Parabolas.gsp** illustrates a nice consequence of this result.

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