Barnsley's Fractal Fern

For GSP5

The Barnsley fern is a fractal created by an *iterated function system*, in which a point (the *seed* or *pre-image*) is repeatedly transformed by using one of four transformation functions. A random process determines which transformation function is used at each step. The final image emerges as the iterations continue.

Affine transformations preserve colinearity and ratios of distances.

The transformations are affine transformations of form

$$x' = ax + cy + e$$

$$y' = bx + dy + f$$

And so each transformation can be specified by six constants (*a, b, c, d, e,* and *f* in the example).

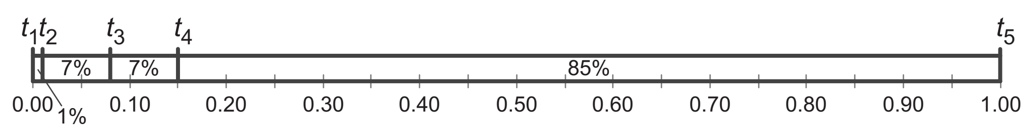
SKETCH

1. Open the sketch **Barnsley's Fractal Fern.gsp**. This sketch contains the constants to be used in the four transformations, and tools to simplify the construction process.

A random process controls which transformation is chosen for each iteration. For the Barnsley fern, the first transformation is chosen 1% of the time, each of the next two is chosen 7% of the time, and the fourth transformation is chosen 85% of the time. You'll start out by constructing a point that can be used to make the random choices.

Select point *C* and choose **Measure | Value of Point.**

2. Construct point *C* on horizontal segment *AB* and measure the value of point *C*. Label the measurement *r*. Drag *C* to be sure that *r* stays between 0 and 1.



3. You'll use parameters to divide the range from 0 to 1 into four parts as shown above, corresponding to the desired probabilities of the transformations. Create five parameters, t_1 through t_5, to define the beginning and end of each part, and assign appropriate values to the parameters.

Q1 What values should you assign to each of the five parameters?

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Press and hold the **Custom** tool icon and choose **Between** from the menu that appears. To use the tool, click first the lower limit, then the value of *r*, and then the upper limit.

4. Choose the **Between** custom tool and use it to generate results m_1 through m_4, using r as the test value and pairs of parameters t_1 through t_5 as the lower and upper limits. (The value of m_1 should be 1 when r is between t_1 and t_2, and 0 otherwise. Similarly, m_2 should be 1 only when r is between t_2 and t_3.) Except at the lower endpoint, exactly one of the values m_1 through m_4 should be 1, and the other three should be 0. Drag C to confirm this behavior.
5. Make sure the origin is near the bottom center of the screen, and that the scale of the axes leaves 10 on the y-axis slightly below the top of the screen.

Make sure these labels are correct or the **Affine Transformation** tool won't work correctly.

6. Construct an independent point P to be the pre-image (seed) for the iteration, and measure its x- and y-coordinates. Label the coordinates x and y, with no subscripts.
7. Choose the **Affine Transformation** tool, and use it to generate the four transformation functions. For each function, click the appropriate values for a, b, c, d, e, and f.

This calculation will select the *x* value for which the corresponding *r* value is 1.

8. To calculate a transformed x value from the four functions, create a calculation that adds the x values of all four transformations, multiplying each x value by the corresponding value from m_1 through m_4.

$$m_1(a_1x + c_1y + e_1) + m_2(a_2x + c_2y + e_2) + m_3(a_3x + c_3y + e_3) + m_4(a_4x + c_4y + e_4)$$

Point *Q* is the transformed image of *P*.

9. Calculate a y value in the same way. Plot the point determined by these two calculations, and label it Q.

As you drag *C*, you should observe *Q* switching among four possible transformed images.

10. The value of r determines which transformation is used. Drag C to change the value of r. Observe that Q moves as each of the transformations is used in turn.

You'll iterate the construction by mapping P to Q and by moving images of C to new random positions on the segment.

11. Iterate the function system: Select points P and C. Choose **Transform | Iterate**, and iterate P to Q and C to itself. Click the **Iterate** button.
12. Delete the table of iterated values that appears. Choose **Edit | Properties | Iteration** for the iterated image of P, and set the number of iterations to 20. Choose **Display | Point Style | Dot** to show the image with small points.

Q2 Observe that the orbit of the iterated images tends toward a fixed point. Drag point C to change the transformation used. Does each transformation function appear to be associated with a single fixed point? Give the approximate coordinates of the fixed point corresponding to each function.

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The image looks better with 10,000 iterations, if this number of iterations doesn't slow your computer down too much.

13. The initial orbit that appears uses only a single transformation because the images of *C* are not yet taking on new random positions. Select the iterated image of *P* and choose **Edit | Properties | Iteration**. Change the number of iterations to 5000, and click the radio button labeled **Random locations on iterated paths.**

14. Drag point *C* so the fourth function is used to generate point *Q* from *P*.

Q3 Move *P* to different locations on the fern, and describe the effect of this transformation function in terms of the shape of the fern.

Q4 Similarly, activate each of the other transformation functions, and describe the effect of these functions.

To find a fixed point of a transformation, drag *P* until *P* and *Q* coincide.

Q5 Where are the fixed points of the four transformations in terms of the shape of the fern?

Q6 Change the *t* parameters so the first transformation is never used. What's the difference in the new shape? Describe the role of the first transformation.

Q7 Determine and describe the role of each of the other transformations in generating the image.

EXPLORE MORE

Q8 Determine the effect of the various constants in specifying the fern's shape.

Q9 Modify the fern transformations so that the leaves of the fern are opposite each other, instead of alternating.

Q10 Make other minor changes to the function system to produce a modified shape. (Be prepared to undo your changes—not all function systems generate interesting results.)

Similar iterated function systems can generate a wide variety of fractals. For instance, consider a system of three functions that have the following effects.

- Dilate the pre-image point by a scale factor of one-half toward the point (6, 0)
- Dilate the pre-image point by a scale factor of one-half toward the point (−6, 0)
- Dilate the pre-image point by a scale factor of one-half toward the point (0, 10)

Q11 Write down the transformation functions for these three transformations.

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Q12 Modify a copy of the fern sketch to use these transformations, with equal probabilities for all three transformations. What figure results?

Q13 Build a similar iterated function system with four transformations, dilating the pre-image by a factor of 0.44 toward one of the points (0, 0), (1, 0), (0, 1), (1, 1) with equal probability. Describe the resulting figure.

Q14 Research Barnsley's method and fractal compression. Report your findings.

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