A Geometric Approach to $e^{i\pi}$

For GSP5

One of the most beautiful—and mysterious—mathematical discoveries has to do with the value of $e^{i\pi}$. What does it mean to raise the number e to an imaginary power?

In this activity you'll explore $e^{i\pi}$ through a geometric approach.

GETTING STARTED

A good place to begin your investigation is with the mathematical constant e. Consider these calculations.

$$\left(1+\frac{1}{10}\right)^{10}, \left(1+\frac{1}{100}\right)^{100}, \left(1+\frac{1}{1{,}000}\right)^{1{,}000}, \left(1+\frac{1}{10{,}000}\right)^{10{,}000}, \ldots$$

1. Open a new sketch. Choose **Edit | Preferences** from the Edit menu. Set the precision of **Others** to **hundred-thousandths.**
2. Choose **Number | Calculate.** Calculate, one at a time, the values of the four expressions.

Q1 What do you notice about your four calculations?

3. Calculate four more expressions that continue the sequence.

Q2 Based on your calculations, approximate the value of $\left(1+\frac{1}{n}\right)^n$ to several decimal places as n grows ever larger.

The mathematical constant e is defined as the limiting value of $\left(1+\frac{1}{n}\right)^n$ as n approaches infinity. Raising e to a power, like e^2 or e^3, involves a similar definition.

$$e^x = \text{the limiting value of } \left(1+\frac{x}{n}\right)^n \text{ as } n \text{ approaches infinity}$$

Q3 Use the definition of e^x and a large n to approximate the value of e^3.

SKETCH AND INVESTIGATE

Now that you know how to approximate e^x when x is a real number, you're ready to consider $e^{i\pi}$. Raising e to an imaginary power certainly seems strange, but let's use our definition of e^x with $x = i\pi$ and see what happens.

$$e^{i\pi} = \text{the limiting value of } \left(1+\frac{i\pi}{n}\right)^n \text{ as } n \text{ approaches infinity}$$

As before, we can get a sense of how this expression behaves by starting with a small value of n. When $n = 10$, we must evaluate $\left(1+\frac{i\pi}{10}\right)^{10}$.

4. Open **eipi.gsp.** The axes represent the complex plane with real numbers on the horizontal axis and imaginary numbers on the vertical axis. Point A is at $(1, 0)$ and represents the value 1. Point B is at $\left(1, \frac{\pi}{10}\right)$ and represents the value $1 + \frac{i\pi}{10}$.

This is a good time to review the activity Multiplication of Complex Numbers.

Q4 The sketch provides two pieces of information about right triangle OAB: the measure of $\angle AOB$ and the length OB. Describe geometrically what it means to multiply the complex number $1 + \frac{i\pi}{10}$ by itself. Describe geometrically what it means to raise $1 + \frac{i\pi}{10}$ to the tenth power.

Sketchpad makes the process you described in Q4 simple to carry out.

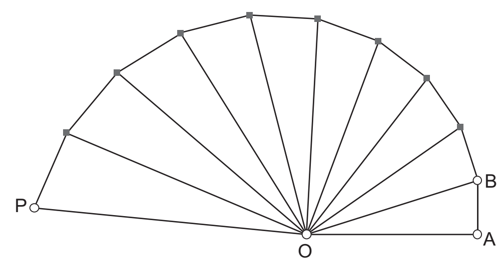
5. Select point A and $n - 1$, and hold down the Shift key. Choose **Transform | Iterate to Depth.**

6. Click point B to map point A to point B. Click **Iterate** to confirm your mappings. The iterated triangles appear, all of which are similar to $\triangle OAB$.

Q5 Identify the locations of $\left(1 + \frac{i\pi}{10}\right)^2, \left(1 + \frac{i\pi}{10}\right)^3, \ldots, \left(1 + \frac{i\pi}{10}\right)^9$, and $\left(1 + \frac{i\pi}{10}\right)^{10}$.

7. Select the iterated point image and choose **Transform | Terminal Point.** Label the terminal point P.

8. With point P selected, choose **Measure | Coordinates.**



Q6 What is the value of $\left(1 + \frac{i\pi}{10}\right)^{10}$?

The third page of the sketch provides an intuitive explanation of what you're observing.

9. Drag slider point n to the right to increase the value of n.

Q7 What do you notice about the value of $\left(1 + \frac{i\pi}{n}\right)^n$ as n grows larger?

EXPLORE MORE

You can generalize the method of finding $e^{i\pi}$ to compute e raised to the $i\theta$ power, where θ is any number.

10. Open page 2 of **eipi.gsp.**

This sketch computes e raised to the power $\frac{i\pi}{k}$ for any value of k.

11. Double-click the parameter k and change its value to 2.

Q8 Use the iteration process from page 1 of the sketch to approximate the value of $e^{i\pi/2}$.

Q9 Approximate the imaginary powers of e for values of k such as 3 and 4.

Q10 For each of your approximations, what do you notice about its distance from the origin?

Q11 When $k = 2$, what angle does the point representing $e^{i\pi/2}$ make with the x-axis? Answer this question for $k = 3$ and for $k = 4$.

Q12 The eighteenth-century mathematician Leonhard Euler developed the identity $e^{i\theta} = \cos\theta + i\sin\theta$ as a way to compute the value of e raised to any imaginary power. Explain how this identity makes sense based on your answers to Q10 and Q11.

Q13 Substitute $\theta = \pi$ into the identity in Q12. What do you get?