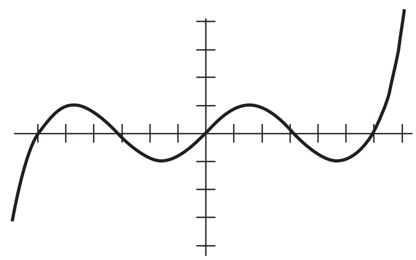
Taylor Series

For GSP5

You can compute the value of a polynomial function directly and easily for any particular value of x using multiplication and addition. But values of other functions, such as the sine function, are much more difficult to compute.

You can learn how to derive the Taylor series when you study calculus.

In this activity you'll approximate the sine function using a series called a *Taylor series* and observe the behavior of the partial sums when the series is evaluated to various depths. The Taylor series approximation for $\sin(x)$ is



$$f(x) = \frac{x^1}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \ldots$$

SKETCH AND INVESTIGATE

1. In a new sketch, create a square grid, construct point A on the x-axis, and measure the point's x-value. Label the x-value x.

2. Create four parameters to use in iterating the series. Label them *i, num, den,* and *sum.*

Q1 Parameter i represents the odd integers, following the sequence 1, 3, 5, ... What rule can you apply to one element of this sequence to calculate the next?

Q2 Parameter *num* represents the numerator, taking on values x, $-x^3$, x^5, $-x^7$, and so forth. What is the rule to calculate a value of this sequence from the previous value?

Q3 Parameter *den* represents the denominator, taking on values 1!, 3!, 5!, and so forth. What's the rule to calculate the next value of this sequence? (Express your answer in terms of the previous values of *den* and *i*.)

Q4 Parameter *sum* represents the sum of all the terms from the first term through the ith term. What value should you use as the initial value of the sum, before adding the very first term? What's the rule to calculate one sum from the previous sum?

Parameters must be independent values in order to be iterated, so you can't set the initial values of the parameter that depends on x until after you construct the iteration.

3. All but one of these parameters have constant initial values that you can assign now. (The initial value for the other isn't constant, but depends on the value of x.) Assign appropriate initial values to the parameters that don't depend on x. Assign an initial value to the other parameter as though the value of x were 2.

Q5 What initial values did you assign to the parameters?

Make sure the calculated values are what you expect.

4. For each of the four parameters, use the rule you described above to calculate the next value of the quantity it represents. (Your calculations should involve only the values of the four parameters and the value of *x*.)

5. Plot the point (*x*, *sum*). The iterated image of this plotted point will allow you to see the graph of each successive expansion of the Taylor series.

*Select all four parameters, and choose **Transform | Iterate.** Then match each parameter to its next value.*

6. Iterate each of the parameters to its calculated next value.

7. The parameter *num* doesn't yet have a correct initial value, because the initial value depends on *x*. Select *num*, choose **Edit | Edit Parameter,** and calculate the initial value so that it depends correctly on *x*.

8. Drag point *A* left and right on the *x*-axis, observing the values in the table and the positions of the plotted points.

The last row of the table should contain $n = 1$.

9. Select the iterated image of the plotted point and press the **–** key on the keyboard twice to set the depth of iteration to 1.

The terminal point *is the very last image of the iterated point, based on the current depth of iteration.*

10. With the iterated image of the plotted point still selected, choose **Transform | Terminal Point.** Then construct the locus of the terminal point as *A* moves along the axis.

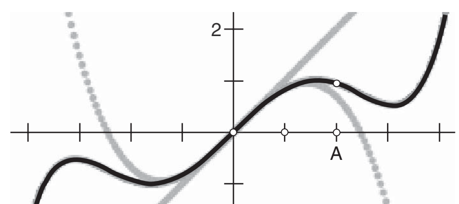
Q6 What is the shape of the locus? Which terms contribute to this shape?

To increase the depth, select either iterated image (the table or the image of the plotted point), and press the + key on the keyboard.

11. Set the depth of iteration to 2.

Q7 How does this change the shape of the locus? Which terms contribute now?

12. Increase the depth to 3. Turn on tracing for both the iterated image of the plotted point and the locus. Animate point *A*, and observe the behavior of the point images.



n	i + 2	−num · x^2	den · (i + 1) · (i + 2)	sum + $\frac{\text{nu}}{\text{de}}$
0	3.00	−8.00	6.00	2.00
1	5.00	32.00	120.00	0.67
2	7.00	−128.00	5040.00	0.93
3	9.00	512.00	362880.00	0.91

Q8 What shapes do the iterated point images trace? Sketch their shapes and explain the role of each trace based on the terms of the series.

You may have to move the origin and change the domain of the locus to see two full periods.

13. While point *A* is moving, increase the depth until the locus accurately approximates the sine curve for at least two periods.

Q9 How many terms are required to give a reasonable approximation for the first period of the sine function? For the first two periods?

EXPLORE MORE

The Taylor series for the cosine function is the following expression.

$$f(x) = \frac{x^0}{1} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \ldots$$

You can change the initial values of your existing iteration to calculate this series. Decide which parameters to change, and calculate and plot the modified series.

