First Edition

Precalculus

High School Edition
Teacher Manual Sample Chapter



Julie Miller Donna Gerken

Precalculus

1st Edition | High School Edition | Teacher Manual



mheonline.com/honorselectives



Front Cover: Anatoli Styf/Shutterstock

Copyright © 2024 McGraw Hill

All rights reserved. No part of this publication may be reproduced or distributed in any form or by any means, or stored in a database or retrieval system, without the prior written consent of McGraw Hill Education, including, but not limited to, network storage or transmission, or broadcast for distance learning.

Send all inquiries to: McGraw Hill 8787 Orion Place Columbus, OH 43240

ISBN: 978-1-26-565888-5 MHID: 1-26-565888-9

Printed in the United States of America.

123456789LHN2928272625242

| Chapter R Review of Prerequisites | R2 |
|--|------|
| Lesson R-1 Sets and the Real Number Line | R3 |
| Lesson R-2 Exponents and Radicals | R16 |
| Lesson R-3 Polynomials and Factoring | R27 |
| Problem Recognition Exercises: Simplifying Algebraic Expressions | R37 |
| Lesson R-4 Rational Expressions and More Operations on Radicals | R39 |
| Lesson R-5 Equations with Real Solutions | R47 |
| Lesson R-6 Complex Numbers and More Quadratic Equations | R57 |
| Lesson R-7 Applications of Equations | R69 |
| Lesson R-8 Linear, Compound, and Absolute Value Inequalities | R83 |
| Problem Recognition Exercises: Recognizing and Solving Equations and Inequalities | R95 |
| Chapter Review | R101 |
| Chapter 1 Functions and Relations | 2 |
| Launch Activity Lesson 1-1 The Rectangular Coordinate System and Graphing Utilities | |
| Lesson 1-2 Circles | |
| Lesson 1-3 Functions and Relations | |
| Lesson 1-4 Linear Equations in Two Variables and Linear Functions | |
| Lesson 1-5 Applications of Linear Equations and Modeling | |
| Problem Recognition Exercises: Comparing Graphs of Equations | |
| Lesson 1-6 Transformations of Graphs | |
| Lesson 1-7 Analyzing Graphs of Functions and Piecewise-Defined Functions | |
| Lesson 1-8 Algebra of Functions and Function Composition | |
| Chapter Review | |
| Chapter 2 Polynomial and Rational Functions Launch Activity | |
| Lesson 2-1 Quadratic Functions and Applications | 128 |
| Lesson 2-2 Introduction to Polynomial Functions | 148 |
| Lesson 2-3 Division of Polynomials and the Remainder and Factor Theorems . | 168 |
| Lesson 2-4 Zeros of Polynomials | 179 |
| Lesson 2-5 Introduction to Rational Functions | 192 |
| Lesson 2-6 Graphs of Rational Functions | 206 |
| Problem Recognition Exercises: Polynomial and Rational Functions | 219 |
| Lesson 2-7 Polynomial and Rational Inequalities | 221 |
| Problem Recognition Exercises: Solving Equations and Inequalities | 234 |
| Lesson 2-8 Variation | 236 |
| Chapter Boyley | 244 |

| Chapter 3 Exponential and Logarithmic Functions | s252 |
|---|-------|
| Launch Activity | |
| Lesson 3-1 Inverse Functions | 254 |
| Lesson 3-2 Exponential Functions | 266 |
| Lesson 3-3 Logarithmic Functions | 284 |
| Problem Recognition Exercises: Analyzing Functions | 298 |
| Lesson 3-4 Properties of Logarithms | 301 |
| Lesson 3-5 Exponential and Logarithmic Equations and Applications | 311 |
| Lesson 3-6 Modeling with Exponential and Logarithmic Functions | 322 |
| Chapter Review | 337 |
| Chapter 4 Trigonometric Functions | 344 |
| Launch Activity | |
| Lesson 4-1 Angles and Their Measure | 346 |
| Lesson 4-2 Trigonometric Functions Defined on the Unit Circle | 361 |
| Lesson 4-3 Right Triangle Trigonometry | 373 |
| Lesson 4-4 Trigonometric Functions of Any Angle | 386 |
| Lesson 4-5 Graphs of Sine and Cosine Functions | 395 |
| Lesson 4-6 Graphs of Other Trigonometric Functions | 418 |
| Problem Recognition Exercises: Comparing Graphical Characteristics of Trigonometric Functions | 431 |
| Lesson 4-7 Inverse Trigonometric Functions | 432 |
| Chapter Review | 446 |
| Chapter 5 Analytic Trigonometry | 452 |
| Launch Activity | |
| Lesson 5-1 Fundamental Trigonometric Identities | 454 |
| Lesson 5-2 Sum and Difference Formulas | 469 |
| Lesson 5-3 Double-Angle, Power-Reducing, and Half-Angle Formulas | 484 |
| Lesson 5-4 Product-to-Sum and Sum-to-Product Formulas | 497 |
| Problem Recognition Exercises: Verifying Trigonometric Identities | 506 |
| Lesson 5-5 Trigonometric Equations | 509 |
| Problem Recognition Exercises: Trigonometric Identities and Trigonometric Equations | 522 |
| Chapter Review | 524 |
| Chapter 6 Applications of Trigonometric Function | .s530 |
| Launch Activity | |
| Lesson 6-1 Applications of Right Triangles | 532 |
| Lesson 6-2 The Law of Sines | 544 |
| Lesson 6-3 The Law of Cosines | 556 |
| Problem Recognition Exercises: Solving Triangles Using a Variety of Tools | 566 |
| Lesson 6-4 Harmonic Motion | 568 |
| Chapter Review | 581 |

| Chapter 7 'I'rigonometry Applied to Polar Coordinate | ì |
|---|------|
| Systems and Vectors | .588 |
| Launch Activity | |
| Lesson 7-1 Polar Coordinates | 590 |
| Lesson 7-2 Graphs of Polar Equations | 601 |
| Problem Recognition Exercises: Comparing Equations in Polar and Rectangular Form | 614 |
| Lesson 7-3 Complex Numbers in Polar Form | 617 |
| Lesson 7-4 Vectors | 630 |
| Lesson 7-5 Dot Product | 645 |
| Chapter Review | 656 |
| Chapter 8 Systems of Equations and Inequalities | .662 |
| Lesson 8-1 Systems of Linear Equations in Two Variables and Applications | 664 |
| Lesson 8-2 Systems of Linear Equations in Three Variables and Applications | 678 |
| Lesson 8-3 Partial Fraction Decomposition | 691 |
| Lesson 8-4 Systems of Nonlinear Equations in Two Variables | 696 |
| Lesson 8-5 Inequalities and Systems of Inequalities in Two Variables | 708 |
| Problem Recognition Exercises: Equations and Inequalities in Two Variables | 721 |
| Lesson 8-6 Linear Programming | 723 |
| Chapter Review | 735 |
| Chapter 9 Matrices and Determinants and | |
| Applications | 742 |
| Launch Activity | |
| Lesson 9-1 Solving Systems of Linear Equations Using Matrices | 744 |
| Lesson 9-2 Inconsistent Systems and Dependent Equations | 756 |
| Lesson 9-3 Operations on Matrices | 768 |
| Lesson 9-4 Inverse Matrices and Matrix Equations | 788 |
| Lesson 9-5 Determinants and Cramer's Rule | 801 |
| Problem Recognition Exercises: Using Multiple Methods to Solve Systems of Linear Equations | 811 |
| Chapter Review | 812 |
| Chapter 10 Analytic Geometry | .820 |
| Launch Activity | |
| Lesson 10-1 The Ellipse | |
| Lesson 10-2 The Hyperbola | 845 |
| Lesson 10-3 The Parabola | 863 |
| Problem Recognition Exercises: Comparing Equations of Conic Sections and the General Equation | 878 |
| Lesson 10-4 Rotation of Axes | 883 |
| Lesson 10-5 Polar Equations of Conics | 896 |
| Lesson 10-6 Plane Curves and Parametric Equations | 906 |
| Chapter Review | 923 |

| Chapter 11 Sequences, Series, Induction, and | |
|--|-----------------|
| Probability | 934 |
| Launch Activity | |
| Lesson 11-1 Sequences and Series | 936 |
| Lesson 11-2 Arithmetic Sequences and Series | 948 |
| Lesson 11-3 Geometric Sequences and Series | 958 |
| Problem Recognition Exercises: Comparing Arithmetic and Geometric Sequences and Series | 970 |
| Lesson 11-4 Mathematical Induction | 97 ² |
| Lesson 11-5 The Binomial Theorem | 985 |
| Lesson 11-6 Principles of Counting | 99 [.] |
| Lesson 11-7 Introduction to Probability | 1002 |
| Chapter Review | 1019 |

Online-Only Chapter 12 Preview of Calculus 🚳



The topics in Chapter 12 are outside the scope of most high school precalculus courses. The chapter is available online as part of the digital Teacher Manual.

USING YOUR TEACHER MANUAL

This teacher manual is intended to guide precalculus teachers through using the tools provided by the *Precalculus* program.

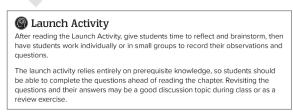
Chapter Opener

Each chapter opens with a pacing chart. This chart indicates the recommended number of class periods that should be devoted to each section of the chapter. The chart also includes the online resources teachers can use as part of their instruction including the Launch Activity, Group Activities, Problem Recognition Worksheets, Corequisite Workbook, and other assessments.

| ress | son | Pacing (class periods |
|------|--|-----------------------|
| 3-1 | Inverse Functions | 4 |
| 3-2 | Exponential Functions | 5 |
| 3-3 | Logarithmic Functions | 5 |
| | Problem Recognition Exercises: Analyzing Functions | - |
| 3-4 | Properties of Logarithms | 3 |
| 3-5 | Exponential and Logarithmic Equations and Applications | 4 |
| 3-6 | Modeling with Exponential and Logarithmic Functions | 3 |
| | ne Resources equisite Workbook Chapter 3 | |
| | oter 3 Launch Activity | |
| | oter 3 Group Activities | |
| | fram Alpha Activities | |
| | elem Recognition Worksheets | |
| | ioned Lesson Quizzes | |
| Vers | | |
| | ioned Chapter Tests | |

Launch Activity

Guidance around and answers to the Launch Activity are provided at the beginning of each chapter of the teacher manual. The activity connects the chapter to topics relevant to students' lives and provides an opportunity to master the diverse mathematical practices.



USING YOUR TEACHER MANUAL (CONTINUED)

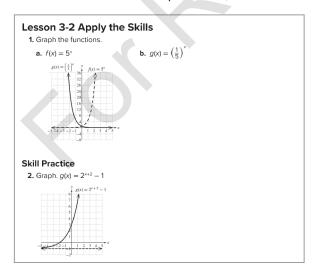
Teaching Tips

A table of teaching tips is provided for each lesson in a chapter. Lesson exercises that are mirrors of worked examples are easily referenced for use in the classroom. If particular skills or problem types are known to be challenging to students, a note is included with additional information, guidance, or alternate explanations.

| | ildəs. Tederler 14 | lotes on specific issues are included where applicable |
|----------|--------------------|--|
| Exercise | Example | Teacher Notes |
| 1 | 8, 10 | |
| 2 | 16, 18, 22 | |
| 3 | 24, 30 | Encourage students to use their knowledge of basic functions and transformations of basic functions to visualize graphs. This is helpful to determine if a function is one-to-one |
| 4 | 32, 34 | |
| 5 | 46 | |
| 6 | 50 | In Example 6, f has vertical asymptote $x=-3$ and horizontal asymptote $y=-1$. The domain is $(-\infty, -3) \cup (-3, \infty)$ and the range is $(-\infty, -1) \cup (-1, \infty)$. Consider asking students to find the domain and range of f^{-1} . |
| 7 | 54 | |
| 8 | 56 | |

Answers to Questions in the Student Edition

The Teacher Manual contains the answers to all questions from each chapter of the Student Edition along with their answers. These include the **Apply the Skills** questions, the lesson-level **Practice** questions, and the **Chapter Review** exercises.



College Colleg

Chapter 4

Trigonometric Functions

| Lesson | Pacing (class periods) |
|---|------------------------|
| 4-1 Angles and Their Measure | 5 |
| 4-2 Trigonometric Functions Defined on the Unit Circle | 4 |
| 4-3 Right Triangle Trigonometry | 4 |
| 4-4 Trigonometric Functions of Any Angle | 5 |
| 4-5 Graphs of Sine and Cosine Functions | 5 |
| 4-6 Graphs of Other Trigonometric Functions | 5 |
| Problem Recognition Exercises: Comparing Graphica Characteristics of Trigonometric Functions | ıl - |
| 4-7 Inverse Trigonometric Functions | 5 |

Online Resources

Corequisite Workbook Chapter 4

Chapter 4 Launch Activity

Chapter 4 Group Activities

Wolfram Alpha Activities

Problem Recognition Worksheets

Versioned Lesson Quizzes

Versioned Chapter Tests

Editable Question Bank

Lesson PowerPoints

Launch Activity

After reading the Launch Activity, give students time to reflect and brainstorm, then have students work individually or in small groups to record their observations and questions.

The launch activity relies entirely on prerequisite knowledge, so students should be able to complete the questions ahead of reading the chapter. Revisiting the questions and their answers may be a good discussion topic during class or as a review exercise.

opyright © McGraw Hill

Make use of Structure: Revolutions of the Wheel

Your robotics team is preparing to compete in a local competition. Once the course for the competition is complete your team can view the layout and measure distances of the path and angles of the turns prior to the event. All coding for this course must be completed before the competition begins. For this competition, the standard radius of each wheel is 3.5 cm.

One teammate from your robotics team measured the first straight segment of the competition path and found the length to be 4 feet 3 inches long. Coding for the robot's movement does not use distance as input but rather revolutions of the wheel.

Talk About It

1. What do you notice?

The robotics course was measured in feet and inches, but the size of the wheel was given in centimeters. Input for coding the robot does not use distance but revolutions. Since coding must be completed before the competition begins, accurate measurements will be important.

2. What questions can you ask?

What is the diameter of the robot's wheel? What is the circumference of robot's wheel? What is the formula for circumference of a wheel? How far is 4 feet 3 inches in centimeters? What is 3.5 centimeters in inches? How many degrees is in one revolution?

You will work with your partner to answer these questions: How many revolutions will the wheel of the robot make in the first straight segment of the course? How many degrees will the wheel travel for this section of the course?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?

I am assuming that the measurements are accurate. I will also assume that when calculating the circumference of the wheel, I will use the stored value for π in the calculator then round as needed (rather than using the rounded 3.14 for calculations). This will allow me to calculate the distance and degrees accurately for the course.

2. What constraints, if any, need to be considered? How might they affect the solution?

Distance for the course must be positive. The number of degrees the wheel travels must be greater than 0° .

3. What type of model best represents the situation?

To represent this situation, I would need to use a nonlinear model. I would use properties of a circle such as the equation for the circumference of a circle (either $C = 2\pi r$ or $C = \pi d$), and the knowledge that one revolution is 360° .

In this situation, C represents the circumference of a circle and d represents the diameter of the circle. The ratio of the circumference of a circle and to its diameter is π such that $\pi = \frac{C}{d}$. Furthermore, to determine the number of revolutions for the straight segment of the course, let the length be I and R be the number of revolutions such that the length divided by the circumference provides the number of revolutions, $\frac{1}{C} = R$. Lastly, let t represent the number degrees traveled by the wheel where $t = 360^{\circ}(R)$.

Lesson 4-1 Angles and Their Measure

Teaching Tips

Each fully worked example is paired with end-of-lesson exercises that mirror the example. These end-of-lesson exercises are excellent problems to work through together in class. Teacher Notes on specific issues are included where applicable.

| Exercise | Example | Teacher Notes |
|----------|---------|---|
| | | Point out to students that the relationships among angular measures of degrees-minutes-seconds is similar to hours-minutes-seconds. |
| 1 | 30 | |
| 2 | 34 | |
| 3 | 40 | |
| 4 | 48 | |
| 5 | 58 | |
| 6 | 68 | Students often have difficulty determining the number of revolutions of an angle greater than 2π , when the angle is in radians. Emphasize the meaning of the quotient and remainder when an angle is divided by 2π . |
| 7 | 74 | To help remember the difference between the terms "latitude" and "longitude", tell students that the Earth is not a perfect sphere, but rather is "fatter" at the equator. Therefore, "longitude" measures the "long" way around the Earth. |
| 8 | 80 | In Example 8, the distance between cities was found by using the difference in latitudes for θ . However, if the cities are due east-west of each other, discuss why taking θ as the difference in longitudes will not work (unless the cities are on the equator). |
| 9 | 92 | |
| 10 | 100 | |

Lesson 4-1 Apply the Skills

1. Convert 131°12′33″ to decimal degrees. Round to 4 decimal places.

131.2092°

2. Convert 26.48° to degree, minute, second form.

26°28'48"

3. Convert from degrees to radians.

a. $300^{\circ} \frac{5\pi}{3}$

b. -70° $-\frac{7\pi}{18}$

4. Convert from radians to degrees.

a. $\frac{\pi}{18}$ 10°

b. $-\frac{7\pi}{4}$ -315°

5. Find an angle coterminal to θ between 0° and 360° .

a. $\theta = 1230^{\circ}$ 150°

b. $\theta = -315^{\circ}$ 45°

6. Find an angle coterminal to θ on the interval $[0, 2\pi)$.

a. $\theta = -\frac{\pi}{8} \quad \frac{15\pi}{8}$

b. $\theta = \frac{19\pi}{4} \quad \frac{3\pi}{4}$

7. Find the length of the arc made by an angle of 220° on a circle of radius 9 in.

 $11\pi \approx 34.6$ in.

8. Lincoln, Nebraska, is located at 40.8°N, 96.7°W and Dallas, Texas, is located at 32.8°N, 96.7°W. Since the longitudes are the same, the cities are north-south of each other. Using the difference in latitudes, approximate the distance between the cities assuming that the radius of the Earth is 3960 mi. Round to the nearest mile.

553 mi

9. A bicycle wheel rotates at 2 revolutions per second.

a. Find the angular speed. $4\pi/\text{sec}$

b. How fast does the bicycle travel (in ft/sec) if the wheel is 2.2 ft in diameter? Round to the nearest tenth.

≈13.8 ft/sec (≈9.4 mph)

10. A sprinkler rotates through an angle of 120° and sprays water a distance of 30 ft. Find the amount of area watered. Round to the nearest whole unit.

 $300 \pi \text{ ft}^2 \approx 942 \text{ ft}^2$

Lesson 4-1 Practice

Prerequisite Review

R.1. Solve for the specified variable.

$$p = hz$$
 for $z = \frac{p}{h}$

R.2. If a plane travels 280 mph for 3.5 hr, find the distance traveled. 980 mi

For Exercises R.3–R.5, convert the unit of time.

R.3. 210 min =
$$3.5$$
 hr

R.5. 7200 sec =
$$\frac{2}{}$$
 hr

- **R.6.** Find the circumference of a circle with a radius of $2\frac{1}{2}$ m.
 - **a.** Give the exact answer in terms of π . 5π m
 - **b.** Approximate the answer by using 3.14 for π . Round to 1 decimal place.

$$\approx 15.7 \text{ m}$$

R.7. Determine the area of a circle with a diameter of 40 ft. Use 3.14 for π . Round to the nearest whole unit. \approx 1256 ft²

Concept Connections

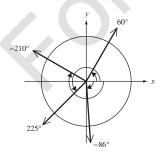
- **1.** The measure of an angle is (positive/negative) <u>negative</u> if its rotation from the initial side to the terminal side is clockwise. If the rotation is counterclockwise, then the measure is <u>positive</u>.
- **2.** An angle with its vertex at the origin of an *xy*-coordinate plane and with initial side on the positive *x*-axis is in <u>standard</u> position.
- **3.** Two common units used to measure angles are <u>degrees</u> and <u>radians</u>
- **4.** One degree is what fractional amount of a full rotation? $\frac{1}{360}$
- **5.** An angle that measures 360° has a measure of $\underline{2\pi}$ radians.
- **6.** Two angles are called <u>complementary</u> if the sum of their measures is 90°. Two angles are called <u>supplementary</u> if the sum of their measures is 180°.
- 8. A(n) __acute angle has a measure between 0° and 90°, whereas a(n) __obtuse angle has a measure between 90° and 180°.
- **9.** One degree is equally divided into 60 parts called <u>minutes</u>
- **10.** One minute is equally divided into 60 parts called <u>seconds</u>
- **11.** 1° = 60 ′ = 3600 ′
- **12.** An angle with its vertex at the center of a circle is called a(n) <u>central</u> angle.

- **13.** A central angle of a circle that intercepts an arc equal in length to the radius of the circle has a measure of <u>1 radian</u>.
- **14.** Which angle has a greater measure, 2° or 2 radians? 2 radians
- **15.** To convert from radians to degrees, multiply by $\frac{100}{\pi}$. To convert from degrees to radians, multiply by $\frac{\pi}{180^{\circ}}$.
- **16.** Two angles are <u>coterminal</u> if they have the same initial side and same terminal side.
- **17.** The measure of all angles coterminal to $\frac{7\pi}{4}$ differ from $\frac{7\pi}{4}$ by a multiple of 2π .
- **18.** The measure of all angles coterminal to 112° differ from 112° by a multiple of 360°
- **19.** The length s of an arc made by an angle θ on a circle of radius r is given by the formula $\underline{s = r\theta}$, where θ is measured in $\underline{radians}$.
- **20.** To locate points on the surface of the Earth that are north or south of the equator, we measure the <u>latitude</u> of the point. To locate points that are east or west of the prime meridian, we measure the <u>longitude</u> of the point.
- **21.** The relationship $v = \frac{\text{arc length}}{\text{time}}$ represents the _____linear___ speed of a point traveling in a circular path.
- **22.** The symbol ω is typically used to denote <u>angular</u> speed and represents the number of radians per unit time that an object rotates.
- **23.** A wedge of a circle, similar in shape to a slice of pie, is called a(n) <u>sector</u> of the circle.
- **24.** The area A of a sector of a circle of radius r with central angle θ is given by the formula $A = \frac{1}{2}r^2\theta$, where θ is measured in <u>radians</u>.

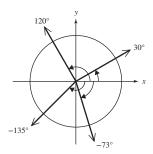
Learn: Find Degree Measure

For Exercises 25–26, sketch the angles in standard position.

- **25. a.** 60°
- **b.** 225°
- **c.** −210°
- **d.** -86°



- **b.** 120°
- **c.** -135°
- **d.** -73°



For Exercises 27–30, convert the given angle to decimal degrees. Round to 4 decimal places. (See Example 1, p. 449)

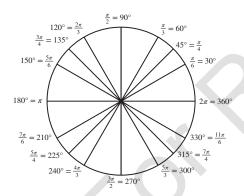
- **27.** 17°34′ 17.5667°
- **28.** 215°47′ 215.7833°
- **29.** 54°36′55″ 54.6153°
- **30.** 23°42′48″ 23.7133°

For Exercises 31-34, convert the given angle to DMS (degree-minutesecond) form. Round to the nearest second if necessary. (See Example 2, p. 449)

- **31.** 46.418° 46°25′5″
- **32.** 82.074° 82°4′26″
- **33.** -84.64° -84°38′24″

Learn: Find Radian Measure

35. Use the conversion factor $\pi = 180^{\circ}$ along with the symmetry of the circle to complete the degree or radian measure of the missing angles.



- **36.** Insert the appropriate symbol <, >, or = in the blank.
 - **a.** $\frac{5\pi}{6}$ 120° >
 - **b.** $-\frac{4\pi}{3} \Box 270^{\circ} >$

For Exercises 37–40, convert from degrees to radians. Give the answers in exact form in terms of π . (See Example 3, p. 452)

- **37.** 75° $\frac{5\pi}{12}$
- **39.** -210° $-\frac{7\pi}{6}$
- **38.** 240° $\frac{4\pi}{3}$ **40.** -195° $-\frac{13\pi}{12}$

For Exercises 45-56, convert from radians to decimal degrees. Round to 1 decimal place if necessary. (See Example 4, p. 453)

45.
$$\frac{\pi}{4}$$
 45°

46.
$$\frac{11\pi}{6}$$
 330°

47.
$$-\frac{5\pi}{3}$$
 -300°

46.
$$\frac{11\pi}{6}$$
 330° **48.** $-\frac{7\pi}{6}$ -210° **50.** $\frac{7\pi}{9}$ 140°

49.
$$\frac{5\pi}{18}$$
 50°

50.
$$\frac{7\pi}{9}$$
 140°

51.
$$-\frac{2\pi}{5}$$
 -72°

52.
$$-\frac{3\pi}{8}$$
 -67.5°

55.
$$\frac{9\pi}{2}$$
 810°

Learn: Determine Coterminal Angles

For Exercises 57-64, find a positive angle and a negative angle that is coterminal to the given angle. (See Examples 5 and 6, pp. 454-455)

61.
$$\frac{5\pi}{6}$$
 For example: $\frac{17\pi}{6}$, $-\frac{7\pi}{6}$ **62.** $\frac{3\pi}{4}$ For example: $\frac{11\pi}{4}$, $-\frac{5\pi}{4}$ **63.** $-\frac{3\pi}{2}$ For example: $\frac{\pi}{2}$, $-\frac{7\pi}{2}$ **64.** $-\pi$ For example: π , -3π

62.
$$\frac{3\pi}{4}$$
 For example: $\frac{11\pi}{4}$, $-\frac{5\pi}{4}$

63.
$$-\frac{3\pi}{2}$$
 For example: $\frac{\pi}{2}$, $-\frac{7\pi}{2}$

64.
$$-\pi$$
 For example: π . -3π

For Exercises 65–70, find an angle between 0° and 360° or between 0 and 2π that is coterminal to the given angle. (See Examples 5 and 6, pp. 454–455)

67.
$$\frac{17\pi}{4}$$

68.
$$\frac{11\pi}{3}$$
 $\frac{5\pi}{3}$

69.
$$-\frac{7\pi}{18}$$
 $\frac{29\pi}{18}$

70.
$$-\frac{5\pi}{9}$$
 $\frac{13\pi}{9}$

Learn: Compute Arc Length of a Sector of a Circle

For Exercises 71–74, find the exact length of the arc intercepted by a central angle θ on a circle of radius r. Then round to the nearest tenth of a unit. (See Example 7, p. 456)

71.
$$\theta = \frac{\pi}{3}$$
, $r = 12$ cm 4π cm ≈ 12.6 cm

72.
$$\theta = \frac{5\pi}{6}$$
, $r = 4 \text{ m} \quad \frac{10\pi}{3} \text{ m} \approx 10.5 \text{ m}$

73.
$$\theta = 135^{\circ}$$
, $r = 10$ in. $\frac{15\pi}{2}$ in. ≈ 23.6 in.

74.
$$\theta = 315^{\circ}$$
, $r = 2$ yd $\frac{7\pi}{2}$ yd ≈ 11.0 yd

- **76.** A gear with a 1.2-cm radius moves through an angle of 220°15′. What distance does a point on the edge of the gear move? Round to the nearest tenth of a centimeter. 4.6 cm
- 77. Which location would best fit the coordinates 12°S, 77°W? b. Lima, Peru
 - a. Paris, France
- b. Lima, Peru
- c. Miami, Florida
- d. Moscow, Russia
- **78. a.** What is the geographical relationship between two points that have the same latitude? The points are due east-west of each other.
 - **b.** What is the geographical relationship between two points that have the same longitude? The points are due north-south of each other.

For Exercises 79–82, assume that the Earth is approximately spherical with radius 3960 mi. Approximate the distances to the nearest mile. (See Example 8, p. 457)

- **79.** Barrow, Alaska (71.3°N, 156.8°W), and Kailua, Hawaii (19.7°N, 156.1°W), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities. \approx 3566 mi
- **80.** Rochester, New York (43.2°N, 77.6°W), and Richmond, Virginia (37.5°N, 77.5°W), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities. \approx 394 mi
- **81.** Raleigh, North Carolina (35.8°N, 78.6°W), is located north of the equator, and Quito, Ecuador (0.3°S, 78.6°W), is located south of the equator. The longitudes are the same, indicating that the cities are due north-south of each other. Use the difference in latitude to approximate the distance between the cities. \approx 2495 mi
- **82.** Trenton, New Jersey (40.2°N, 74.8°W), is located north of the equator, and Ayacucho, Peru (13.2°S, 74.2°W), is located south of the equator. The longitudes are nearly the same, indicating that the cities are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities. \approx 3691 mi
- 83. A pulley is 16 cm in diameter.
 - **a.** Find the distance the load will rise if the pulley is rotated 1350°. Find the exact distance in terms of π and then round to the nearest centimeter. 60π cm \approx 188 cm
 - **b.** Through how many degrees should the pulley rotate to lift the load 100 cm? Round to the nearest degree. 716°

- 84. A pulley is 1.2 ft. in diameter.
 - a. Find the distance the load will rise if the pulley is rotated 630°. Find the exact distance in terms of π and then round to the nearest tenth of a foot. 2.1π ft ≈ 6.6 ft
 - b. Through how many degrees should the pulley rotate to lift the load 24 ft? Round to the nearest degree. 2292°
- 85. A hoist is used to lift a pallet of bricks. The drum on the hoist is 15 in. in diameter. How many degrees should the drum be rotated to lift the pallet a distance of 6 ft? Round to the nearest degree. 550°
- 86. A winch on a sailboat is 8 in. in diameter and is used to pull in the "sheets" (ropes used to control the corners of a sail). To the nearest degree, how far should the winch be turned to pull in 2 ft of rope? 344°

Before the widespread introduction of electronic devices to measure distances, surveyors used a subtense bar to measure a distance x that is not directly measurable. A subtense bar is a bar of known length h with marks or "targets" at either end. The surveyor measures the angle heta formed by the location of the surveyor's scope and the top and bottom of the bar (this is the angle subtended by the bar). Since the angle and height of the bar are known, right triangle trigonometry can be used to find the horizontal distance. Alternatively, if the distance from the surveyor to the bar is large, then the distance can be approximated by the radius r of the arc s intercepted by the bar. Use this information for Exercises 87–88.

- 87. A surveyor uses a subtense bar to find the distance across a river. If the angle of sight between the bottom and top marks on a 2-m bar is 57'18", approximate the distance across the river between the surveyor and the bar. Round to the nearest meter. 120 m
- 88. A surveyor uses a subtense bar to find the distance across a canyon. If the angle of sight between the bottom and top marks on a 2-m bar is 24'33", approximate the distance across the river between the surveyor and the bar. Round to the nearest meter. 280 m

Learn: Compute Linear and Angular Speed

- 89. A circular paddle wheel of radius 3 ft is lowered into a flowing river. The current causes the wheel to rotate at a speed of 12 rpm. To 1 decimal place,
 - a. What is the angular speed? (See Example 9, p. 459) $24\pi/\min \approx 75.4/\min$
 - **b.** Find the speed of the current in ft/min. 72π ft/min ≈ 226.2 ft/min
 - **c.** Find the speed of the current in mph. $\frac{9\pi}{11}$ mph ≈ 2.6 mph

- **a.** What is the angular speed? $8400\pi/\text{min}$
- **b.** How fast in in./min does a point on the edge of the hard drive spin? Give the exact speed and the speed rounded to the nearest in./min.

 $10,500\pi$ in./min $\approx 32,987$ in./min

- **91.** A $7\frac{1}{4}$ -in.-diameter circular saw has 24 teeth and spins at 5800 rpm.
 - **a.** What is the angular speed? $11,600\pi/\text{min}$
 - **b.** What is the linear speed of one of the "teeth" on the outer edge of the blade? Round to the nearest inch per minute. 132,104 in./min
- 92. On a weed-cutting device, a thick nylon line rotates on a spindle at 3000 rpm.
 - **a.** Determine the angular speed. $6000\pi/\text{min}$
 - **b.** Determine the linear speed (to the nearest inch per minute) of a point on the tip of the line if the line is 5 in. 94,248 in./min
- 93. A truck has 2.5-ft tires (in diameter).
 - **a.** What distance will the truck travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of a foot.

 2.5π ft ≈ 7.9 ft

b. How far will the truck travel with 10,000 rotations of the wheels? Give the exact distance and an approximation to the nearest foot.

 $25,000\pi \text{ ft} \approx 78,540 \text{ ft}$

- **c.** If the wheels turn at 672 rpm, what is the angular speed? $1344\pi/\text{min}$
- **d.** If the wheels turn at 672 rpm, what is the linear speed in feet per minute? Give the exact distance and an approximation to the nearest whole unit.

 1680π ft/min ≈ 5278 ft/min

- e. If the wheels turn at 672 rpm, what is the linear speed in miles per hour?
 Round to the nearest mile per hour. (*Hint*: 1 mi = 5280 ft and 1 hr = 60 min.)
 60 mph
- 94. A bicycle has 25-in. wheels (in diameter).
 - **a.** What distance will the bicycle travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of an inch.

 25π in. ≈ 78.5 in.

b. How far will the bicycle travel with 200 rotations of the wheels? Give the exact distance and approximations to the nearest inch and nearest foot.

 5000π in. $\approx 15,708$ in. ≈ 1309 ft

d. If the wheels turn at 80 rpm, what is the linear speed in inches per minute? Give the exact speed and an approximation to the nearest inch per minute.

 2000π in./min ≈ 6283 in./min

e. If the wheels turn at 80 rpm, what is the linear speed in miles per hour? Round to the nearest mile per hour. (*Hint*: 1 ft = 12 in., 1 mi = 5280 ft, and 1 hr = 60 min.) 6 mph

Learn: Compute the Area of a Sector of a Circle

For Exercises 95–98, find the exact area of the sector. Then round the result to the nearest tenth of a unit. (See Example 10, p. 460)

- **95.** $30\pi \text{ m}^2 \approx 94.2 \text{ m}^2$
- **96.** $\frac{3\pi}{25}$ ft² ≈ 0.4 ft²
- **97.** $3\pi \text{ cm}^2 \approx 9.4 \text{ cm}^2$
- **98.** $10\pi \text{ m}^2 \approx 31.4 \text{ m}^2$
- **99.** A slice of a circular pizza 12 in. in diameter is cut into a wedge with a 45° angle. Find the area and round to the nearest tenth of a square inch. (See Example 10, p. 460) 14.1 in.²
- **100.** A circular cheesecake 9 in. in diameter is cut into a slice with a 20° angle. Find the area and round to the nearest tenth of a square inch. 3.5 in.²
- **101.** The back wiper blade on an SUV extends 3 in. from the pivot point to a distance of 17 in. from the pivot point. If the blade rotates through an angle of 175°, how much area does it cover? Round to the nearest square inch.

 428 in.²
- 102. A robotic arm rotates through an angle of 160°. It sprays paint between a distance of 0.5 ft and 3 ft from the pivot point. Determine the amount of area that the arm makes. Round to the nearest square foot. 12 ft²

Mixed Exercises

For Exercises 103–108, find the (a) complement and (b) supplement of the given angle.

- **103.** 16.21°
- **a.** 73.79°
- **b.** 163.79°

- **104.** 49.87°
- **a.** 40.13°
- **b.** 130.13°

- **105.** 18°13′37″
- **a.** 71°46′23″
- **b.** 161°46′23″

- **106.** 22°9′54″
- **a.** 67°50′6″
- **b.** 157°50′6″

- **107.** 9°42′7″
- **a.** 80°17′53″
- **b.** 170°17′53″

- **108.** 82°15′3″
- **a.** 7°44′57″
- **b.** 97°44′57″

- 109. The second hand of a clock moves from 12:10 to 12:30.
 - **a.** How many degrees does it move during this time? 120°
 - **b.** How many radians does it move during this time? $\frac{2\pi}{3}$
 - **c.** If the second hand is 10 in. in length, determine the exact distance that the tip of the second hand travels during this time. $\frac{20\pi}{3}$ in.
 - **d.** Determine the exact angular speed of the second hand in radians per second. $\frac{\pi}{30}/\text{sec}$
 - **e.** What is the exact linear speed (in inches per second) of the tip of the second hand? $\frac{\pi}{3}$ in./sec
 - **f.** What is the amount of area that the second hand sweeps out during this time? Give the exact area in terms of π and then approximate to the nearest square inch. $\frac{100\pi}{3}$ in.² \approx 105 in.²
- 110. The minute hand of a clock moves from 12:10 to 12:15.
 - **a.** How many degrees does it move during this time? 30°
 - **b.** How many radians does it move during this time? $\frac{\pi}{6}$
 - **c.** If the minute hand is 9 in. in length, determine the exact distance that the tip of the minute hand travels during this time. $\frac{3\pi}{2}$ in.
 - **d.** Determine the exact angular speed of the minute hand in radians per minute. $\frac{\pi}{30}$ /min
 - **e.** What is the exact linear speed (in inches per minute) of the tip of the minute hand? $\frac{3\pi}{10}$ in./min
 - **f.** What is the amount of area that the minute hand sweeps out during this time? Give the exact area in terms of π and then approximate to the nearest square inch. $\frac{27\pi}{4}$ in.² ≈ 21 in.²
 - **111.** The Earth's orbit around the Sun is elliptical (oval shaped). However, the elongation is small, and for our discussion here, we take the orbit to be circular with a radius of approximately 93,000,000 mi.
 - **a.** Find the linear speed (in mph) of the Earth through its orbit around the Sun. Round to the nearest hundred miles per hour. 66,700 mph
 - **b.** How far does the Earth travel in its orbit in one day? Round to the nearest thousand miles. 1,601,000 mi
- 112. The Earth completes one full rotation around its axis (poles) each day.
 - **a.** Determine the angular speed (in radians per hour) of the Earth during its rotation around its axis. $\frac{\pi}{12}/hr$
 - b. The Earth is nearly spherical with a radius of approximately 3960 mi. Find the linear speed of a point on the surface of the Earth rounded to the nearest mile per hour. 1037 mph

Copyright © McGraw Hill

- **113.** Two gears are calibrated so that the smaller gear drives the larger gear. For each rotation of the smaller gear, how many degrees will the larger gear rotate? 144°
- **114.** Two gears are calibrated so that the larger gear drives the smaller gear. The larger gear has a 6-in. radius, and the smaller gear has a 1.5-in. radius. For each rotation of the larger gear, by how many degrees will the smaller gear rotate? 1440°
- **115.** A spinning-disc confocal microscope contains a rotating disk with multiple small holes arranged in a series of nested Archimedean spirals. An intense beam of light is projected through the holes, enabling biomedical researchers to obtain detailed video images of live cells. The spinning disk has a diameter of 55 mm and rotates at a rate of 1800 rpm. At the edge of the disk,
 - **a.** Find the angular speed. $3600\pi/\text{min}$
 - **b.** Find the linear speed. Round to the nearest whole unit. 311,018 mm/min

For Exercises 116–119, approximate the area of the shaded region to 1 decimal place. In the figure, *s* represents arc length, and *r* represents the radius of the circle.

```
116. s = 28 \text{ in.} 320.9 in.<sup>2</sup>

117. s = 9 \text{ cm} 21.5 cm<sup>2</sup>

118. s = 18 \text{ m} 40.5 m<sup>2</sup>

119. s = 4.5 \text{ in.} 3.4 in.<sup>2</sup>
```

Write About It

120. Explain what is meant by 1 radian. Explain what is meant by 1°.

One radian is the angular measure of a central angle that intercepts an arc equal in length to the radius of the circle. One degree is the angular measure equal to $\frac{1}{360}$ of a full rotation.

121. For an angle drawn in standard position, explain how to determine in which quadrant the terminal side lies.

The four quadrants are divided by 0°, 90°, 180°, 270°, and 360° angles (likewise $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$, and 2π radians). Compare the measure of the given angle to these benchmarks. It may first be necessary to find an angle between 0° and 360° (or 0 and 2π radians) coterminal to the given angle.

122. As the fan rotates (see figure), which point *A* or *B* has a greater angular speed? Which point has a greater linear speed? Why?

Points *A* and *B* travel in circular paths around the center of the fan and have the same angular speed. The radius for the path of *B* is greater than the radius for the path of *A*. Therefore, point *B* travels a greater distance per unit time, and has the greater linear speed.

- 123. If an angle of a sector is held constant, but the radius is doubled, how will the arc length of the sector and area of the sector be affected?
 - The arc length will also double because it varies directly as the radius ($s = r\theta$). However, the area will increase by four times because the area varies directly as the *square* of the radius $\left(A = \frac{1}{2}r^2\theta\right)$.
- 124. If an angle of a sector is doubled, but the radius is held constant, how will the arc length of the sector and the area of the sector be affected?
 - Both the arc length ($s = r\theta$) and the area $\left(A = \frac{1}{2}r^2\theta\right)$ will double because they vary directly as the angle θ .

Expanding Your Skills

- 125. When a person pedals a bicycle, the front sprocket moves a chain that drives the back wheel and propels the bicycle forward. For each rotation of the front sprocket, the chain moves a distance equal to the circumference of the front sprocket. The back sprocket is smaller, so it will simultaneously move through a greater rotation. Furthermore, since the back sprocket is rigidly connected to the back wheel, each rotation of the back sprocket generates a rotation of the wheel.
 - Suppose that the front sprocket of a bicycle has a 4-in. radius and the back sprocket has a 2-in. radius.
 - a. How much chain will move with one rotation of the pedals (front sprocket)? 8π in. ≈ 25.1 in.
 - b. How many times will the back sprocket turn with one rotation of the pedals? 2 times
 - c. How many times will the wheels turn with one rotation of the pedals? 2 times
 - d. If the wheels are 27 in. in diameter, how far will the bicycle travel with one rotation of the pedals?
 - 54π in. \approx 169.6 in. or equivalently 4.5π ft \approx 14.1 ft
 - e. If the bicyclist pedals 80 rpm, what is the linear speed (in ft/min) of the bicycle? 360π ft/min \approx 1131 ft/min
 - **f.** If the bicyclist pedals 80 rpm, what is the linear speed (in mph) of the bicycle? (*Hint*: 1 mi = 5280 ft, and 1 hr = 60 min) $\frac{45\pi}{11}$ mph \approx 12.9 mph

- **126.** In the third century B.C., the Greek astronomer Eratosthenes approximated the Earth's circumference. On the summer solstice at noon in Alexandria, Egypt, Eratosthenes measured the angle α of the Sun relative to a line perpendicular to the ground. At the same time in Syene (now Aswan), located on the Tropic of Cancer, the Sun was directly overhead.
 - **a.** If $\alpha = \frac{1}{50}$ of a circle, find the measure of α in degrees. (In Eratosthenes' time, the degree measure had not yet been defined.) 7.2°
 - **b.** If the distance between Alexandria and Syene is 5000 stadia, find the circumference of the Earth measured in stadia. 250,000 stadia
 - c. If 10 stadia \approx 1 mi, find Eratosthenes' approximation of the circumference of the Earth in miles (the modern-day approximation at the equator is 24,900 mi). 25,000 mi
- **127.** The space shuttle program involved 135 crewed space flights in 30 yr. In addition to supplying and transporting astronauts to the International Space Station, space shuttle missions serviced the Hubble Space Telescope and deployed satellites. For a particular mission, a space shuttle orbited the Earth in 1.5 hr at an altitude of 200 mi.
 - **a.** Determine the angular speed (in radians per hour) of the shuttle. $\frac{4\pi}{3}$ /hr
 - b. Determine the linear speed of the shuttle in miles per hour. Assume that the Earth's radius is 3960 mi. Round to the nearest hundred miles per hour. 17,400 mph
- **128.** What is the first time (to the nearest second) after 12:00 midnight for which the minute hand and hour hand of a clock make a 120° angle? 12:21:49

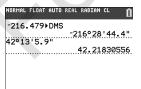
Technology Connections

For Exercises 129–130, use the functions in the ANGLE menu on your calculator to

- a. Convert from decimal degrees to DMS (degree-minute-second) form and
- b. Convert from DMS form to decimal degrees.
- **129. a.** -216.479°

130. a. −14.908°

b. 42°13′5.9″



b. 71°19′4.7″

| NORMAL FLOAT AUTO | REAL RADIA | AN CL | Û |
|-------------------|------------|-------|-----|
| -14.908 DMS | | | |
| 71°19'4.7" | -14°5 | 4 28 | 8". |
| | 71.3 | 1797 | 222 |
| | | | |
| | | | |
| | | | |

For Exercises 131–132, use a calculator to convert from degrees to radians.

- **131. a.** 147°26′9″
 - **b.** −228.459°

| NORMAL FLOAT AUTO R | EAL RADIAN CL 👖 |
|--|-----------------|
| (147°26'9")°r | |
| ,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,,, | 2.573240727 |
| (-228.459)°r | -3.987361756 |
| | |
| | |
| | |
| | |

Calculator in radian mode.

- **132. a.** 36°4′47″
 - **b.** -25.716°

| (36°4'47")°r | |
|--------------|-----------|
| (-25.716)°r | .62970994 |
| (-25./16)** | 448828870 |
| | |
| | |
| | |

Calculator in radian mode.

For Exercises 133–134, use a calculator to convert from radians to degrees.

- **133.** a. $\frac{4\pi}{9}$
 - **b.** -5.718

Calculator in degree mode.

- **134.** a. $\frac{11\pi}{18}$
 - **b.** -1.356

| (11π/18) ^r ° | | |
|-------------------------|------------|-------|
| | | 110 |
| (-1.356)r° | 77.693 | 07702 |
| | | |
| | | |
| | | |

Calculator in degree mode.

Lesson 4-2 Trigonometric Functions Defined on the Unit Circle

Teaching Tips

Each fully worked example is paired with end-of-lesson exercises that mirror the example. These end-of-lesson exercises are excellent problems to work through together in class.

| Example | Exercise |
|---------|----------|
| 1 | 16 |
| 2 | 22 |
| 3 | 40 |
| 4 | 44 |
| 5 | 50 |
| 6 | 56 |
| 7 | 60 |
| 8 | 70, 72 |
| 9 | 76 |

Lesson 4-2 Apply the Skills

1. Suppose that the real number t corresponds to the point $P\left(\frac{\sqrt{5}}{5}, \frac{2\sqrt{5}}{5}\right)$ on the unit circle. Evaluate the six trigonometric functions of t.

$$\sin t = \frac{2\sqrt{5}}{5}, \cos t = \frac{\sqrt{5}}{5},$$

$$\tan t = 2, \csc t = \frac{\sqrt{5}}{2},$$

$$\sec t = \sqrt{5}$$
, $\cot t = \frac{1}{2}$

2. Evaluate the six trigonometric functions of the real number t.

a.
$$t = \frac{5\pi}{6}$$

$$\sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\tan\frac{5\pi}{6} = -\frac{\sqrt{3}}{3}$$

$$\csc\frac{5\pi}{6} = 2$$

$$\sec\frac{5\pi}{6} = -\frac{2\sqrt{3}}{3}$$

$$\cot\frac{5\pi}{6} = -\sqrt{3}$$

b.
$$t = -\frac{3\pi}{4}$$

$$\sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

$$\tan\left(-\frac{3\pi}{4}\right) = 1$$

$$\csc\left(-\frac{3\pi}{4}\right) = -\sqrt{2}$$

$$\sec\left(-\frac{3\pi}{4}\right) = -\sqrt{2}$$

$$\cot\left(-\frac{3\pi}{4}\right) = 1$$

- **3.** Evaluate the six trigonometric functions of the real number t.
 - **a.** $t = -2\pi$

$$\sin(-2\pi) = 0$$
, $\cos(-2\pi) = 1$, $\tan(-2\pi) = 0$, $\csc(-2\pi)$ is undefined, $\sec(-2\pi) = 1$, $\cot(-2\pi)$ is undefined

b.
$$\frac{3\pi}{2}$$

$$\sin \frac{3\pi}{2} = -1$$
, $\cos \frac{3\pi}{2} = 0$, $\tan \frac{3\pi}{2}$ is undefined, $\csc \frac{3\pi}{2} = -1$, $\sec \frac{3\pi}{2}$ is undefined, $\cot \frac{3\pi}{2} = 0$

$$\cot\frac{3\pi}{2} = 0$$

4. Given
$$\sin t = \frac{3}{7}$$
 and $\cos t = \frac{2\sqrt{10}}{7}$ use the reciprocal and quotient identities to find the values of the other trigonometric functions of t .

$$\tan t = \frac{3\sqrt{10}}{20}$$
, $\cot t = \frac{2\sqrt{10}}{3}$, $\csc t = \frac{7}{3}$, $\sec t = \frac{7\sqrt{10}}{20}$

5. Given that $\csc t = \frac{5}{4}$ for $\frac{\pi}{2} < t < \pi$, use an appropriate identity to find the value of cot t.

$$\cot t = -\frac{3}{4}$$

6. For a given real number t, express tant in terms of sect.

$$\tan t = \pm \sqrt{\sec^2 t - 1}$$

7. Given
$$\sin \frac{\pi}{8} = \frac{\sqrt{2 - \sqrt{2}}}{2}$$
, determine the value of $\sin \left(-\frac{15\pi}{8} \right)$.

$$\frac{\sqrt{2-\sqrt{2}}}{2}$$

8. Use the properties of the trigonometric functions to simplify.

a.
$$\sec t - 2\sec(-t)$$
 $-\sec t$

b.
$$\sin(2t + 2\pi) - \sin(-2t)$$
 2 $\sin 2t$

- 9. Use a calculator to approximate the function values. Round to 4 decimal places.
 - **a.** tan 1.4 5.7979
- **b.** $\cot \frac{\pi}{8}$ 2.4142

Lesson 4-2 Practice

Prerequisite Review

For Exercises R.1–R.3, determine if the function is even, odd, or neither.

R.1.
$$k(x) = 11x^3 + 12x$$
 Odd

R.2.
$$r(x) = \sqrt{25 - (x + 2)^2}$$
 Neither

R.3.
$$q(x) = \sqrt{23 + x^2}$$
 Even

For Exercises R.4–R.5, write the domain of the function in set-builder

R.4.
$$p(v) = \frac{v+3}{v+4}$$
 {v| v is a real number and $v \neq -4$ }

R.4.
$$p(v) = \frac{v+3}{v+4}$$
 { $v \mid v$ is a real number and $v \neq -4$ }
R.5. $k(q) = \frac{q^2}{2q^2 + 3q - 27}$ { $q \mid q$ is a real number and $q \neq -\frac{9}{2}$, $q \neq 3$ }

Concept Connections

- **1.** The graph of $x^2 + y^2 = 1$ is known as the _____ circle. It has a radius of length _____1 and center at the ___origin__.
- **2.** The circumference of the unit circle is $\underline{2\pi}$. Thus, the value $t = \frac{\pi}{2}$ represents $\frac{1}{4}$ of a revolution.
- **3.** When $\tan t = 0$, the value of $\cot t$ is <u>undefined</u>.
- **4.** The value of $\sec t = \frac{1}{\cos t}$. Therefore, when $\cos t = 0$, the value of $\sec t$ is
- **5.** On the interval $[0, 2\pi)$, $\sin t = 0$ for t = 0 and $t = \pi$. Since $\csc t = \frac{1}{\sin t}$, the value of $\csc t$ is <u>undefined</u> for these values of t.
- **6.** The domain of the trigonometric functions ____sine__ and __cosine__ is all real numbers.
- 7. Which trigonometric functions have domain

$$\left\{t \mid t \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n\right\}$$
? tangent and secant

- **8.** A function f is <u>periodic</u> if f(t + p) = f(t) for some constant p.
- **9.** The period of the tangent and cotangent functions is $\frac{\pi}{2\pi}$. The period of the sine, cosine, cosecant, and secant functions is $\frac{2\pi}{2\pi}$.
- <u>even</u> function because cos(-t) = cos t. The **10.** The cosine function is an _ sine function is an <u>odd</u> function because $\sin(-t) = -\sin t$.

Learn: Evaluate Trigonometric Functions Using the Unit Circle

For Exercises 11–14, determine if the point lies on the unit circle.

11.
$$\left(\frac{\sqrt{10}}{5}, \frac{\sqrt{15}}{5}\right)$$
 Yes

12.
$$\left(\frac{\sqrt{61}}{8}, -\frac{\sqrt{2}}{8}\right)$$
 No

13.
$$\left(-\frac{\sqrt{7}}{10}, -\frac{2\sqrt{23}}{10}\right)$$
 No

11.
$$\left(\frac{\sqrt{10}}{5}, \frac{\sqrt{15}}{5}\right)$$
 Yes **12.** $\left(\frac{\sqrt{61}}{8}, -\frac{\sqrt{2}}{8}\right)$ No **13.** $\left(-\frac{\sqrt{7}}{10}, -\frac{2\sqrt{23}}{10}\right)$ No **14.** $\left(-\frac{2\sqrt{34}}{17}, \frac{3\sqrt{17}}{17}\right)$ Yes

For Exercises 15–18, the real number t corresponds to the point P on the unit circle. Evaluate the six trigonometric functions of t. (See Example 1, p. 472)

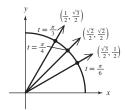
15.
$$\sin t = \frac{\sqrt{7}}{4}$$
, $\cos t = -\frac{3}{4}$, $\tan t = -\frac{\sqrt{7}}{3}$, $\csc t = \frac{4\sqrt{7}}{7}$, $\sec t = -\frac{4}{3}$, $\cot t = -\frac{3\sqrt{7}}{7}$

16.
$$\sin t = \frac{7}{25}$$
, $\cos t = -\frac{24}{25}$, $\tan t = -\frac{7}{24}$. $\csc t = \frac{25}{7}$, $\sec t = -\frac{25}{24}$, $\cot t = -\frac{24}{7}$

17.
$$\sin t = -\frac{15}{17}$$
, $\cos t = -\frac{8}{17}$, $\tan t = \frac{15}{8}$, $\csc t = -\frac{17}{15}$, $\sec t = -\frac{17}{8}$, $\cot t = \frac{8}{15}$

18.
$$\sin t = -\frac{3\sqrt{38}}{19}$$
, $\cos t = -\frac{\sqrt{19}}{19}$, $\tan t = 3\sqrt{2}$, $\csc t = -\frac{\sqrt{38}}{6}$, $\sec t = -\sqrt{19}$, $\cot t = \frac{\sqrt{2}}{6}$

19. Fill in the ordered pairs on the unit circle corresponding to each real number t.



For Exercises 20–23, identify the coordinates of point P. Then evaluate the six trigonometric functions of t. (See Example 2, p. 477)

20.
$$P\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$
; $\sin t = \frac{1}{2}$, $\cos t = -\frac{\sqrt{3}}{2}$, $\tan t = -\frac{\sqrt{3}}{3}$,

$$\csc\theta = 2, \ \sec\theta = -\frac{2\sqrt{3}}{3}, \ \cot\theta = -\sqrt{3}$$
 21. $P\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$$\sin t = -\frac{\sqrt{3}}{2}$$
, $\cos t = -\frac{1}{2}$, $\tan t = \sqrt{3}$,
 $\csc t = -\frac{2\sqrt{3}}{3}$, $\sec t = -2$, $\cot t = \frac{\sqrt{3}}{3}$

22.
$$P\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

 $\sin t = -\frac{\sqrt{2}}{2}, \cos t = -\frac{\sqrt{2}}{2}, \tan t = 1,$
 $\csc t = -\sqrt{2}, \sec t = -\sqrt{2}, \cot t = 1$

23.
$$P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$$

 $\sin t = \frac{\sqrt{2}}{2}, \cos t = \frac{\sqrt{2}}{2}, \tan t = 1,$
 $\csc t = \sqrt{2}, \sec t = \sqrt{2}, \cot t = 1$

For Exercises 24–26, identify the ordered pairs on the unit circle corresponding to each real number t.

24. a.
$$t = \frac{2\pi}{3} \left(-\frac{1}{2}, \frac{\sqrt{3}}{2} \right)$$

24. a.
$$t = \frac{2\pi}{3} \quad \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$
 b. $t = -\frac{5\pi}{4} \quad \left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ c. $t = \frac{5\pi}{6} \quad \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

c.
$$t = \frac{5\pi}{6} \quad \left(-\frac{\sqrt{3}}{2}, \frac{1}{2} \right)$$

25. a.
$$t = \frac{7\pi}{6} \quad \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$
 b. $t = -\frac{3\pi}{4} \quad \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ c. $t = \frac{4\pi}{3} \quad \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

b.
$$t = -\frac{3\pi}{4} \quad \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2} \right)$$

c.
$$t = \frac{4\pi}{3} \quad \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2} \right)$$

26. a.
$$t = \frac{5\pi}{3}$$
 $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ b. $t = \frac{7\pi}{4}$ $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ c. $t = -\frac{\pi}{6}$ $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

b.
$$t = \frac{7\pi}{4} \quad \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$$

c.
$$t = -\frac{\pi}{6}$$
 $\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

For Exercises 27-32, evaluate the trigonometric function at the given real number.

27.
$$f(t) = \cos t$$
; $t = \frac{2\pi}{3}$ $f(\frac{2\pi}{3}) = -\frac{1}{2}$

28.
$$g(t) = \sin t$$
; $t = \frac{5\pi}{4}$ $g(\frac{5\pi}{4}) = -\frac{\sqrt{2}}{2}$

29.
$$h(t) = \cot t$$
; $t = \frac{11\pi}{6}$ $h(\frac{11\pi}{6}) = -\sqrt{3}$

30.
$$s(t) = \sec t$$
; $t = \frac{5\pi}{3}$ $s(\frac{5\pi}{3}) = 2$

31.
$$z(t) = \csc t$$
; $t = \frac{7\pi}{4}$ $z(\frac{7\pi}{4}) = -\sqrt{2}$

32.
$$r(t) = \tan t$$
; $t = \frac{5\pi}{6}$ $r(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{3}$

Learn: Identify the Domains of the Trigonometric Functions

For Exercises 33-38, select the domain of the trigonometric function.

b.
$$\left\{ t | t \neq \frac{(2n+1)\pi}{2} \text{ for all integers } n \right\}$$

c. $\left\{ t | t \neq n\pi \text{ for all integers } n \right\}$

c.
$$\{t|t \neq n\pi \text{ for all integers } n\}$$

33.
$$f(t) = \sin t$$
 a

34.
$$f(t) = \tan t$$
 b

35.
$$f(t) = \cot t$$
 c

36.
$$f(t) = \cos t$$
 a

37.
$$f(t) = \sec t$$
 b

38.
$$f(t) = \csc t$$
 c

For Exercises 39-42, evaluate the function if possible. (See Example 3, p. 479)

b.
$$\cot \pi$$
 Undefined **c.** $\tan 3\pi$ 0

c.
$$tan 3\pi$$

$$\tau$$
. $\cos \pi$ –

d.
$$\sec \pi - 1$$
 e. $\csc 0$ Undefined **f.** $\cos \pi - 1$ **40. a.** $\cos \left(-\frac{\pi}{2} \right)$ 0 **b.** $\csc \frac{\pi}{2}$ 1 **c.** $\cot \frac{5\pi}{2}$ 0

b.
$$\csc \frac{\pi}{2}$$

c.
$$\cot \frac{3\pi}{2} = 0$$

d.
$$\tan \frac{\pi}{2}$$
 Undefined

e.
$$\sec \frac{3\pi}{2}$$
 Und

f.
$$\sin \frac{\pi}{2}$$

41. a.
$$\sin \frac{3\pi}{2}$$
 -1

b.
$$\cos \frac{7\pi}{2}$$
 (

d.
$$\tan \frac{\pi}{2}$$
 Undefined e. $\sec \frac{3\pi}{2}$ Undefined f. $\sin \frac{\pi}{2}$ 1

41. a. $\sin \frac{3\pi}{2}$ -1 b. $\cos \frac{7\pi}{2}$ 0 c. $\tan \frac{3\pi}{2}$ Undefined d. $\csc \left(-\frac{\pi}{2}\right)$ -1 e. $\sec 1.5\pi$ Undefined f. $\cot \frac{\pi}{2}$ 0

d.
$$\csc\left(-\frac{\pi}{2}\right)$$
 -1

e. Sec
$$1.5\pi$$
 Under

f.
$$\cot \frac{\pi}{2}$$
 0

42. a. cot 0 Undefined **b.** cos
$$2\pi$$
 1

b.
$$\cos 2\pi$$

c.
$$\csc \pi$$
 Undefined

e.
$$\sin(-3\pi)$$
 0

Learn: Use Fundamental Trigonometric Identities

For Exercises 43–46, given the values for $\sin t$ and $\cos t$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of t. (See Example 4, p. 480)

43.
$$\sin t = \frac{\sqrt{5}}{3}$$
 and $\cos t = \frac{2}{3}$ $\tan t = \frac{\sqrt{5}}{2}$, $\csc t = \frac{3\sqrt{5}}{5}$, $\sec t = \frac{3}{2}$, $\cot t = \frac{2\sqrt{5}}{5}$

44.
$$\sin t = \frac{3}{4}$$
 and $\cos t = \frac{\sqrt{7}}{4}$ $\tan t = \frac{3\sqrt{7}}{7}$, $\csc t = \frac{4}{3}$, $\sec t = \frac{4\sqrt{7}}{7}$, $\cot t = \frac{\sqrt{7}}{3}$

45.
$$\sin t = -\frac{\sqrt{39}}{8}$$
 and $\cos t = -\frac{5}{8}$ $\tan t = \frac{\sqrt{39}}{5}$, $\sec t = -\frac{8}{5}$, $\csc t = -\frac{8\sqrt{39}}{39}$, $\cot t = \frac{5\sqrt{39}}{39}$

46.
$$\sin t = \frac{28}{53}$$
 and $\cos t = -\frac{45}{53}$ $\tan t = -\frac{28}{45}$, $\sec t = -\frac{53}{45}$, $\csc t = \frac{53}{28}$, $\cot t = -\frac{45}{28}$

For Exercises 47–48, derive the given identity from the Pythagorean identity, $\sin^2 t + \cos^2 t = 1$.

47.
$$tan^2t + 1 = sec^2t$$

Divide both sides of $\sin^2 t + \cos^2 t = 1$ by $\cos^2 t$.

$$\frac{\sin^2 t}{\sin^2 t} + \frac{\cos^2 t}{\sin^2 t} = \frac{1}{\sin^2 t}$$

$$1 + \cot^2 t = \csc^2 t$$

48.
$$1 + \cot^2 t = \csc^2 t$$

Divide both sides of
$$\sin^2 t + \cos^2 t = 1$$
 by $\sin^2 \theta$.
$$\frac{\sin^2 t}{\cos^2 t} + \frac{\cos^2 t}{\cos^2 t} = \frac{1}{\cos^2 t}$$

$$\tan^2 t + 1 = \sec^2 t$$

For Exercises 49–54, use an appropriate Pythagorean identity to find the indicated value. (See Example 5, p. 481)

49. Given
$$\cos t = -\frac{7}{25}$$
 for $\frac{\pi}{2} < t < \pi$, find the value of $\sin t$. $\frac{24}{25}$

50. Given
$$\sin t = -\frac{8}{17}$$
 for $\pi < t < \frac{3\pi}{2}$, find the value of $\cos t$. $-\frac{15}{17}$

51. Given
$$\cot t = \frac{45}{28}$$
 for $\pi < t < \frac{3\pi}{2}$, find the value of csc t . $-\frac{53}{28}$

52. Given
$$\csc t = -\frac{41}{40}$$
 for $\frac{3\pi}{2} < t < 2\pi$, find the value of $\cot t$. $-\frac{9}{40}$

53. Given
$$\tan t = -\frac{11}{60}$$
 for $\frac{3\pi}{2} < t < 2\pi$, find the value of $\sec t$. $\frac{61}{60}$

54. Given
$$\sec t = -\frac{37}{35}$$
 for $\frac{\pi}{2} < t < \pi$, find the value of $\tan t$. $-\frac{12}{35}$

a.
$$t$$
 in Quadrant I. (See Example 6, p. 482) $\sin t = \sqrt{1 - \cos^2 t}$

b. t in Quadrant III.
$$\sin t = -\sqrt{1 - \cos^2 t}$$

- **a.** t in Quadrant II. $\tan t = -\sqrt{\sec^2 t 1}$
- **b.** t in Quadrant IV. $\tan t = -\sqrt{\sec^2 t 1}$
- **57.** Write cot *t* in terms of csc *t* for
 - **a.** t in Quadrant I. $\cot t = \sqrt{\csc^2 t 1}$
 - **b.** t in Quadrant III. $\cot t = \sqrt{\csc^2 t 1}$
- **58.** Write cos *t* in terms of sin *t* for
 - **a.** t in Quadrant II. $\cos t = -\sqrt{1 \sin^2 t}$
 - **b.** t in Quadrant IV. $\cos t = \sqrt{1 \sin^2 t}$

Learn: Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions

- **59.** Given that $\cos \frac{29\pi}{12} = \frac{\sqrt{6} \sqrt{2}}{4}$, determine the value of $\cos \frac{5\pi}{12}$. (See Example 7,
- **60.** Given that $\sec \frac{11\pi}{12} = \sqrt{2} \sqrt{6}$, determine the value of $\sec \left(-\frac{13\pi}{12}\right)$. $\sqrt{2} \sqrt{6}$
- **61.** Given $\tan\left(-\frac{\pi}{8}\right) = 1 \sqrt{2}$, determine the value of $\cot\frac{7\pi}{8}$. $-1 \sqrt{2}$
- **62.** Given that $\sin \frac{\pi}{10} = \frac{\sqrt{5} 1}{4}$, determine the value of $\csc \frac{21\pi}{10}$. $1 + \sqrt{5}$

For exercises 63-66, use the periodic properties of the trigonometric functions to simplify each expression to a single function of t.

- **63.** $\sin(t + 2\pi) \cdot \cot(t + \pi) = \cos t$
- **64.** $\sin(t + 2\pi) \cdot \sec(t + 2\pi) \quad \tan t$
- **65.** $\csc(t + 2\pi) \cdot \cos(t + 2\pi) = \cot t$
- **66.** tan $(t + \pi) \cdot \csc(t + 2\pi)$ sec t

For Exercises 67–74, use the even-odd and periodic properties of the trigonometric functions to simplify. (See Example 8, p. 485)

- **67.** $\csc t 4 \csc (-t)$ 5 $\csc t$
- **68.** tan(-t) 3tan t 4tan t
- **69.** $-\cot(-t + \pi) \cot t$
- **70.** $\sec(t + 2\pi) \sec(-t)$ 0
- **71.** $-2\sin(3t + 2\pi) 3\sin(-3t) \sin 3t$ **72.** $\cot(-3t) 3\cot(3t + \pi) 4\cot 3t$
- **73.** $\cos(-2t) \cos 2t$ 0
- **74.** $\sec(-2t) + 3\sec 2t$ 4 $\sec 2t$

Learn: Approximate Trigonometric Functions on a Calculator

Use a calculator to approximate the function values. Round to 4 decimal places. (See Example 9, p. 486)

- **75. a.** sin (-0.15) -0.1494
- **b.** $\cos \frac{2\pi}{5}$ 0.3090
- **76. a.** $\cos\left(-\frac{7\pi}{11}\right)$ -0.4154 **b.** $\sin 0.96$ 0.8192

77. a.
$$\cot \frac{12\pi}{7}$$
 -0.7975 **b.** $\sec 5.43$ 1.5207

b.
$$\tan \frac{3\pi}{8}$$
 2.4142

Mixed Exercises

For Exercises 79–80, evaluate $\sin t$, $\cos t$, and $\tan t$ for the real number t.

79. a.
$$t = \frac{2\pi}{3}$$
 $\sin t = \frac{\sqrt{3}}{2}$, $\cos t = -\frac{1}{2}$, $\tan t = -\sqrt{3}$

b.
$$t = -\frac{4\pi}{3}$$
 $\sin t = \frac{\sqrt{3}}{2}$, $\cos t = -\frac{1}{2}$, $\tan t = -\sqrt{3}$

80. a.
$$t = -\frac{11\pi}{6}$$
 $\sin t = \frac{1}{2}$, $\cos t = \frac{\sqrt{3}}{2}$, $\tan t = \frac{\sqrt{3}}{3}$

b.
$$t = \frac{\pi}{6}$$
 $\sin t = \frac{1}{2}$, $\cos t = \frac{3}{2}$, $\tan t = \frac{\sqrt{3}}{3}$

For Exercises 81–86, identify the values of t on the interval $[0, 2\pi]$ that make the function undefined (if any).

81.
$$y = \sin t$$
 None

82.
$$y = \cot t = 0, \pi, 2$$

82.
$$y = \cot t$$
 0, π , 2π **83.** $y = \tan t$ $\frac{\pi}{2}$, $\frac{3\pi}{2}$

84.
$$y = \cos t$$
 None

85.
$$y = \csc t + 0, \pi, 2\pi$$
 86. $y = \sec t + \frac{\pi}{2}$,

86.
$$y = \sec t + \frac{\pi}{2}, \frac{3\pi}{2}$$

For Exercises 87-92, select all properties that apply to the trigonometric function.

- a. The function is even.
- **b.** The function is odd.
- **c.** The period is 2π .
- **d.** The period is π .
- e. The domain is all real numbers.
- **f.** The domain is all real numbers excluding odd multiples of $\frac{\pi}{2}$.
- **g.** The domain is all real numbers excluding multiples of π .

87.
$$f(t) = \sin(t)$$
 b, c, e

88.
$$f(t) = \tan(t)$$
 b, d, f

89.
$$f(t) = \sec(t)$$
 a, c, f

90.
$$f(t) = \cot(t)$$
 b, d, g

91.
$$f(t) = \csc(t)$$
 b, c, g

92.
$$f(t) = \cos(t)$$
 a, c, e

For Exercises 93–98, simplify using properties of trigonometric functions.

93.
$$\sin^2(t + 2\pi) + \cos^2 t + \tan^2(t + \pi)$$
 $\sec^2 t$

94.
$$\cot(-t) \cdot \sin(t + 2\pi) + \cos^2 t \cdot \sec(-t + 2\pi)$$
 0

95.
$$\sec^2\left(-\frac{\pi}{3}\right) + \tan^2\left(-\frac{\pi}{3}\right)$$

96.
$$\sin^2\left(\frac{\pi}{6}\right) - \cos^2\left(-\frac{\pi}{6}\right) - \frac{1}{2}$$

97.
$$\sec^2\left(-\frac{\pi}{4}\right) - \csc^2\left(\frac{7\pi}{6}\right)$$
 -

95.
$$\sec^2\left(-\frac{\pi}{3}\right) + \tan^2\left(-\frac{\pi}{3}\right)$$
 7 **96.** $\sin^2\left(\frac{\pi}{6}\right) - \cos^2\left(-\frac{\pi}{6}\right)$ - $\frac{1}{2}$ **97.** $\sec^2\left(-\frac{\pi}{4}\right) - \csc^2\left(\frac{7\pi}{6}\right)$ - 2 **98.** $\tan^2\left(\frac{2\pi}{3}\right) + \csc^2\left(-\frac{5\pi}{4}\right)$ 5

- **99.** In a small coastal town, the monthly revenue received from tourists rises and falls throughout the year. The tourist revenue peaks during the months when the town holds seafood cooking competitions. The function $f(t) = 5.6 \cos\left(\frac{\pi}{3}t\right) + 11.2$ represents the monthly revenue f(t), in tens of thousands of dollars, for a month t, where t = 0 represents April.
 - **a.** Complete the table and give the period of the function. Round to 1 decimal place. Period = 6 months

| t | 0 | 1 | 2 | 3 | 4 | 5 |
|------|------|----|-----|-----|-----|----|
| f(t) | 16.8 | 14 | 8.4 | 5.6 | 8.4 | 14 |

| t | 6 | 7 | 8 | 9 | 10 | 11 |
|------|------|----|-----|-----|-----|----|
| f(t) | 16.8 | 14 | 8.4 | 5.6 | 8.4 | 14 |

- **b.** What are the months of peak revenue, and what is the revenue for those months? Peak revenue is in month 0 (April) and month 6 (October). The revenue is \$168,000.
- **100.** The fluctuating brightness of a distant star is given by the function $f(d) = 3.8 + 0.25 \sin\left(\frac{2\pi}{3}d\right)$ where d is the number of days and f(d) is the apparent brightness. Complete the table and give the period of the function. Round to 2 decimal places. Period = 3 days

| d | 0 | 1 | 2 | 3 | 4 | 5 |
|------|------|------|------|------|------|------|
| f(d) | 3.80 | 4.02 | 3.58 | 3.80 | 4.02 | 3.58 |

For Exercises 101–104, identify each function as even, odd, or neither.

- **101.** $f(t) = t^2 \sin t$ Odd
- **102.** $g(t) = t \cos t$ Odd
- **103.** $z(t) = t^3 \tan t$ Even
- **104.** $h(t) = t^3 + \sec t$ Neither

Write About It

105. Describe the changes in $\sin t$ and $\cos t$ as t increases from 0 to $\frac{\pi}{2}$.

As t increases from 0 to $\frac{\pi}{2}$, the value of $\cos t$ decreases from 1 to 0 and the value of $\sin t$ increases from 0 to 1.

106. Describe the changes in cos t and sect as t increases from 0 to $\frac{\pi}{2}$.

As t increases from 0 to $\frac{\pi}{2}$, the value of cos t decreases from 1 to 0. The value of sec t increases from 1 when t=0 and becomes arbitrarily large as t approaches $\frac{\pi}{2}$. The value of sec t is undefined at $\frac{\pi}{2}$.

107. Explain why $-1 \le \cos t \le 1$ and $-1 \le \sin t \le 1$ for all real numbers t.

 $\cos t = x$ and $\sin t = y$ for (x, y), satisfying the equation $x^2 + y^2 = 1$. It follows that $|x| = \sqrt{1 - y^2}$ and $|y| = \sqrt{1 - x^2}$, which implies that $-1 \le x \le 1$ and $-1 \le y \le 1$.

108. Do the trigonometric functions satisfy the definition of a function learned in Chapter 1? Explain your answer.

Yes; each value of the independent variable corresponds to one and only one location on the unit circle, which is then used to evaluate the function.

Expanding Your Skills

For Exercises 109–114, use the figure to estimate the value of (a) $\sin t$ and (b) $\cos t$ for the given value of t.

109. t = 0.5 **a.** 0.48 **b.** 0.88 **110.** t = 1.25 **a.** 0.95 **b.** 0.32

111. t = 3.75 **a.** -0.57 **b.** -0.82 **112.** t = 2.75 **a.** 0.38 **b.** -0.92

113. t = 5 **a.** -0.96 **b.** 0.28 **114.** t = 3 **a.** 0.14 **b.** -0.99

For Exercises 115–120, use the figure from Exercises 109–114 to approximate the solutions to the equation over the interval $[0, 2\pi)$.

115. $\sin t = 0.2$ $t \approx 0.2$ and $t \approx 2.9$ **116.** $\sin t = -0.4$ $t \approx 3.6$ and $t \approx 5.9$

117. $\cos t = -0.4$ $t \approx 2.0$ and $t \approx 4.3$ **118.** $\cos t = 0.6$ $t \approx 0.9$ and $t \approx 5.4$

119. Show that $tan(t + \pi) = tan t$.

Let the real number t correspond to the point P(x, y) on the unit circle. Then $t + \pi$ corresponds to the point P(-x, -y) on the unit circle. By definition, $\tan t = \frac{y}{x}$ and $\tan (t + \pi) = \frac{-y}{-x} = \frac{y}{x}$. So we have shown that $\tan (t + \pi) = \tan t$.

120. Show that $\cot(t + \pi) = \cot t$

Let the real number t correspond to the point P(x, y) on the unit circle. Then $t + \pi$ corresponds to the point P(-x, -y) on the unit circle. By definition, $\cot t = \frac{x}{y}$ and $\cot (t + \pi) = \frac{-x}{-y} = \frac{x}{y}$. So we have shown that $\cot (t + \pi) = \cot t$.

121. Prove that the period of $f(t) = \sin t$ is 2π .

Let the real number t correspond to the point P(x, y) on the unit circle. Since the circumference of the unit circle is 2π , adding 2π to t results in the same terminal point (x, y). Consequently, $\sin(t + 2\pi) = \sin t$ for all t.

To show that 2π is the *smallest* positive value p for which $\sin(t+p)=\sin t$, we will do a proof by contradiction. Assume the contrary; that is, assume that there exists a value p on the interval $0 such that <math>\sin(t+p)=\sin t$. Substituting t=0 results in $\sin(0+p)=\sin 0$, or equivalently $\sin p=0$. The only solution on the interval $0 is <math>p=\pi$. Now substitute $t=\frac{\pi}{2}$ into the equation $\sin(t+p)=\sin t$. We have $\sin\left(\frac{\pi}{2}+\pi\right)=\sin\frac{\pi}{2}$. This is a contradiction because $\sin\frac{3\pi}{2}=-1$ but $\sin\frac{\pi}{2}=1$. Therefore, our original assumption that there exists a value p on the interval $0 such that <math>\sin(t+p)=\sin t$ is incorrect. The smallest possible value of p is 2π .

Let the real number t correspond to the point P(x, y) on the unit circle. Since the circumference of the unit circle is 2π , adding 2π to t results in the same terminal point (x, y). Consequently, $\cos(t + 2\pi) = \cos t$ for all t.

To show that 2π is the *smallest* positive value p for which $\cos(t+p)=\cos t$, we will do a proof by contradiction. Assume the contrary. That is, assume that there exists a value p on the interval $0 such that <math>\cos(t+p)=\cos t$. Substituting t=0 results in $\cos(0+p)=\cos 0$ or equivalently $\cos p=1$. This is a contradiction because there is no solution to this equation on the interval 0 . (The values of <math>p that would satisfy this equation are multiples of 2π which do not fall strictly between 0 and 2π .) Therefore, our original assumption that there exists a value p on the interval $0 such that <math>\cos(t+2\pi)=\cos t$ is incorrect. The smallest possible value of p is 2π .

123. Given that $\tan t = a$ for t in Quadrant II, write expressions for $\sin t$ and $\cos t$.

$$\sin t = -\frac{a}{\sqrt{a^2 + 1}}, \cos t = -\frac{1}{\sqrt{a^2 + 1}}$$

124. Given that $\cos t = a$ and that t is in Quadrant III, find the values of $\sin t$ and $\tan t$.

$$\sin t = -\sqrt{1-\alpha^2}$$
, $\tan t = -\frac{\sqrt{1-\alpha^2}}{\alpha}$

125. For an arbitrary value of t, write an expression for $\sin t$ and $\cos t$ in terms of $\tan t$.

$$\sin t = \pm \frac{\tan t}{\sqrt{1 + \tan^2 t}}, \cos t = \pm \frac{1}{\sqrt{1 + \tan^2 t}}$$

126. For an arbitrary value of t, write an expression for $\sin t$ and $\tan t$ in terms of $\cos t$.

$$\sin t = \pm \sqrt{1 - \cos^2 t}, \tan t = \pm \frac{\sqrt{1 - \cos^2 t}}{\cos t}$$

Technology Connections

127. a. Complete the table to show that $\sin t \approx t$ for values of t close to zero. Round to 5 decimal places.

| t | sin t | sin t t | |
|-------|---------|------------|--|
| 0.2 | 0.19867 | 0.99335 | |
| 0.1 | 0.09983 | 0.99833 | |
| 0.01 | 0.01000 | 0.99998 | |
| 0.001 | 0.00100 | 1.00000 | |

- **b.** What value does the ratio $\frac{\sin t}{t}$ seem to approach as $t \to 0$?
- **128.** Consider the expression $\frac{1-\cos t}{t}$.
 - **a.** What is the value of the numerator for t = 0? 0
 - **b.** What is the value of the denominator for t = 0?

c. The expression $\frac{1-\cos t}{t}$ is undefined at t=0.

Complete the table to investigate the value of the expression close to t=0. Round to 5 decimal places.

| t | $\frac{1-\cos t}{t}$ |
|-------|----------------------|
| 0.2 | 0.09967 |
| 0.1 | 0.04996 |
| 0.01 | 0.00500 |
| 0.001 | 0.00050 |

Lesson 4-3 Right Triangle Trigonometry

Teaching Tips

Each fully worked example is paired with end-of-lesson exercises that mirror the example. These end-of-lesson exercises are excellent problems to work through together in class. Teacher Notes on specific issues are included where applicable.

| Example | Exercise | Teacher Notes | |
|---------|----------|---|--|
| 1 | 20 | Ask your students to label opp, adj, and hyp for angle A. Then ask them to find the values of the six trigonometric functions of angle A. | |
| 2 | 26 | | |
| 4 | 32 | | |
| 5 | 36 | | |
| 6 | 46, 48 | | |
| 7 | 56 | | |
| 8 | 60 | | |

Lesson 4-3 Apply the Skills

1. Suppose that a right triangle $\triangle ABC$ has legs of length 6 cm and 3 cm. Evaluate the six trigonometric functions for angle B, where angle B is the larger acute angle.

$$\sin B = \frac{2\sqrt{5}}{5}, \cos B = \frac{\sqrt{5}}{5}, \tan B = 2,$$

 $\csc B = \frac{\sqrt{5}}{2}, \sec B = \sqrt{5}, \cot B = \frac{1}{2}$

- **2.** Suppose that $\sin\theta = \frac{3}{4}$ for the acute angle θ . Evaluate $\sec\theta$. $\sec\theta = \frac{4\sqrt{7}}{7}$
- 3. Evaluate csc 45°, sec 45°, and cot 45°.

$$\csc 45^\circ = \sqrt{2}$$
, $\sec 45^\circ = \sqrt{2}$, and $\cot 45^\circ = 1$

4. Evaluate sin 30°, cos 30°, and tan 30°.

$$\sin 30^{\circ} = \frac{1}{2}$$
, $\cos 30^{\circ} = \frac{\sqrt{3}}{2}$, and $\tan 30^{\circ} = \frac{\sqrt{3}}{3}$

5. Simplify the expressions.

a.
$$\cot 60^{\circ} - \cot 30^{\circ} - \frac{2\sqrt{3}}{3}$$

b.
$$\sin \frac{\pi}{3} \cos \frac{\pi}{6} + \cos \frac{\pi}{3} \sin \frac{\pi}{6}$$
 1

- **6.** For each function value, find a cofunction with the same value.
 - **a.** $\tan 22.5^{\circ} = \sqrt{2} 1$ $\cot 67.5^{\circ} = \sqrt{2} 1$
 - **b.** $\sec \frac{\pi}{3} = 2$ $\csc \frac{\pi}{6} = 2$
- **7.** Suppose that the distance along the ground from a farm worker to the base of a palm tree is 16 ft. If the angle of elevation from an eye level of 5 ft to the top of the tree is 45.9°, is the tree tall enough to harvest (at least 20 ft tall)?

Yes; the tree is approximately 21.5 ft.

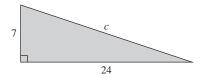
8. If a 15-ft ladder is leaning against a wall at an angle of 62° with the ground, how high up the wall will the ladder reach? Round to the nearest tenth of a foot.

13.2 ft

Lesson 4-3 Practice

Prerequisite Review

- **R.1.** Simplify the radical. $\sqrt{45}$ $3\sqrt{5}$
- **R.2.** Rationalize the denominator. $\frac{14}{\sqrt{7}}$ $2\sqrt{7}$
- **R.3.** Rationalize the denominator. $\frac{9}{9}$ $\frac{3\sqrt{2}}{\sqrt{18}}$
- **R.4.** Use the Pythagorean theorem to find the length of the missing side c. 25



- **R.5.** In a right triangle, one leg measures 24 ft and the hypotenuse measures 26 ft. Find the length of the missing side. 10 ft
- **R.6.** Find the measure of angle a. 43°



Concept Connections

- **1.** In a right triangle with an acute angle θ , the longest side in the triangle is the <u>hypotenuse</u> and is opposite the <u>right</u> angle.
- **2.** The leg of a right triangle that lies on one ray of angle θ is called the <u>adjacent</u> leg, and the leg that lies across the triangle from θ is called the <u>opposite</u> leg.

- **3.** The mnemonic device "SOH-CAH-TOA" stands for the ratios $\sin \theta = \frac{\Box}{\Box}$, $\cos \theta = \frac{\Box}{\Box}$, and $\tan \theta = \frac{\Box}{\Box}$.
- **4.** Complete the reciprocal and quotient identities. $\csc\theta = \frac{1}{\Box}$, $\sec\theta = \frac{1}{\Box}$, $\cot\theta = \frac{1}{\Box}$ or $\frac{\Box}{\Box}$ $\sin\theta$, $\cos\theta$, $\tan\theta$ or $\frac{\cos\theta}{\sin\theta}$
- **5.** Given the lengths of two sides of a right triangle, we can find the length of the third side by using the Pythagorean theorem.
- **6.** An <u>isosceles</u> right triangle is a right triangle in which the two legs are of equal length. The two acute angles in this triangle each measure <u>45</u> °
- **7.** The length of the shorter leg of a 30° – 60° – 90° triangle is always <u>one-half</u> the length of the hypotenuse.
- **8.** For the six trigonometric functions $\sin \theta$, $\cos \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, and $\cot \theta$, identify the three reciprocal pairs.

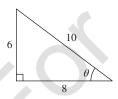
 $\sin \theta$ and $\csc \theta$, $\cos \theta$ and $\sec \theta$, and $\tan \theta$ and $\cot \theta$

- **9.** $\tan \theta$ is the ratio of $\sin \theta$ and $\cos \theta$.
- **10.** Complete the Pythagorean identities. $\sin^2\theta + \underline{\cos^2\theta} = 1$, $\underline{\tan^2\theta} + 1 = \sec^2\theta$, $1 + \cot^2\theta = \csc^2\theta$.
- **11.** The two acute angles in a right triangle are complementary because the sum of their measures is $\underline{90}$ °.
- **12.** The sine of angle θ equals the cosine of $\frac{90^{\circ} \theta \text{ or } \frac{\pi}{2} \theta}{2}$. For this reason, the sine and cosine functions are called <u>cofunctions</u>.

Learn: Evaluate Trigonometric Functions of Acute Angles

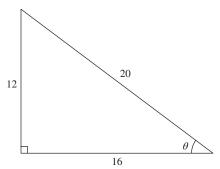
For Exercises 13–14, find the exact values of the six trigonometric functions for angle θ .

13. a.



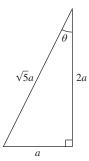
$$\sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5}, \tan \theta = \frac{3}{4},$$
$$\csc \theta = \frac{5}{3}, \sec \theta = \frac{5}{4}, \cot \theta = \frac{4}{3}$$

b.



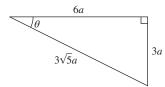
$$\sin \theta = \frac{3}{5}, \cos \theta = \frac{4}{5}, \tan \theta = \frac{3}{4},$$
$$\csc \theta = \frac{5}{3}, \sec \theta = \frac{5}{4}, \cot \theta = \frac{4}{3}$$

14. a.



$$\sin \theta = \frac{\sqrt{5}}{5}, \cos \theta = \frac{2\sqrt{5}}{5}, \tan \theta = \frac{1}{2},$$
$$\csc \theta = \sqrt{5}, \sec \theta = \frac{\sqrt{5}}{2}, \cot \theta = 2$$

b



$$\sin \theta = \frac{\sqrt{5}}{5}, \cos \theta = \frac{2\sqrt{5}}{5}, \tan \theta = \frac{1}{2},$$
$$\csc \theta = \sqrt{5}, \sec \theta = \frac{\sqrt{5}}{2}, \cot \theta = 2$$

For Exercises 15–18, first use the Pythagorean theorem to find the length of the missing side. Then find the exact values of the six trigonometric functions for angle θ . (See Example 1, p. 496)

15.



Hypotenuse: 13;

$$\sin \theta = \frac{5}{13}, \cos \theta = \frac{12}{13}, \tan \theta = \frac{5}{12},$$

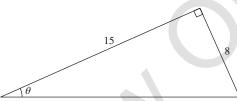
 $\csc \theta = \frac{13}{5}, \sec \theta = \frac{13}{12}, \cot \theta = \frac{12}{5}$



Hypotenuse: $\sqrt{61}$;

$$\sin \theta = \frac{5\sqrt{61}}{61}, \cos \theta = \frac{6\sqrt{61}}{61}, \tan \theta = \frac{5}{6},$$
$$\csc \theta = \frac{\sqrt{61}}{5}, \sec \theta = \frac{\sqrt{61}}{6}, \cot \theta = \frac{6}{5}$$

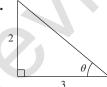
16



Hypotenuse: 17:

$$\sin \theta = \frac{8}{17}, \cos \theta = \frac{15}{17}, \tan \theta = \frac{8}{15}$$

 $\csc \theta = \frac{17}{8}, \sec \theta = \frac{17}{15}, \cot \theta = \frac{15}{8}$



Hypotenuse: √13;

$$\sin \theta = \frac{2\sqrt{13}}{13}, \cos \theta = \frac{3\sqrt{13}}{13}, \tan \theta = \frac{2}{3}, \\ \csc \theta = \frac{\sqrt{13}}{2}, \sec \theta = \frac{\sqrt{13}}{3}, \cot \theta = \frac{3}{2}$$

For Exercises 19–22, first use the Pythagorean theorem to find the length of the missing side of the right triangle. Then find the exact values of the six trigonometric functions for the angle θ opposite the shortest side. (See Example 1, p. 496)

19. Leg =
$$2 \text{ cm}$$
, leg = 6 cm

Hypotenuse =
$$2\sqrt{10}$$
 cm;

$$\sin\theta = \frac{\sqrt{10}}{10}, \cos\theta = \frac{3\sqrt{10}}{10}, \tan\theta = \frac{1}{3},$$

$$\csc \theta = \sqrt{10}$$
, $\sec \theta = \frac{\sqrt{10}}{3}$, $\cot \theta = 3$

$$\sin \theta = \frac{\sqrt{26}}{26}, \cos \theta = \frac{5\sqrt{26}}{26}, \tan \theta = \frac{1}{5},$$

$$\csc\theta = \sqrt{26}$$
, $\sec\theta = \frac{\sqrt{26}}{5}$, $\cot\theta = 5$

21. Leg = $2\sqrt{7}$ m, hypotenuse = $2\sqrt{11}$ m

$$Leg = 4 m;$$

$$\sin \theta = \frac{2\sqrt{11}}{11}, \cos \theta = \frac{\sqrt{77}}{11}, \tan \theta = \frac{2\sqrt{7}}{7},$$

$$\csc\theta = \frac{\sqrt{11}}{2}$$
, $\sec\theta = \frac{\sqrt{77}}{7}$, $\cot\theta = \frac{\sqrt{7}}{2}$

22. Leg = $5\sqrt{3}$ in., hypotenuse = $2\sqrt{21}$ in.

$$Leg = 3 in.;$$

$$\sin \theta = \frac{\sqrt{21}}{14}$$
, $\cos \theta = \frac{5\sqrt{7}}{14}$, $\tan \theta = \frac{\sqrt{3}}{5}$

$$\csc\theta = \frac{2\sqrt{21}}{3}, \sec\theta = \frac{2\sqrt{7}}{5}, \cot\theta = \frac{5\sqrt{3}}{3}$$

In Exercises 23-24, given the value of one trigonometric function of an acute angle θ , find the values of the remaining five trigonometric functions of θ . (See Example 2, p. 498)

23.
$$\tan \theta = \frac{4}{7}$$

$$\sin \theta = \frac{4\sqrt{65}}{65}, \cos \theta = \frac{7\sqrt{65}}{65}$$

$$\sin \theta = \frac{4\sqrt{65}}{65}, \cos \theta = \frac{7\sqrt{65}}{65},$$

$$\csc \theta = \frac{\sqrt{65}}{4}, \sec \theta = \frac{\sqrt{65}}{7}, \cot \theta = \frac{7}{4}$$

24.
$$\cos \theta = \frac{\sqrt{10}}{10}$$

$$\sin\theta = \frac{3\sqrt{10}}{10}, \tan\theta = 3,$$

$$\csc\theta = \frac{\sqrt{10}}{3}$$
, $\sec\theta = \sqrt{10}$, $\cot\theta = \frac{1}{3}$

For Exercises 25–30, assume that θ is an acute angle. (See Example 2, p. 498)

25. If
$$\cos \theta = \frac{\sqrt{21}}{7}$$
, find $\csc \theta$. $\frac{\sqrt{7}}{2}$

26. If
$$\sin \theta = \frac{\sqrt{17}}{17}$$
, find $\cot \theta$. 4

27. If
$$\sec \theta = \frac{3}{2}$$
, find $\sin \theta$. $\frac{\sqrt{5}}{3}$

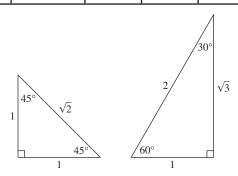
28. If
$$\csc \theta = 3$$
, find $\cos \theta$. $\frac{2\sqrt{2}}{3}$

29. If
$$\tan \theta = \frac{\sqrt{15}}{9}$$
, find $\cos \theta$. $\frac{3\sqrt{6}}{8}$

30. If
$$\cot \theta = \frac{\sqrt{3}}{2}$$
, find $\cos \theta$. $\frac{\sqrt{21}}{7}$

For Exercise 31, use the isosceles right triangle and the 30°-60°-90° triangle to complete the table. (See Examples 3-4, pp. 498-499)

| 31. | θ | $\sin 	heta$ | $\cos \theta$ | an	heta | $csc\theta$ | $\sec 	heta$ | $\cot 	heta$ |
|-----|----------------------------|----------------------|----------------------|----------------------|-----------------------|-----------------------|----------------|
| | $30^\circ = \frac{\pi}{6}$ | 1/2 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | 2 | $\frac{2\sqrt{3}}{3}$ | $\sqrt{3}$ |
| | $45^\circ = \frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | √2 | √2 | 1 |
| | $60^\circ = \frac{\pi}{3}$ | <u>√3</u> 2 | 1/2 | √3 | $\frac{2\sqrt{3}}{3}$ | 2 | <u>√3</u> 3 |



- **32. a.** Evaluate $\sin 60^{\circ}$. $\frac{\sqrt{3}}{2}$
 - **b.** Evaluate $\sin 30^{\circ} + \sin 30^{\circ}$. 1
 - c. Are the values in parts (a) and (b) the same? No

For Exercises 33-38, find the exact value of each expression without the use of a calculator. (See Example 5, p. 501)

33.
$$\sin \frac{\pi}{4} \cdot \cot \frac{\pi}{3} = \frac{\sqrt{6}}{6}$$

34.
$$\tan \frac{\pi}{6} \cdot \csc \frac{\pi}{4} = \frac{\sqrt{6}}{3}$$

35.
$$3\cos\frac{\pi}{3} + 4\sin\frac{\pi}{6} + \frac{7}{2}$$

34.
$$\tan \frac{\pi}{6} \cdot \csc \frac{\pi}{4} = \frac{\sqrt{6}}{3}$$

36. $2\cos \frac{\pi}{6} - 5\sin \frac{\pi}{3} = -\frac{3\sqrt{3}}{2}$
38. $\cos^2 45^\circ + \tan^2 60^\circ = \frac{7}{2}$

37.
$$\csc^2 60^\circ - \sin^2 45^\circ$$

38.
$$\cos^2 45^\circ + \tan^2 60^\circ \frac{7}{2}$$

Learn: Use Fundamental Trigonometric Identities

For Exercises 39-44, determine whether the statement is true or false for an acute angle $\boldsymbol{\theta}$ by using the fundamental identities. If the statement is false, provide a counterexample by using a special angle: $\frac{\pi}{3}$, $\frac{\pi}{4}$, or $\frac{\pi}{6}$.

39.
$$\sin \theta \cdot \tan \theta = 1$$
 False; For example, $\sin \frac{\pi}{4} \tan \frac{\pi}{4} = \frac{\sqrt{2}}{2} \cdot 1 = \frac{\sqrt{2}}{2} \neq 1$

40.
$$\cos^2 \theta \cdot \tan^2 \theta = \sin^2 \theta$$
 True

41.
$$\sin^2 \theta + \tan^2 \theta + \cos^2 \theta = \sec^2 \theta$$
 True

42.
$$\csc\theta \cdot \cot\theta = \sec\theta$$

$$\csc\frac{\pi}{3} \cdot \cot\frac{\pi}{3} = \frac{2\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{3} = \frac{2}{3}$$

$$\sec \frac{\pi}{3} = 2$$

$$\frac{2}{3} \neq 2$$

43.
$$\frac{1}{\tan \theta} \cdot \cot \theta + 1 = \csc^2 \theta$$
 True

44.
$$\sin \theta \cdot \cos \theta \cdot \tan \theta + 1 = \cos^2 \theta$$

False; For example,
$$\sin \frac{\pi}{6} \cos \frac{\pi}{6} \tan \frac{\pi}{6} + 1 = \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{3} + 1 = \frac{5}{4}$$

$$\cos^2\left(\frac{\pi}{6}\right) = \left(\cos\frac{\pi}{6}\right)^2 = \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{3}{4}$$

$$\frac{5}{4} \neq \frac{3}{4}$$

For Exercises 45–50, given the function value, find a cofunction of another angle with the same value. (See Example 6, p. 503)

45.
$$\tan 75^\circ = 2 + \sqrt{3} \cot 15^\circ$$

46.
$$\sec \frac{\pi}{12} = \sqrt{6} - \sqrt{2} \csc \frac{5\pi}{12}$$

47.
$$\csc \frac{\pi}{3} = \frac{2\sqrt{3}}{3}$$
 $\sec \frac{\pi}{6}$

47.
$$\csc \frac{\pi}{3} = \frac{2\sqrt{3}}{3}$$
 $\sec \frac{\pi}{6}$ **48.** $\sin 15^{\circ} = \frac{\sqrt{6} - \sqrt{2}}{4}$ $\cos 75^{\circ}$

49.
$$\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2} \sin \frac{\pi}{4}$$

50.
$$\cot \frac{\pi}{6} = \sqrt{3} \tan \frac{\pi}{3}$$

For Exercises 51-54, use a calculator to approximate the function values to 4 decimal places. Be sure that your calculator is in the correct mode.

c.
$$\tan \frac{3\pi}{8}$$
 2.4142

b.
$$\tan 19^{\circ}36'18''$$
 0.3562 **c.** $\cos \left(\frac{5\pi}{22}\right)$ 0.7557

53. a. csc 39.84° 1.5609 **b.** sec
$$\frac{\pi}{18}$$
 1.0154

b.
$$\sec \frac{\pi}{18}$$
 1.0154

54. a. cot 18.46° 2.9956 **b.** csc
$$\frac{2\pi}{9}$$
 1.5557

b.
$$\csc \frac{2\pi}{9}$$
 1.5557

Learn: Use Trigonometric Functions in Applications

- 55. An observer at the top of a 462-ft cliff measures the angle of depression from the top of a cliff to a point on the ground to be 5°. What is the distance from the base of the cliff to the point on the ground? Round to the nearest foot. (See Example 7, p. 504) 5281 ft
- 56. A lamppost casts a shadow of 18 ft when the angle of elevation of the Sun is 33.7°. How high is the lamppost? Round to the nearest foot. 12 ft
- **57.** A 30-ft boat ramp makes a 7° angle with the water. What is the height of the ramp above the water at the ramp's highest point? Round to the nearest tenth of a foot. (See Example 8, p. 506) 3.7 ft
- 58. A backyard slide is designed for a child's playground. If the top of the 10-ft slide makes an angle of 58° from the vertical, how far out from the base of the steps will the slide extend? Round to the nearest tenth of a foot. 8.5 ft

- **59.** The Lookout Mountain Incline Railway, located in Chattanooga, Tennessee, is 4972 ft long and runs up the side of the mountain at an average incline of 17°. What is the gain in altitude? Round to the nearest foot. 1454 ft
- **60.** A 12-ft ladder leaning against a house makes a 64° angle with the ground. Will the ladder reach a window sill that is 10.5 ft up from the base of the house?

Yes. The top of the ladder reaches the house approximately 10.8 ft above ground level.

61. According to National Football League (NFL) rules, all crossbars on goalposts must be 10 ft from the ground. However, teams are allowed some freedom on how high the vertical posts on each end may extend, as long as they measure at least 30 ft. A measurement on an NFL field taken 100 yd from the goalposts yields an angle of 7.8° from the ground to the top of the posts. If the crossbar is 10 ft from the ground, do the goalposts satisfy the NFL rules?

Yes. The tops of the goalposts are approximately 41 ft high (rounded to the nearest foot). This means that the vertical posts are approximately 31 ft.

- **62.** A scientist standing at the top of a mountain 2 mi above sea level measures the angle of depression to the ocean horizon to be 1.82°. Use this information to approximate the radius of the Earth to the nearest mile. (*Hint*: The line of sight \overrightarrow{AB} is tangent to the Earth and forms a right angle with the radius at the point of tangency.) 3963 mi
- **63.** A scenic overlook along the Pacific Coast Highway in Big Sur, California, is 280 ft above sea level. A 6-ft- tall hiker standing at the overlook sees a sailboat and estimates the angle of depression to be 30°. Approximately how far off the coast is the sailboat? Round to the nearest foot. 495 ft
- **64.** An airplane traveling 400 mph at a cruising altitude of 6.6 mi begins its descent. If the angle of descent is 2° from the horizontal, determine the new altitude after 15 min. Round to the nearest tenth of a mile. 3.1 mi

Mixed Exercises

For Exercises 65–70, given that $\sin A = \frac{4}{5}$, $\tan B = \frac{5}{12}$, $\cos C = \frac{8}{17}$, and $\cot D = \frac{24}{7}$, evaluate each expression. Assume that A, B, C, and D are all on the interval $\left(0, \frac{\pi}{2}\right)$ or $\left(0^{\circ}, 90^{\circ}\right)$.

65.
$$\cos A \cdot \sec D \cdot \cot C = \frac{1}{3}$$

66.
$$\sin B \cdot \csc C \cdot \cos D = \frac{136}{325}$$

67.
$$\sin A \cdot \csc B \cdot \sec D \cdot \cos B$$
 2

68.
$$\csc^2 A \cdot \sin D \cdot \cot B = \frac{21}{20}$$

69.
$$\sin(90^\circ - B) \cdot \tan D \cdot \cot C \cdot \csc A = \frac{7}{39}$$

70.
$$\cos\left(\frac{\pi}{2} - A\right) \cdot \tan C \cdot \cos D \cdot \sin B = \frac{36}{65}$$

For Exercises 71–75, use the fundamental trigonometric identities as needed.

- **71.** Given that $\sin x^{\circ} \approx 0.3746$, approximate the given function values. Round to 4 decimal places.
 - **a.** $\cos(90 x)^{\circ}$ 0.3746
- **b.** $\cos x^{\circ}$ 0.9272
- **c.** $\tan x^{\circ}$ 0.4040

- **d.** $\sin(90 x)^{\circ}$ 0.9272 **e.** $\cot(90 x)^{\circ}$ 0.4040
- **f.** $\csc x^{\circ}$ 2.6695
- **72.** Given that $\cos x \approx 0.6691$, approximate the given function values. Round to 4 decimal places.

 - **a.** $\sin x = 0.7432$ **b.** $\sin \left(\frac{\pi}{2} x\right) = 0.6691$ **c.** $\tan x = 1.1107$ **d.** $\cos \left(\frac{\pi}{2} x\right) = 0.7432$ **e.** $\sec x = 1.4945$ **f.** $\cot \left(\frac{\pi}{2} x\right) = 1.1107$

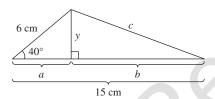
- **73.** Given that $\cos \frac{\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}$, give the exact function values. **a.** $\sin \frac{5\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}$ **b.** $\sin \frac{\pi}{12} = \frac{\sqrt{2} \sqrt{3}}{2}$ **c.** $\sec \frac{\pi}{12} = \sqrt{6} \sqrt{2}$

- **74.** Given that $\tan 36^{\circ} = \sqrt{5 2\sqrt{5}}$, give the exact function values.

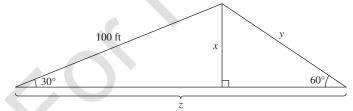
 - **a.** $\sec 36^{\circ} \quad \sqrt{6-2\sqrt{5}}$ **b.** $\csc 54^{\circ} \quad \sqrt{6-2\sqrt{5}}$
- 75. Simplify each expression to a single trigonometric function.

 - **a.** $\sec \theta \cdot \tan \left(\frac{\pi}{2} \theta\right) \quad \csc \theta$ **b.** $\cot^2 \theta \cdot \csc (90^\circ \theta) \cdot \sin \theta \quad \cot \theta$
- **76.** Find the lengths a, b, y, and c. Round to the nearest tenth of a centimeter.

 $a \approx 4.6$ cm, $b \approx 10.4$ cm, $y \approx 3.9$ cm, $c \approx 11.1$ cm

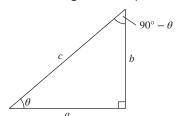


77. Find the exact lengths *x*, *y*, and *z*. x = 50 ft, $y = \frac{100\sqrt{3}}{3}$ ft, $z = \frac{200\sqrt{3}}{3}$ ft



Write About It

78. Use the figure to explain why $\tan \theta = \cot(90^{\circ} - \theta)$.



Relative to angle θ , side b is the opposite side and side a is the adjacent side.

So
$$\tan \theta = \frac{\text{opp}}{\text{adj}} = \frac{b}{a}$$
.

Relative to angle $(90^{\circ} - \theta)$, side *b* is the adjacent side and side *a* is the opposite side.

So
$$\cot(90^{\circ} - \theta) = \frac{\text{adj}}{\text{opp}} = \frac{b}{a}$$
.

This shows that $\tan \theta = \cot(90^{\circ} - \theta)$.

79. Explain the difference between an angle of elevation and an angle of depression.

An angle of elevation is an angle measured upward from a horizontal line of reference. An angle of depression is an angle measured downward from a horizontal line of reference.

Expanding Your Skills

For Exercises 80–85, simplify each expression to obtain one of the following expressions.

80.
$$\frac{\cos\theta\csc\theta}{\cot\theta}$$

81.
$$\frac{1}{\csc\theta\tan\theta}$$
 k

82.
$$\frac{\sin\theta\cos\theta}{1-\sin^2\theta}$$

83.
$$\sec \theta - \tan \theta \sin \theta$$
 b

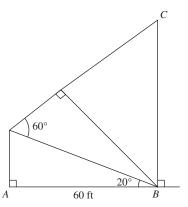
84.
$$\frac{\tan^2\theta + 1}{\tan\theta + \cot\theta}$$

85.
$$\frac{\sin^2\theta\sec\theta}{1+\sec\theta}+\cos\theta$$
 a

86. An athlete is in a boat at point A, $\frac{1}{4}$ mi from the nearest point D on a straight shoreline. She can row at a speed of 3 mph and run at a speed of 6 mph. Her planned workout is to row to point D and then run to point C farther down the shoreline. However, the current pushes her at an angle of 24° from her original path so that she comes ashore at point B, D mi from her final destination at point D. How many minutes will her trip take? Round to the nearest minute.

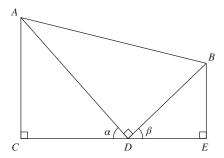
25 minutes

87. Given that the distance from A to B is 60 ft, find the distance from B to C. Round to the nearest foot. 72 ft

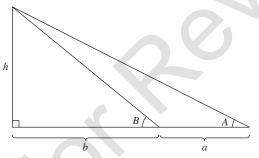


- **88.** In the figure, $\overline{CD} = 15$, $\overline{DE} = 8$, $\tan \alpha = \frac{4}{3}$, and $\sin \beta = \frac{3}{5}$. Find the lengths of
 - **a.** \overline{AC} $\overline{AC} = 20$
- **b.** \overline{AD} $\overline{AD} = 25$
- **c.** \overline{DB} \overline{DB} = 10

- **d.** \overline{BE} $\overline{BE} = 6$
- **e.** \overline{AB} $\overline{AB} = 5\sqrt{29}$

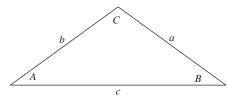


89. For the given triangle, show that $a = h \cot A - h \cot B$.



 $\cot A = \frac{a+b}{h}$ and $\cot B = \frac{b}{h}$.

Solving each equation for b yields, $b = h \cot A - a$ and $b = h \cot B$. Equating the expressions for b, gives $h \cot A - a = h \cot B$, which implies that $h \cot A - h \cot B = a$. **90.** Show that the area A of the triangle is given by $A = \frac{1}{2}bc \sin A$.



Let h represent the height of the perpendicular line segment from vertex C to side c.

Then $\sin A = \frac{h}{b}$, which implies that $h = b \sin A$.

The area of the triangle is

$$A = \frac{1}{2}(base)(height) = \frac{1}{2}c(b\sin A) = \frac{1}{2}bc\sin A.$$

91. Use a cofunction relationship to show that the product (tan1°)(tan2°)(tan3°) · · · · · (tan87°)(tan88°)(tan89°) is equal to 1.

$$(\tan 1^\circ)(\tan 2^\circ)(\tan 3^\circ) \cdot \cdots \cdot (\tan 88^\circ)(\tan 89^\circ)$$

- = $(\tan 1^\circ)(\tan 89^\circ)(\tan 2^\circ)(\tan 88^\circ)(\tan 3^\circ)(\tan 87^\circ)$ ··· $(\tan 45^\circ)$
- = $[(\tan 1^\circ)(\cot 1^\circ)][(\tan 2^\circ)(\cot 2^\circ)][(\tan 3^\circ)(\cot 3^\circ)]$ ··· $(\tan 45^\circ)$
- $= [1][1][1] \cdots (\tan 45^{\circ})$
- = 1

Technology Connections

For Exercises 92–95, use a calculator to approximate the values of the left- and right-hand sides of each statement for $A=30^{\circ}$ and $B=45^{\circ}$. Based on the approximations from your calculator, determine if the statement appears to be true or false.

92. a.
$$\sin(A + B) = \sin A + \sin B$$
 False

b.
$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$
 True

93. a.
$$tan(A - B) = tan A - tan B$$
 False

b.
$$tan(A - B) = \frac{tanA - tanB}{1 + tanA tanB}$$
 True

94. a.
$$\cos(\frac{A}{2}) = \sqrt{\frac{1 + \cos A}{2}}$$
 True

b.
$$\cos\left(\frac{A}{2}\right) = \frac{1}{2}\cos A$$
 False

95. a.
$$\tan\left(\frac{B}{2}\right) = \frac{1-\cos B}{\sin B}$$
 True

b.
$$\tan\left(\frac{B}{2}\right) = \frac{\sin B}{1 + \cos B}$$
 True

Point of Interest

- **96.** A zip line is to be built between two towers labeled A and B across a wetland area. To approximate the distance of the zip line, a surveyor marks a third point C, a distance of 175 ft from one end of the zip line and perpendicular to the zip line. The measure of $\angle ACB$ is 74.5°. How long is the zip line? Round to the nearest foot. 631 ft
- **97.** To determine the width of a river from point A to point B, a surveyor walks downriver 50 ft along a line perpendicular to \overline{AB} to a new position at point C. The surveyor determines that the measure of $\angle ACB$ is 60°. Find the exact width of the river from point A to point B. $50\sqrt{3}$ ft

Lesson 4-4 Trigonometric Functions of Any Angle

Teaching Tips

Each fully worked example is paired with end-of-lesson exercises that mirror the example. These end-of-lesson exercises are excellent problems to work through together in class.

| Example | Exercise |
|---------|------------|
| 1 | 18 |
| 2 | 24, 26 |
| 3 | 34, 36, 38 |
| 4 | 44, 54 |
| 5 | 64 |
| 6 | 70 |

Lesson 4-4 Apply the Skills

1. Let P(-5, -7) be a point on the terminal side of angle θ drawn in standard position. Find the values of the six trigonometric functions of θ .

$$\sin\theta = -\frac{7\sqrt{74}}{74}, \cos\theta = -\frac{5\sqrt{74}}{74},$$

$$\tan\theta = \frac{7}{5}, \csc\theta = -\frac{\sqrt{74}}{7},$$

$$\sec \theta = -\frac{\sqrt{74}}{5}, \cot \theta = \frac{5}{7}$$

2. Find the reference angle θ' .

a.
$$\theta = 150^{\circ}$$
 $\theta' = 30^{\circ}$

b.
$$\theta = -157.5^{\circ}$$
 $\theta' = 22.5^{\circ}$

c.
$$\theta = 5$$
 $\theta' = 2\pi - 5$

d.
$$\theta = \frac{13\pi}{3}$$
 $\theta' = \frac{\pi}{3}$

3. Evaluate the functions.

a.
$$\cos \frac{5\pi}{6} - \frac{\sqrt{3}}{2}$$
c. $\csc \frac{7\pi}{4} - \sqrt{2}$

b. cot(-120°)
$$\frac{\sqrt{3}}{3}$$

c.
$$\csc \frac{7\pi}{4} - \sqrt{2}$$

4. Evaluate the functions.

a.
$$csc(11\pi)$$
 Undefined

b.
$$\cos(-600^{\circ})$$
 $-\frac{1}{2}$

5. Given $\cos \theta = -\frac{3}{8}$ and $\sin \theta < 0$, find $\sin \theta$ and $\tan \theta$.

$$\sin \theta = -\frac{\sqrt{55}}{8}$$
 and $\tan \theta = \frac{\sqrt{55}}{3}$

6. Given $\sin \theta = -\frac{5}{13}$ for θ in Quadrant IV, find $\cos \theta$ and $\tan \theta$.

$$\cos\theta = \frac{12}{13}, \tan\theta = -\frac{5}{12}$$

Lesson 4-4 Practice

Prerequisite Review

R.1. Find the length of the missing side of the right triangle. c = 29 m

R.2. Find the length of the third side using the Pythagorean theorem. Round the answer to the nearest tenth of a foot. 6.4 ft

R.3. Use the distance formula to find the distance from (2, 5) to (-3, 10). $5\sqrt{2}$

Concept Connections

1. The distance from the origin to a point P(x, y) is given by $\sqrt{x^2 + y^2}$

2. Angles with terminal sides on the coordinate axes are referred to as quadrantal angles.

3. If θ is an angle in standard position, the <u>reference</u> angle for θ is the acute angle θ' formed by the terminal side of θ and the horizontal axis.

4. The values $\tan \theta$ and $\sec \theta$ are undefined for odd multiples of $\frac{\pi}{2}$ or 90°

5. The values $\cot \theta$ and $\csc \theta$ are undefined for multiples of π or 180°

6. For what values of θ is $\sin \theta$ greater than 1?

There are no values for which $\sin \theta$ is greater than 1.

Learn: Evaluate Trigonometric Functions of Any Angle

7. Let P(x, y) be a point on the terminal side of an angle θ drawn in standard position and let r be the distance from P to the origin. Fill in the boxes to form the ratios defining the six trigonometric functions.

a.
$$\sin \theta = \frac{\Box}{\Box} \frac{y}{r}$$

b.
$$\cos \theta = \frac{\Box}{\Box}$$

c.
$$\tan \theta = \frac{\Box}{\Box}$$

a.
$$\sin \theta = \frac{\Box}{\Box} \frac{y}{r}$$
 b. $\cos \theta = \frac{\Box}{\Box} \frac{x}{r}$ **c.** $\tan \theta = \frac{\Box}{\Box} \frac{y}{x}$ **d.** $\csc \theta = \frac{\Box}{\Box} \frac{r}{y}$ **e.** $\sec \theta = \frac{\Box}{\Box} \frac{r}{x}$ **f.** $\cot \theta = \frac{\Box}{\Box} \frac{x}{y}$

e.
$$\sec \theta = \frac{\Box}{\Box} \frac{r}{x}$$

f.
$$\cot \theta = \frac{\Box}{\Box} \frac{x}{y}$$

8. Fill in the cells in the table with the appropriate sign for each trigonometric function for θ in Quadrants I, II, III, and IV. The signs for the sine function are done for you.

| | Quadrant | | | | |
|---------------|----------|----|---|----|--|
| | | II | Ш | IV | |
| $\sin \theta$ | + | + | - | _ | |
| $\cos \theta$ | + | _ | _ | + | |
| an	heta | + | _ | + | _ | |
| $\csc \theta$ | + | + | _ | _ | |
| $\sec \theta$ | + | _ | _ | + | |
| $\cot \theta$ | + | _ | + | _ | |

For Exercises 9–14, given the stated conditions, identify the quadrant in which θ lies.

9. $\sin \theta < 0$ and $\tan \theta > 0$ Quadrant III

10. $\csc \theta > 0$ and $\cot \theta < 0$ Quadrant II

11. $\sec \theta < 0$ and $\tan \theta < 0$ Quadrant II

12. $\cot \theta < 0$ and $\sin \theta < 0$ Quadrant IV

13. $\cos \theta > 0$ and $\cot \theta < 0$ Quadrant IV

14. $\cot \theta < 0$ and $\sec \theta < 0$ Quadrant II

For Exercises 15–20, a point is given on the terminal side of an angle θ drawn in standard position. Find the values of the six trigonometric functions of θ . (See Example 1, p. 516)

15. (5, -12)

$$\sin \theta = -\frac{12}{13}$$
, $\cos \theta = \frac{5}{13}$, $\tan \theta = -\frac{12}{5}$, $\csc \theta = -\frac{13}{12}$, $\sec \theta = \frac{13}{5}$, $\cot \theta = -\frac{5}{12}$

16. (-8, 15)

$$\sin \theta = \frac{15}{17}, \cos \theta = -\frac{8}{17}, \tan \theta = -\frac{15}{8}$$

 $\csc \theta = \frac{17}{15}, \sec \theta = -\frac{17}{8}, \cot \theta = -\frac{8}{15}$

17. (-3, -5)

$$\sin \theta = -\frac{5\sqrt{34}}{34}, \cos \theta = -\frac{3\sqrt{34}}{34}, \tan \theta = \frac{5}{3},$$
 $\csc \theta = -\frac{\sqrt{34}}{5}, \sec \theta = -\frac{\sqrt{34}}{3}, \cot \theta = \frac{3}{5}$

18 /2 _3

$$\sin \theta = -\frac{3\sqrt{13}}{13}, \cos \theta = \frac{2\sqrt{13}}{13}, \tan \theta = -\frac{3}{2},$$

$$\csc \theta = -\frac{\sqrt{13}}{3}, \sec \theta = \frac{\sqrt{13}}{2}, \cot \theta = -\frac{2}{3}$$

19. $\left(-\frac{3}{2}, 2\right)$

$$\sin \theta = \frac{4}{5}, \cos \theta = -\frac{3}{5}, \tan \theta = -\frac{4}{3},$$
$$\csc \theta = \frac{5}{4}, \sec \theta = -\frac{5}{3}, \cot \theta = -\frac{3}{4}$$

20. $(5, -\frac{8}{3})$

$$\sin \theta = -\frac{8}{17}$$
, $\cos \theta = \frac{15}{17}$, $\tan \theta = -\frac{8}{15}$.
 $\csc \theta = -\frac{17}{8}$, $\sec \theta = \frac{17}{15}$, $\cot \theta = -\frac{15}{8}$

21. Complete the table for the given angles.

| θ | $\sin	heta$ | $\cos \theta$ | $tan\hspace{.01in} 	heta$ | $\csc \theta$ | $\sec 	heta$ | $\cot 	heta$ |
|------------------------------|-------------|---------------|---------------------------|---------------|--------------|--------------|
| 0° = 0 | 0 | 1 | 0 | undefined | 1 | undefined |
| $90^\circ = \frac{\pi}{2}$ | 1 | 0 | undefined | 1 | undefined | 0 |
| $180^{\circ} = \pi$ | 0 | -1 | 0 | undefined | -1 | undefined |
| $270^\circ = \frac{3\pi}{2}$ | -1 | 0 | undefined | -1 | undefined | 0 |
| $360^{\circ} = 2\pi$ | 0 | 1 | 0 | undefined | 1 | undefined |

22. Evaluate the expression for $A = 90^{\circ}$, $B = 180^{\circ}$, and $C = 270^{\circ}$.

 $\frac{\sin A + 2\cos B}{\sec B - 3\csc C}$

Learn: Determine Reference Angles

For Exercises 23–30, find the reference angle for the given angle. (See Example 2, p. 517)

- **23.** a. 135° 45°
- **b.** -330° 30° **c.** 660° 60°
- **d.** -690° 30°

- **24.** a. -120° 60° b. 225° 45°

- **25.** a. $\frac{2\pi}{3}$ $\frac{\pi}{3}$ b. $-\frac{5\pi}{6}$ $\frac{\pi}{6}$ c. $\frac{13\pi}{4}$ $\frac{\pi}{4}$ d. $-\frac{10\pi}{3}$ $\frac{\pi}{3}$ **26.** a. $-\frac{5\pi}{4}$ $\frac{\pi}{4}$ b. $\frac{11\pi}{6}$ $\frac{\pi}{6}$ c. $\frac{17\pi}{3}$ $\frac{\pi}{3}$ d. $\frac{19\pi}{6}$ $\frac{\pi}{6}$ **27.** a. $\frac{20\pi}{17}$ $\frac{3\pi}{17}$ b. $-\frac{99\pi}{20}$ $\frac{\pi}{20}$ c. 110° 70° d. -422° 62° **28.** a. $\frac{25\pi}{11}$ $\frac{3\pi}{11}$ b. $-\frac{27\pi}{14}$ $\frac{\pi}{14}$ c. -512° 28° d. 1280° 20°

- **29.** a. $1.8 \quad \pi 1.8$
- **b.** 1.8π 0.2 π **c.** 5.1 $2\pi 5.1$

- **30. a.** 0.6 0.6
- **b.** 0.6π 0.4 π **c.** 100 $32\pi 100$ **d.** 100° 80°

Learn: Evaluate Trigonometric Functions Using Reference Angles

For Exercises 31–54, use reference angles to find the exact value. (See Examples 3 and 4, pp. 519-520)

- 31. $\sin 120^{\circ} \frac{\sqrt{3}}{2}$ 32. $\cos 225^{\circ} \frac{\sqrt{2}}{2}$ 33. $\cos \frac{4\pi}{3} \frac{1}{2}$ 34. $\sin \frac{5\pi}{6} = \frac{1}{2}$ 35. $\sec (-330^{\circ}) = \frac{2\sqrt{3}}{3}$ 36. $\csc (-225^{\circ}) = \sqrt{2}$ 37. $\sec \frac{13\pi}{3} = 2$ 38. $\csc \frac{5\pi}{3} = -\frac{2\sqrt{3}}{3}$ 39. $\cot 240^{\circ} = \frac{\sqrt{3}}{3}$ 40. $\tan (-150^{\circ}) = \frac{\sqrt{3}}{3}$ 41. $\tan \frac{5\pi}{4} = 1$ 42. $\cot \left(-\frac{3\pi}{4}\right) = 1$

- **43.** cos (-630°) 0
- **44.** $\sin 630^{\circ}$ -1 **45.** $\sin \frac{17\pi}{6}$ $\frac{1}{2}$
- **46.** $\cos\left(-\frac{11\pi}{4}\right)$ $-\frac{\sqrt{2}}{2}$ **47.** $\sec 1170^{\circ}$ Undefined **48.** $\csc 750^{\circ}$ 2
- **49.** $\csc(-5\pi)$ Undefined **50.** $\sec 5\pi$ -1 **51.** $\tan (-2400^{\circ})$ $-\sqrt{3}$
- **52.** cot 900° Undefined **53.** cot $\frac{42\pi}{8}$ 1 **54.** tan $\frac{18\pi}{4}$ Undefined

For Exercises 55-58, find two angles between 0° and 360° for the given condition.

55.
$$\sin \theta = \frac{1}{2}$$
 30° and 150°

55.
$$\sin \theta = \frac{1}{2}$$
 30° and 150° **56.** $\cos \theta = -\frac{\sqrt{2}}{2}$ 135° and 225°

57.
$$\cot \theta = -\sqrt{3}$$
 150° and 330° **58.** $\tan \theta = \sqrt{3}$ 60° and 240°

58.
$$\tan \theta = \sqrt{3}$$
 60° and 240

For Exercises 59–62, find two angles between 0 and 2π for the given condition.

59.
$$\sec \theta = \sqrt{2} \quad \frac{\pi}{4} \text{ and } \frac{7\pi}{4}$$

59.
$$\sec \theta = \sqrt{2} \quad \frac{\pi}{4} \text{ and } \frac{7\pi}{4}$$
 60. $\csc \theta = -\frac{2\sqrt{3}}{3} \quad \frac{4\pi}{3} \text{ and } \frac{5\pi}{3}$ **61.** $\tan \theta = -\frac{\sqrt{3}}{3} \quad \frac{5\pi}{6} \text{ and } \frac{11\pi}{6}$ **62.** $\cot \theta = 1 \quad \frac{\pi}{4} \text{ and } \frac{5\pi}{4}$

61.
$$\tan \theta = -\frac{\sqrt{3}}{3} \quad \frac{5\pi}{6} \text{ and } \frac{11\pi}{6}$$

62.
$$\cot \theta = 1$$
 $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

In Exercises 63-68, find the values of the trigonometric functions from the given information. (See Example 5, p. 521)

63. Given
$$\tan \theta = -\frac{20}{21}$$
 and $\cos \theta < 0$, find $\sin \theta$ and $\cos \theta$. $\sin \theta = \frac{20}{29}$, $\cos \theta = -\frac{21}{29}$

64. Given
$$\cot \theta = \frac{11}{60}$$
 and $\sin \theta < 0$, find $\cos \theta$ and $\sin \theta$. $\cos \theta = -\frac{11}{61}$, $\sin \theta = -\frac{60}{61}$

65. Given
$$\sin \theta = -\frac{3}{10}$$
 and $\tan \theta > 0$, find $\cos \theta$ and $\cot \theta$. $\cos \theta = -\frac{\sqrt{91}}{10}$, $\cot \theta = \frac{\sqrt{91}}{3}$

66. Given
$$\cos \theta = -\frac{5}{8}$$
 and $\csc \theta > 0$, find $\sin \theta$ and $\tan \theta$. $\sin \theta = \frac{\sqrt{39}}{8}$, $\tan \theta = -\frac{\sqrt{39}}{5}$

67. Given
$$\sec \theta = \frac{\sqrt{58}}{7}$$
 and $\cot \theta < 0$, find $\csc \theta$ and $\cos \theta$.

$$\csc\theta = -\frac{\sqrt{58}}{3}, \cos\theta = \frac{7\sqrt{58}}{58}$$

$$\csc\theta = -\frac{\sqrt{58}}{3}, \ \cos\theta = \frac{7\sqrt{58}}{58}$$
68. Given $\csc\theta = -\frac{\sqrt{26}}{5}$ and $\cos\theta < 0$, find $\sin\theta$ and $\cot\theta$. $\sin\theta = -\frac{5\sqrt{26}}{26}$, $\cot\theta = \frac{1}{5}$

For Exercises 69-74, use fundamental trigonometric identities to find the values of the functions. (See Example 6, p. 522)

69. Given
$$\cos \theta = \frac{20}{29}$$
 for θ in Quadrant IV, find $\sin \theta$ and $\tan \theta$.

$$\sin \theta = -\frac{21}{29}$$
, $\tan \theta = -\frac{21}{20}$

70. Given
$$\sin \theta = -\frac{8}{17}$$
 for θ in Quadrant III, find $\cos \theta$ and $\cot \theta$.

$$\cos\theta = -\frac{15}{17}, \cot\theta = \frac{15}{8}$$

71. Given
$$\tan \theta = -4$$
 for θ in Quadrant II, find $\sec \theta$ and $\cot \theta$.

$$\sec \theta = -\sqrt{17}$$
, $\cot \theta = -\frac{1}{4}$

72. Given
$$\sec \theta = 5$$
 for θ in Quadrant IV, find $\csc \theta$ and $\cos \theta$.

$$\csc\theta = -\frac{5\sqrt{6}}{12}, \cos\theta = \frac{1}{5}$$

73. Given
$$\cot \theta = \frac{5}{4}$$
 for θ in Quadrant III, find $\csc \theta$ and $\sin \theta$.

$$\csc\theta = -\frac{\sqrt{41}}{4}, \sin\theta = -\frac{4\sqrt{41}}{41}$$

74. Given
$$\csc \theta = \frac{7}{3}$$
 for θ in Quadrant II, find $\cot \theta$ and $\cos \theta$.

$$\cot \theta = -\frac{2\sqrt{10}}{3}, \cos \theta = -\frac{2\sqrt{10}}{7}$$

Mixed Exercises

For Exercises 75–76, find the sign of the expression for θ in each quadrant.

| 75 . | | | I | П | III | IV |
|-------------|----|-----------------------------------|---|---|-----|----|
| | a. | $\sin\theta\cos\theta$ | + | - | + | _ |
| | b. | $\frac{\tan \theta}{\cos \theta}$ | + | + | - | 1 |

| 76 . | | | I | Ш | Ш | IV |
|-------------|----|---------------------------------|---|---|---|----|
| | a. | $\frac{\cot\theta}{\sin\theta}$ | + | ı | 1 | + |
| | b. | $\tan 	heta$ sec $	heta$ | + | + | - | - |

For Exercises 77–84, suppose that θ is an acute angle. Identify each statement as true or false. If the statement is false, rewrite the statement to give the correct answer for the right side.

77.
$$\cos(180^{\circ} - \theta) = -\cos\theta$$
 True

78.
$$tan(180^{\circ} - \theta) = tan \theta$$
 False; $tan(180^{\circ} - \theta) = -tan \theta$

79.
$$\tan (180^{\circ} + \theta) = \tan \theta$$
 True

80.
$$\sin(180^{\circ} + \theta) = -\sin\theta$$
 True

81.
$$\csc(\pi - \theta) = -\csc\theta$$
 False; $\csc(\pi - \theta) = \csc\theta$

82.
$$sec(\pi + \theta) = sec\theta$$
 False; $sec(\pi + \theta) = -sec\theta$

83.
$$cos(\pi + \theta) = -cos\theta$$
 True

84.
$$\sin(\pi + \theta) = -\sin\theta$$
 True

For Exercises 85–90, find the value of each expression.

85.
$$\sin 30^{\circ} \cdot \cos 150^{\circ} \cdot \sec 60^{\circ} \cdot \csc 120^{\circ}$$
 -1

86.
$$\cos 45^{\circ} \cdot \sin 240^{\circ} \cdot \tan 135^{\circ} \cdot \cot 60^{\circ} \quad \frac{\sqrt{2}}{4}$$

87.
$$\cos^2 \frac{5\pi}{4} - \sin^2 \frac{2\pi}{3} - \frac{1}{4}$$

88.
$$\sin^2 \frac{11\pi}{6} + \cos^2 \frac{4\pi}{3} = \frac{1}{2}$$

89.
$$\frac{2 \tan \frac{11\pi}{6}}{1 - \tan^2 \frac{11\pi}{6}} - \sqrt{3}$$

90.
$$\frac{\cot^2 \frac{4\pi}{3} - 1}{2 \cot \frac{4\pi}{3}} - \frac{\sqrt{3}}{3}$$

For Exercises 91–94, verify the statement for the given values.

91.
$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$
; $A = 240^{\circ}$, $B = 120^{\circ}$

$$\sin(240^{\circ} - 20^{\circ}) = \sin 120^{\circ} = \frac{\sqrt{3}}{2};$$

$$\sin 240^{\circ} \cos 120^{\circ} - \cos 240^{\circ} \sin 120^{\circ} = \left(-\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right)\left(\frac{3}{2}\right) = \frac{\sqrt{3}}{2}$$

92. $\cos(B - A) = \cos B \cos A + \sin B \sin A$; $A = 330^{\circ}$, $B = 120^{\circ}$

$$\cos(120^{\circ} - 330^{\circ}) = \cos(-210^{\circ}) = -\frac{\sqrt{3}}{2};$$

$$\cos 120^{\circ}\cos 330^{\circ} + \sin 120^{\circ}\sin 330^{\circ} = \left(-\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(-\frac{1}{2}\right) = -\frac{\sqrt{3}}{2}$$

93. $\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$; $A = 210^{\circ}$, $B = 120^{\circ}$

$$\tan(210^\circ + 120^\circ) = \tan 330^\circ = -\frac{\sqrt{3}}{3}; \frac{\tan 210^\circ + \tan 120^\circ}{1 - \tan 210^\circ \tan 120^\circ} = \frac{\frac{\sqrt{3}}{3} - \sqrt{3}}{1 - \left(\frac{\sqrt{3}}{3}\right)(-\sqrt{3})} = -\frac{\sqrt{3}}{3}$$

94. $\cot(A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$; $A = 300^{\circ}$, $B = 150^{\circ}$

$$\cot(300^{\circ} + 150^{\circ}) = \cot 450^{\circ} = 0; \frac{\cot 300^{\circ} \cot 150^{\circ} - 1}{\cot 300^{\circ} + \cot 150^{\circ}} = \frac{\left(-\frac{\sqrt{3}}{3}\right)(-\sqrt{3}) - 1}{-\frac{\sqrt{3}}{3} + (-\sqrt{3})} = 0$$

For Exercises 95-96, give the exact values if possible. Otherwise, use a calculator and approximate the result to 4 decimal places.

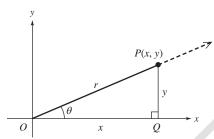
95. a.
$$\sin 30^{\circ} \frac{1}{2}$$

96. a.
$$\cos(0.25\pi)$$
 $\frac{\sqrt{2}}{2}$ **b.** $\frac{\cos\pi}{0.25}$ -4 **c.** $\cos(25^\circ)$ 0.9063

b.
$$\frac{\cos \pi}{0.25}$$
 -4

Write About It

97. Explain why neither $\sin \theta$ nor $\cos \theta$ can be greater than 1. Refer to the figure for your explanation.



Let P(x, y) be a point on the terminal side of θ in standard position, and let $r = \sqrt{x^2 + y^2}$ be the distance between P and the origin.

We know that $r = \sqrt{x^2 + y^2} \ge y$.

Therefore, $\sin \theta = \frac{y}{r} \le 1$.

Likewise, we know that $r = \sqrt{x^2 + y^2} \ge x$. Therefore, $\cos \theta = \frac{x}{r} \le 1$.

98. Explain why $\tan \theta$ is undefined at $\theta = \frac{\pi}{2}$ but $\cot \theta$ is defined at $\theta = \frac{\pi}{2}$.

When $\theta = \frac{\pi}{2}$, $\cos \theta = 0$ and $\sin \theta = 1$. In the case of $\tan \theta$, we would be dividing by zero, which is undefined. In the case of $\cot \theta$, we have zero in the numerator and 1 in the denominator, so the quotient is defined.

Expanding Your Skills

For Exercises 99–100, for part (a), find four angles between 0° and 720° that satisfy the given condition. For part (b), find four angles between 0° and 360° that satisfy the given condition.

99. a.
$$\cos \theta = -\frac{\sqrt{2}}{2}$$
 135°, 225°, 495°, 585°
b. $\cos 2\theta = -\frac{\sqrt{2}}{2}$ 67.5°, 112.5°, 247.5°, 292.5°

b.
$$\cos 2\theta = -\frac{\sqrt{2}}{2}$$
 67.5°, 112.5°, 247.5°, 292.5°

100. a.
$$\tan \theta = \sqrt{3}$$
 60°, 240°, 420°, 600°

b.
$$\tan 2\theta = \sqrt{3}$$
 30°, 120°, 210°, 300°

For Exercises 101–102, for part (a), find four angles between 0 and 3π that satisfy the given condition. For part (b), find four angles between 0 and π that satisfy the given condition.

101. a.
$$\csc \theta = \sqrt{2}$$
 $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{11\pi}{4}$ **b.** $\csc 3\theta = \sqrt{2}$ $\frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$ **102. a.** $\sin \theta = \frac{\sqrt{3}}{2}$ $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$ **b.** $\sin 3\theta = \frac{\sqrt{3}}{2}$ $\frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$

b.
$$\csc 3\theta = \sqrt{2} \quad \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{12}$$

102. a.
$$\sin \theta = \frac{\sqrt{3}}{2} \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$$

b.
$$\sin 3\theta = \frac{\sqrt{3}}{2} \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \frac{8\pi}{9}$$

103. a. Graph the point (3, 4) on a rectangular coordinate system and draw a line segment connecting the point to the origin. Find the slope of the line segment.



Slope =
$$\frac{4}{3}$$

b. Draw another line segment from the point (3, 4) to meet the x-axis at a right angle, thus forming a right triangle with the x-axis as one side. Find the tangent of the acute angle that has the x-axis as its initial side.

Tangent of the angle
$$=\frac{4}{3}$$

- c. Compare the results in part (a) and part (b). The results are the same.
- 104. The circle shown (on page 527) is centered at the origin with a radius of 1. The segment \overline{BD} is tangent to the circle at D. Match the length of each segment with the appropriate trigonometric function.

a.
$$\overline{AC}$$
 iii

i.
$$tan \theta$$

ii.
$$\cos \theta$$

c.
$$\overline{OB}$$
 iv

iii.
$$\sin \theta$$

d.
$$\overline{OC}$$
 ii

iv.
$$\sec \theta$$

- **105.** Circle A, with radius a, and circle B, with radius b, are tangent to each other and to \overline{PQ} (see figure). \overline{PR} passes through the center of each circle. Let x be the distance from point P to a point S where \overline{PR} intersects circle A on the left. Let θ denote $\angle RPQ$.
 - **a.** Show that $\sin \theta = \frac{a}{x+a}$ and $\sin \theta = \frac{b}{x+2a+b}$.

Using the right triangle formed by point P and the center of circle A, $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{a}{x+a}$.

Using the right triangle formed by point P and the center of circle B, $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{b}{x + 2a + b}$.

b. Use the results from part (a) to show that $\sin \theta = \frac{b-a}{b+a}$.

$$\frac{a}{x+a} = \frac{b}{x+2a+b}$$

$$a(x+2a+b) = b(x+a)$$

$$ax + 2a^2 + ab = bx + ba$$

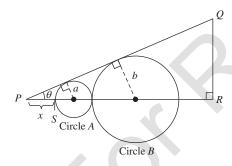
$$2a^2 = bx - ax$$

$$2a^2 = x(b-a)$$

$$\frac{2a^2}{b-a} = x$$

$$\sin \theta = \frac{a}{x+a} = \frac{a}{\left(\frac{2a^2}{b-a} + a\right)} \cdot \frac{(b-a)}{(b-a)}$$
$$= \frac{a(b-a)}{2a^2 + ab - a^2} = \frac{a(b-a)}{a^2 + ab}$$

$$=\frac{a(b-a)}{a(a+b)}=\frac{b-a}{a+b}$$



Lesson 4-5 Graphs of Sine and Cosine Functions

Teaching Tips

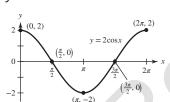
Each fully worked example is paired with end-of-lesson exercises that mirror the example. These end-of-lesson exercises are excellent problems to work through together in class. Teacher Notes on specific issues are included where applicable.

| Example | Exercise | Teacher Notes |
|---------|----------|---|
| 1 | 20, 22 | Before beginning this section, consider assigning a few problems where students review the transformations of the six basic graphs presented in Lesson 1-6. |
| 2 | 28 | |
| 3 | 40 | |
| 4 | 56 | |
| 5 | 68 | |
| 6 | 74 | |

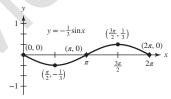
Lesson 4-5 Apply the Skills

1. Graph the function and identify the key points on one full period.

a.
$$y = 2\cos x$$



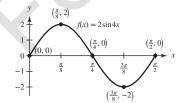
b.
$$y = -\frac{1}{3}\sin x$$



- **2.** Given $f(x) = 2 \sin 4x$,
 - a. Identify the amplitude and period.

Amplitude: 2, Period:
$$\frac{\pi}{2}$$

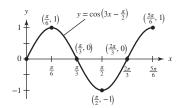
b. Graph the function and identify the key points on one full period.



- **3.** Given $y = \cos\left(3x \frac{\pi}{2}\right)$,
 - a. Identify the amplitude and period.

Amplitude: 1, Period:
$$\frac{2\pi}{3}$$

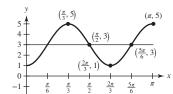
b. Graph the function and identify the key points on one full period.



- **4.** Given $y = 2\cos(3x \pi) + 3$,
 - a. Identify the amplitude, period, phase shift, and vertical shift.

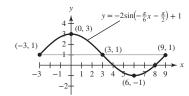
Amplitude: 2, Period: $\frac{2\pi}{3}$, Phase shift: $\frac{\pi}{3}$, Vertical shift: 3

b. Graph the function and identify the key points on one full period.



- **5.** Given $y = -2\sin\left(-\frac{\pi}{6}x \frac{\pi}{2}\right) + 1$,
 - a. Identify the amplitude, period, phase shift, and vertical shift.

Amplitude: 2, Period: 12, Phase shift: -3, Vertical shift: 1



6. A mechanical metronome uses an inverted pendulum that makes a regular, rhythmic click as it swings to the left and right. With each swing, the pendulum moves 3 in. to the left and right of the center position. The pendulum is initially pulled to the right 3 in. and then released. It returns to its starting position in 0.8 sec. Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $x(t) = A\cos(Bt - C) + D$. The value x(t) is the horizontal position (in inches) relative to the center line of the pendulum.

$$x(t) = 3\cos\left(\frac{5\pi}{2}t\right)$$

Lesson 4-5 Practice

Prerequisite Review

For Exercises R.1–R.5, use translations to graph the function.

R.1.
$$h(x) = |x| - 1$$



R.2.
$$n(x) = |x + 1|$$



R.3.
$$b(x) = 2|x|$$



R.4.
$$h(x) = \frac{1}{3}|x|$$



R.5.
$$p(x) = -|x|$$



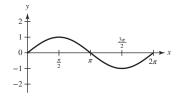
Concept Connections

- **1.** The value of $\sin x$ (increases/decreases) <u>increases</u> on $\left(0, \frac{\pi}{2}\right)$ and (increases/decreases) <u>decreases</u> on $\left(\frac{\pi}{2}, \pi\right)$.
- **2.** The value of $\cos x$ (increases/decreases) <u>decreases</u> on $\left(0, \frac{\pi}{2}\right)$ and (increases/decreases) <u>decreases</u> on $\left(\frac{\pi}{2}, \pi\right)$.
- **3.** The graph of $y = \sin x$ and $y = \cos x$ differ by a horizontal shift of $\frac{\pi}{2}$ units.
- **4.** Given $y = A\sin(Bx C) + D$ or $y = A\cos(Bx C) + D$, for B > 0 the amplitude is $A = A\cos(Bx C) + D$, the period is $A = A\cos(Bx C) + D$, for B > 0 the amplitude is $A = A\cos(Bx C) + D$, for $A = A\cos(Bx C) + D$
- **5.** The sine function is an (even/odd) <u>odd</u> function because $\sin(-x) = \frac{-\sin(x)}{}$. The cosine function is an (even/odd) <u>even</u> function because $\cos(-x) = \frac{\cos(x)}{}$.
- **6.** Given B > 0, how would the equation $y = A\sin(-Bx C) + D$ be rewritten to obtain a positive coefficient on x? $y = -A\sin(Bx + C) + D$
- **7.** Given B > 0, how would the equation $y = A\cos(-Bx C) + D$ be rewritten to obtain a positive coefficient on x? $y = A\cos(Bx + C) + D$
- **8.** Given $y = \sin(Bx)$ and $y = \cos(Bx)$, for B > 1, is the period less than or greater than 2π ? If 0 < B < 1, is the period less than or greater than 2π ?

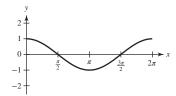
less than 2π ; greater than 2π

Learn: Graph $y = \sin x$ and $y = \cos x$

9. From memory, sketch $y = \sin x$ on the interval $[0, 2\pi]$.



10. From memory, sketch $y = \cos x$ on the interval $[0, 2\pi]$.



11. For $y = \cos x$,

- **a.** The domain is $(-\infty, \infty)$.
- **b.** The range is $_{-1, 1]}$.
- **c.** The amplitude is _____1___.
- **d.** The period is $\underline{2\pi}$
- **e.** The cosine function is symmetric to the $\frac{y}{y}$ -axis
- **f.** On the interval $[0, 2\pi]$, the *x*-intercepts are $(\frac{\pi}{2}, 0), (\frac{3\pi}{2}, 0)$.
- **g.** On the interval $[0, 2\pi]$, the maximum points are ____(0, 1) _ and ___(2\pi, 1) _ and the minimum point is ____(\pi, -1) _.

12. For $y = \sin x$,

- **a.** The domain is $(-\infty, \infty)$
- **b.** The range is [-1, 1].
- **c.** The amplitude is ______1
- **d.** The period is $\underline{2\pi}$.
- e. The sine function is symmetric to the <u>origin</u>.
- **f.** On the interval $[0, 2\pi]$, the *x*-intercepts are $\underline{(0, 0)}, (\pi, 0), (2\pi, 0)$.
- **g.** On the interval $[0, 2\pi]$, the maximum point is $\frac{\left(\frac{\pi}{2}, 1\right)}{2\pi}$ and the minimum point is $\frac{\left(\frac{3\pi}{2}, -1\right)}{2\pi}$.
- **13. a.** Over what interval(s) taken between 0 and 2π is the graph of $y = \sin x$ increasing? $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$
 - **b.** Over what interval(s) taken between 0 and 2π is the graph of $y = \sin x$ decreasing? $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$

- **14. a.** Over what interval(s) taken between 0 and 2π is the graph of $y = \cos x$ increasing? $(\pi, 2\pi)$
 - **b.** Over what interval(s) taken between 0 and 2π is the graph of $y = \cos x$ decreasing? $(0, \pi)$

Learn: Graph $y = A\sin x$ and $y = A\cos x$

For Exercises 15–16, identify the amplitude of the function.

15. a.
$$y = 7\sin x$$
 7

b.
$$y = \frac{1}{7} \sin x$$
 $\frac{1}{7}$

c.
$$y = -7\sin x$$
 7

16. a.
$$y = 2\cos x + 2$$

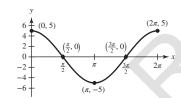
b.
$$y = \frac{1}{2} \cos x + \frac{1}{2}$$

c.
$$y = -2\cos x + 2$$

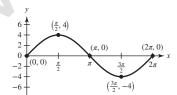
- **17.** By how many units does the graph of $y = \frac{1}{4}\cos x$ deviate from the x-axis? $\frac{1}{4}$ unit
- **18.** By how many units does the graph of $y = -5 \sin x$ deviate from the x-axis? 5 units

For Exercises 19–24, graph the function and identify the key points on one full period. (See Example 1, p. 531)

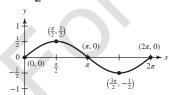
19.
$$y = 5 \cos x$$



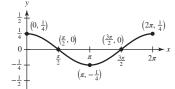
20.
$$y = 4 \sin x$$



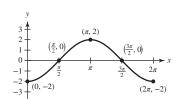
21.
$$y = \frac{1}{2} \sin x$$



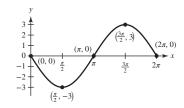
22.
$$y = \frac{1}{4}\cos x$$



23.
$$y = -2\cos x$$



24.
$$y = -3 \sin x$$



Learn: Graph $y = A \sin Bx$ and $y = A \cos Bx$

For Exercises 25–26, identify the period.

25. a.
$$\sin 2x \pi$$

b.
$$\sin 2\pi x$$
 1

c.
$$\sin(-\frac{2}{3}x)$$
 3π **c.** $\cos\frac{1}{3}\pi x$ 6

26. a.
$$\cos \frac{1}{3}x + 6\pi$$

b.
$$\cos(-3\pi x)$$
 $\frac{2}{3}$

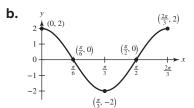
c.
$$\cos \frac{1}{3} \pi x$$

For Exercises 27–32

- **a.** Identify the amplitude and period.
- **b.** Graph the function and identify the key points on one full period. (See Example 2, p. 533)

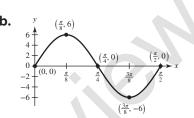
27.
$$y = 2 \cos 3x$$

a. Amplitude = 2, Period =
$$\frac{2\pi}{3}$$



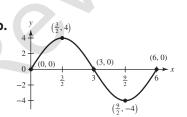
28.
$$y = 6 \sin 4x$$

a. Amplitude = 6, Period =
$$\frac{\pi}{2}$$



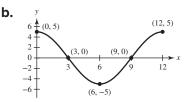
29.
$$y = 4 \sin \frac{\pi}{3} x$$

a. Amplitude
$$= 4$$
, Period $= 6$

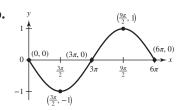


30.
$$y = 5\cos\frac{\pi}{6}x$$

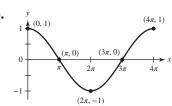
a. Amplitude =
$$5$$
, Period = 12



- **31.** $y = \sin(-\frac{1}{3}x)$
 - **a.** Amplitude = 1, Period = 6π

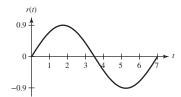


- **32.** $y = \cos(-\frac{1}{2}x)$
 - **a.** Amplitude = 1, Period = 4π



- **33.** Write a function of the form $f(x) = A\cos Bx$ for the given graph. $f(x) = 2\cos\frac{1}{2}x$
- **34.** Write a function of the form $g(x) = A\sin Bx$ for the given graph. $g(x) = -\frac{3}{2}\sin\frac{1}{4}x$
- **35.** The graph shows the percentage in decimal form of the Moon illuminated for the first 40 days of a recent year. (*Source*: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)
 - **a.** On approximately which days during this time period did a full moon occur? (A full moon corresponds to 100% or 1.0.) Day 5 and Day 35
 - **b.** On which day was there a new moon (no illumination)? Day 20
 - c. From the graph, approximate the period of a synodic month. A synodic month is the period of one lunar cycle (full moon to full moon). Approximately 30 days
- **36.** A respiratory cycle is defined as the beginning of one breath to the beginning of the next breath. The rate of air intake r (in L/sec) during a respiratory cycle for a physically fit male can be approximated by $r(t) = 0.9 \sin \frac{\pi}{3.5} t$, where t is the number of seconds into the cycle. A positive value for r represents inhalation and a negative value represents exhalation.
 - a. How long is the respiratory cycle? 7 sec
 - **b.** What is the maximum rate of air intake? 0.9 L/sec
 - **c.** Graph one cycle of the function. On what interval does inhalation occur? On what interval does exhalation occur?

Inhalation: (0, 3.5); Exhalation: (3.5, 7)



Learn: Graph $y = A\sin(Bx - C) + D$ and $y = A\cos(Bx - C) + D$

For Exercises 37–38, identify the phase shift and indicate whether the shift is to the left or to the right.

37. a. $\cos(x - \frac{\pi}{3})$ $\frac{\pi}{3}$, right

b.
$$\cos\left(2x - \frac{\pi}{3}\right) = \frac{\pi}{6}$$
, right

c.
$$\cos(3\pi x + \frac{\pi}{3}) - \frac{1}{9}$$
, left

38. a. $\sin(x + \frac{\pi}{8})$ $-\frac{\pi}{8}$, left

b.
$$\sin(2\pi x - \frac{\pi}{8}) = \frac{1}{16}$$
, right

c.
$$\sin\left(4x - \frac{\pi}{8}\right) = \frac{\pi}{32}$$
, right

For Exercises 39-44,

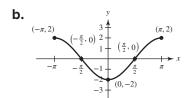
a. Identify the amplitude, period, and phase shift.

b. Graph the function and identify the key points on one full period.

(See Example 3, p. 534)

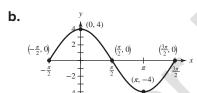
39.
$$y = 2\cos(x + \pi)$$

a. Amplitude = 2, Period =
$$2\pi$$
, Phase shift = $-\pi$



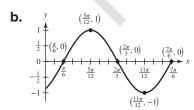
40.
$$y = 4 \sin(x + \frac{\pi}{2})$$

a. Amplitude = 4, Period =
$$2\pi$$
, Phase shift = $-\frac{\pi}{2}$



41.
$$y = \sin(2x - \frac{\pi}{3})$$

a. Amplitude = 1, Period =
$$\pi$$
, Phase shift = $\frac{\pi}{6}$



- **42.** $y = \cos(3x \frac{\pi}{4})$
 - **a.** Amplitude = 1, Period = $\frac{2\pi}{3}$, Phase shift = $\frac{\pi}{12}$
- **43.** $y = -6\cos\left(\frac{1}{2}x + \frac{\pi}{4}\right)$
 - **a.** Amplitude = 6, Period = 4π , Phase shift = $-\frac{\pi}{2}$
 - **b.** $\begin{array}{c}
 y \\
 6 \\
 4 \\
 (\frac{\pi}{2}, 0)
 \end{array}$ $\begin{array}{c}
 (\frac{3\pi}{2}, 6) \\
 (\frac{5\pi}{2}, 0)
 \end{array}$ $\begin{array}{c}
 (\frac{5\pi}{2}, 0) \\
 -\frac{\pi}{2}
 \end{array}$ $\begin{array}{c}
 \frac{\pi}{2}
 \end{array}$ $\begin{array}{c}
 \frac{3\pi}{2}
 \end{array}$ $\begin{array}{c}
 \frac{3\pi}{2}
 \end{array}$ $\begin{array}{c}
 \frac{7\pi}{2}
 \end{array}$ $\begin{array}{c}
 (\frac{7\pi}{2}, -6)
 \end{array}$
- **44.** $y = -5\sin(\frac{1}{3}x + \frac{\pi}{6})$
 - **a.** Amplitude = 5, Period = 6π , Phase shift = $-\frac{\pi}{2}$
- **45.** Write a function of the form $f(x) = A\cos(Bx C)$ for the given graph. $y = 2\cos\left(x \frac{\pi}{3}\right)$
- **46.** Write a function of the form $f(x) = A\sin(Bx C)$ for the given graph. $y = \frac{1}{2}\sin(x \frac{\pi}{4})$
- **47.** Given $y = -2\sin(2x \frac{\pi}{6}) 7$,
 - **a.** Is the period less than or greater than 2π ? Less than
 - **b.** Is the phase shift to the left or right? Right
 - c. Is the vertical shift upward or downward? Downward
- **48.** Given $y = \cos(\frac{1}{2}x + \pi) + 4$,
 - **a.** Is the period less than or greater than 2π ? Greater than
 - **b.** Is the phase shift to the left or right? Left
 - c. Is the vertical shift upward or downward? Upward

For Exercises 49–52, rewrite the equation so that the coefficient on x is positive.

49.
$$y = \cos\left(-2x + \frac{\pi}{6}\right) - 4$$
 $y = \cos\left(2x - \frac{\pi}{6}\right) - 4$

50.
$$y = 4\cos(-3x - \pi) + 5$$
 $y = 4\cos(3x + \pi) + 5$

51.
$$y = \sin(-2x + \frac{\pi}{6}) - 4$$
 $y = -\sin(2x - \frac{\pi}{6}) - 4$

52.
$$y = 4\sin(-3x - \pi) + 5$$
 $y = -4\sin(3x + \pi) + 5$

53. Given
$$y = 2\sin\left(-\frac{\pi}{6}x + \frac{\pi}{2}\right) - 3$$
, is the phase shift to the right or left? Right

54. Given
$$y = -4\cos\left(-\frac{\pi}{6}x - \frac{\pi}{2}\right) + 1$$
, is the phase shift to the right or left? Left

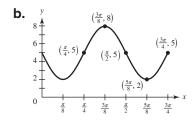
For Exercises 55-68,

a. Identify the amplitude, period, phase shift, and vertical shift.

b. Graph the function and identify the key points on one full period.(See Examples 4–5, p. 536, 538)

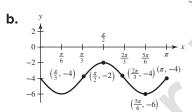
55.
$$h(x) = 3\sin(4x - \pi) + 5$$

a. Amplitude = 3, Period =
$$\frac{\pi}{2}$$
, Phase shift = $\frac{\pi}{4}$, Vertical shift = 5



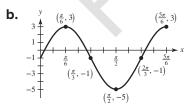
56.
$$g(x) = 2\sin(3x - \pi) - 4$$

a. Amplitude = 2, Period =
$$\frac{2\pi}{3}$$
, Phase shift = $\frac{\pi}{3}$, Vertical shift = -4

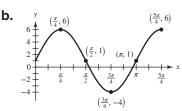


57.
$$f(x) = 4\cos\left(3x - \frac{\pi}{2}\right) - 1$$

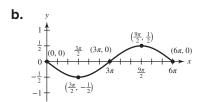
a. Amplitude = 4, Period =
$$\frac{2\pi}{3}$$
, Phase shift = $\frac{\pi}{6}$, Vertical shift = -1



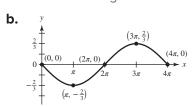
- **58.** $k(x) = 5\cos\left(2x \frac{\pi}{2}\right) + 1$
 - **a.** Amplitude = 5, Period = π , Phase shift = $\frac{\pi}{4}$, Vertical shift = 1



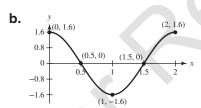
- **59.** $y = \frac{1}{2} \sin(-\frac{1}{3}x)$
 - **a.** Amplitude $=\frac{1}{2}$, Period $=6\pi$, Phase shift =0, Vertical shift =0



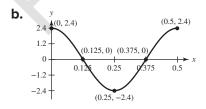
- **60.** $y = \frac{2}{3} \sin(-\frac{1}{2}x)$
 - **a.** Amplitude $=\frac{2}{3}$, Period $=4\pi$, Phase shift =0, Vertical shift =0



- **61.** $v(x) = 1.6 \cos(-\pi x)$
 - **a.** Amplitude = 1.6, Period = 2, Phase shift = 0, Vertical shift = 0



- **62.** $m(x) = 2.4 \cos(-4\pi x)$
 - **a.** Amplitude = 2.4, Period = 0.5, Phase shift = 0, Vertical shift = 0



a. Amplitude = 2, Period = π , Phase shift = $-\frac{\pi}{2}$, Vertical shift = 5

64. $y = 3\sin(-4x - \pi) - 7$

a. Amplitude = 3, Period = $\frac{\pi}{2}$, Phase shift = $-\frac{\pi}{4}$, Vertical shift = -7

b.y $\frac{2}{4}$ $-\frac{\pi}{4}$ $-\frac{\pi}{8}$ -2 $-\frac{\pi}{8}$ -4

65. $p(x) = -\cos(-\frac{\pi}{2}x - \pi) + 2$

a. Amplitude = 1, Period = 4, Phase shift = -2, Vertical shift = 2

b. (-1,2) (-2,1) 1 -2 -1 1 2 (2,1) 1 2 x

66. $q(x) = -\cos(-\frac{\pi}{3}x - \pi) - 2$

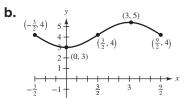
a. Amplitude = 1, Period = 6, Phase shift = -3, Vertical shift = -2

b.y (-3, -3) (0, -1) (3, -3)y (0, -1) (3, -3) (3, -3)

67. $y = \sin\left(-\frac{\pi}{4}x - \frac{\pi}{2}\right) - 3$

a. Amplitude = 1, Period = 8, Phase shift = -2, Vertical shift = -3

- **68.** $y = \sin\left(-\frac{\pi}{3}x \frac{\pi}{2}\right) + 4$
 - **a.** Amplitude = 1, Period = 6, Phase shift = $-\frac{3}{2}$, Vertical shift = 4



- **69.** The temperature T(in °F) for a midwestern city over a several day period in April can be approximated by $T(t) = -5.9 \cos(0.262t 1.245) + 48.2$, where t is the number of hours since midnight on day 1.
 - a. What is the period of the function? Round to the nearest hour. 24 hr
 - **b.** What is the significance of the term 48.2 in this model? 48.2°F represents the average temperature for the day.
 - **c.** What is the significance of the factor 5.9 in this model?

The factor 5.9 is the amplitude of the curve and represents the amount that the temperature deviates above or below the daily average.

- **d.** What was the minimum temperature for the day? When did it occur? 42.3° at approximately 4:45 A.M.
- **e.** What was the maximum temperature for the day? When did it occur? 54.1° at approximately 4:45 P.M.
- **70.** The duration of daylight and darkness varies during the year due to the angle of the Sun in the sky. The model $d(t) = 2.65 \sin(0.51t 1.32) + 12$ approximates the amount of daylight d(t) (in hours) for a northern city as a function of the time t (in months) after January 1 for a recent year; that is, t = 0 is January 1, t = 1 is February 1, and so on. The model y = n(t) represents the amount of darkness as a function of t.
 - **a.** Describe the relationship between the graphs of the functions and the line y = 12.

The two graphs are symmetric with respect to the line y = 12.

- **b.** Use the result of part (a) and a transformation of y = d(t) to write an equation representing n as a function of t. $n(t) = -2.65 \sin(0.51t 1.32) + 12$
- c. What do the points of intersection of the two graphs represent?

The points of intersection represent the numbers of the months since January 1 when the duration of daylight and darkness are each 12 hr. In the northern hemisphere, these correspond to the first day of spring (March equinox) and first day of fall (September equinox).

d. What do the relative minima and relative maxima of the graphs represent?

d(t) has a relative maximum and n(t) has a relative minimum on the first day of summer (the longest day). d(t) has a relative minimum and n(t) has a relative maximum on the first day of winter (the shortest day).

e. What does T(t) = d(t) + n(t) represent?

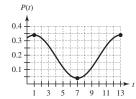
The value of T(t) is the sum of the number of hours of daylight and the number of hours of darkness. This is the constant function T(t) = 24.

Learn: Model Sinusoidal Behavior

- **71.** The probability of precipitation in Modesto, California, varies from a peak of 0.34 (34%) in January to a low of 0.04 (4%) in July. Assume that the percentage of precipitation varies monthly and behaves like a cosine curve.
 - **a.** Write a function of the form $P(t) = A \cos(Bt C) + D$ to model the precipitation probability. The value P(t) is the probability of precipitation (as a decimal), for month t, with January as t = 1.

$$P(t) = 0.15 \cos\left(\frac{\pi}{6}t - \frac{\pi}{6}\right) + 0.19$$

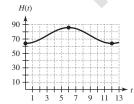
b. Graph the function from part (a) on the interval [0, 13] and plot the points (1, 0.34), (7, 0.04), and (13, 0.34) to check the accuracy of your model.



- **72.** The monthly high temperature for Atlantic City, New Jersey, peaks at an average high of 86° in July and goes down to an average high of 64° in January. Assume that this pattern for monthly high temperatures continues indefinitely and behaves like a cosine wave.
 - **a.** Write a function of the form $H(t) = A\cos(Bt C) + D$ to model the average high temperature. The value H(t) is the average high temperature for month t, with January as t = 0.

$$H(t) = -11\cos\left(\frac{\pi}{6}t\right) + 75$$

b. Graph the function from part (a) on the interval [0, 13] and plot the points (0, 64), (6, 86), and (12, 64) to check the accuracy of your model.



- 73. An adult human at rest inhales and exhales approximately 500 mL of air (called the tidal volume) in approximately 5 sec. However, at the end of each exhalation, the lungs still contain a volume of air, called the functional residual capacity (FRC), which is approximately 2000 mL. (See Example 6, p. 540)
 - a. What volume of air is in the lungs after inhalation? 2500 mL
 - **b.** What volume of air is in the lungs after exhalation? 2000 mL
 - c. What is the period of a complete respiratory cycle? 5 sec
 - **d.** Write a function $V(t) = A \cos Bt + D$ to represent the volume of air in the lungs t seconds after the end of an inhalation. $V(t) = 250 \cos \frac{2\pi}{5}t + 2250$
 - e. What is the average amount of air in the lungs during one breathing cycle?
 2250 mL
 - **f.** During hyperventilation, breathing is more rapid with deep inhalations and exhalations. What parts of the equation from part (d) change?

The period would shorten and the amplitude would increase.

- **74.** The times for high and low tides are given in the table for a recent day in Jacksonville Beach, Florida. The times are rounded to the nearest hour, and the tide levels are measured relative to mean sea level (MSL).
 - **a.** Write a model $h(t) = A\cos(Bt C)$ to represent the tide level h(t) (in feet) in terms of the amount of time t elapsed since midnight. $h(t) = 3.4\cos\frac{\pi t}{6}$
 - **b.** Use the model from part (a) to estimate the tide level at 3:00 P.M. 0 ft

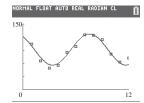
| | Time (hr after midnight) | Height Relative to MSL (ft) |
|-----------|-----------------------------|-----------------------------|
| High tide | 0 | 3.4 |
| Low tide | 6 | -3.4 |
| High tide | 12 | 3.4 |
| Low tide | 18 | -3.4 |
| High tide | 24 | 3.4 |

- **75.** The data in the table represent the monthly power bills (in dollars) for a homeowner.
 - **a.** Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $A(t) = a\sin(bt + c) + d$, where A(t) represents the amount of the bill for month t(t = 1) represents January, t = 2 represents February, and so on). $A(t) = 35.4 \sin(0.753t + 2.18) + 92.2$

b. Graph the data and the resulting function.

| Month, t | 1 | 2 | 3 | 4 | 5 | 6 |
|-------------|--------|-------|-------|-------|-------|-------|
| Amount (\$) | 104.73 | 66.13 | 48.99 | 56.04 | 85.51 | 98.57 |

| Month, t | 7 | 8 | 9 | 10 | 11 | 12 |
|-------------|--------|--------|--------|-------|-------|-------|
| Amount (\$) | 125.08 | 124.48 | 113.93 | 81.06 | 63.30 | 71.85 |



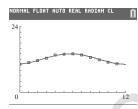
- **76.** The data in the table represent the duration of daylight *d*(*t*) (in hours) for Houston, Texas, for the first day of the month, *t* months after January 1 for a recent year. (*Source*: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)
 - **a.** Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $d(t) = a \sin(bt + c) + d$.

$$d(t) = 1.9\sin(0.5t - 1.3) + 12.2$$

b. Graph the data and the resulting function.

| Month, t | 0 | 1 | 2 | 3 | 4 | 5 |
|-----------|-------|-------|-------|-------|-------|-------|
| Time (hr) | 10.28 | 10.80 | 11.57 | 12.48 | 13.65 | 13.95 |

| Month, t | 6 | 7 | 8 | 9 | 10 | 11 |
|-----------|-------|-------|-------|-------|-------|-------|
| Time (hr) | 14.02 | 13.55 | 12.73 | 11.87 | 11.00 | 10.38 |



Mixed Exercises

For Exercises 77–78, write the range of the function in interval notation.

77. a.
$$y = 8\cos(2x - \pi) + 4$$
 [-4, 12]

b.
$$y = -3\cos(x + \frac{\pi}{3}) - 5$$
 [-8, -2]

78. a.
$$y = -6\sin(3x - \frac{\pi}{2}) - 2$$
 [-8, 4]

b.
$$y = 2 \sin(3x + 2\pi) + 12$$
 [10, 14]

79. Given $f(x) = \cos x$ and h(x) = 3x + 2, find $(h \circ f)(x)$ and for $(h \circ f)(x)$,

$$(h \circ f)(x) = 3\cos x + 2$$

- **a.** Find the amplitude. 3
- **b.** Find the period. 2π
- **c.** Write the domain in interval notation. $(-\infty, \infty)$
- **d.** Write the range in interval notation. [-1, 5]
- **80.** Given $g(x) = \sin x$ and k(x) = 6x, find $(g \circ k)(x)$ and for $(g \circ k)(x)$,

$$(g \circ k)(x) = \sin 6x$$

- a. Find the amplitude. 1
- **b.** Find the period. $\frac{\pi}{3}$
- **c.** Write the domain in interval notation. $(-\infty, \infty)$
- **d.** Write the range in interval notation. [-1, 1]
- **81.** Write a function of the form $y = A\sin(Bx C) + D$ that has period $\frac{\pi}{3}$, amplitude 4, phase shift $\frac{\pi}{2}$, and vertical shift 5.

$$y = 4\sin(6x - 3\pi) + 5$$
 or $y = -4\sin(6x - 3\pi) + 5$

82. Write a function of the form $y = A\cos(Bx - C) + D$ that has period $\frac{\pi}{4}$, amplitude 2, phase shift $-\frac{\pi}{3}$, and vertical shift 7.

$$y = 2\cos\left(8x + \frac{8\pi}{3}\right) + 7 \text{ or } y = -2\cos\left(8x + \frac{8\pi}{3}\right) + 7$$

83. Write a function of the form $y = A\cos(Bx - C) + D$ that has period 16, phase shift -4, and range $3 \le y \le 7$.

$$y = 2\cos(\frac{\pi}{8}x + \frac{\pi}{2}) + 5$$
 or $y = -2\cos(\frac{\pi}{8}x + \frac{\pi}{2}) + 5$

84. Write a function of the form $y = A\sin(Bx - C) + D$ that has period 8, phase shift -2, and range $-14 \le y \le -6$.

$$y = 4\sin(\frac{\pi}{4}x + \frac{\pi}{2}) - 10$$
 or $y = -4\sin(\frac{\pi}{4}x + \frac{\pi}{2}) - 10$

- **85.** Describe the transformation applied to the dashed curve to make the solid curve. Choose from: change in amplitude, change in period, phase shift, or vertical shift. Then write an equation for the solid curve in terms of the cosine function.
 - **a.** Vertical shift; $y = \cos x + 2$
 - **b.** Change in amplitude; $y = 2\cos x$
 - **c.** Change in period; $y = \cos 2x$
 - **d.** Phase shift; $y = \cos\left(x + \frac{\pi}{4}\right)$

- **86.** Describe the transformations (there may be more than one) applied to the dashed curve to make the solid curve. Choose from: change in amplitude, change in period, phase shift, or vertical shift. Then write an equation for the solid curve in terms of the sine function.
 - **a.** Change in amplitude and period; $y = 2 \sin 2x$
 - **b.** Change in amplitude and vertical shift; $y = 2 \sin x + 2$
 - **c.** Change in period and phase shift; $y = \sin(2x \frac{\pi}{4})$
 - **d.** Phase shift and vertical shift; $y = \sin(x + \frac{\pi}{4}) 2$
- **87.** Write an equation of the form $y = A \sin Bx$ to model the graph.

$$y = \frac{\pi}{2} \sin\left(\frac{\pi}{3}x\right)$$

88. Write an equation of the form $y = A\cos Bx$ to model the graph.

$$y = \frac{\pi}{2} \cos\left(\frac{\pi}{4}x\right)$$

For Exercises 89-90,

- **a.** Write an equation of the form $y = A\cos(Bx C) + D$ with A > 0 to model the graph.
- **b.** Write an equation of the form $y = A\sin(Bx C) + D$ with A > 0 to model the graph.
- **89. a.** $y = 2\cos\left(x \frac{\pi}{4}\right) + 3$
 - **b.** $y = 2\sin(x + \frac{\pi}{4}) + 3$
- **90. a.** $y = 3\cos(2x \frac{\pi}{2}) + 1$
 - **b.** $y = 3 \sin 2x + 1$

For Exercises 91–94, explain how to graph the given function by performing transformations on the "parent" graphs $y = \sin x$ and $y = \cos x$.

- **91. a.** $y = \sin 2x$ The graph of $y = \sin x$ is compressed horizontally by a factor of 2.
 - **b.** $y = 2\sin x$ The graph of $y = \sin x$ is vertically stretched by a factor of 2.
- **92. a.** $y = \frac{1}{3}\cos x$ The graph of $y = \cos x$ is vertically compressed by a factor of $\frac{1}{3}$.
 - **b.** $y = \cos \frac{1}{3}x$ The graph of $y = \cos x$ is stretched horizontally by a factor of 3.
- **93. a.** $y = \sin(x + 2)$ The graph of $y = \sin x$ is shifted 2 units to the left.
 - **b.** $y = \sin x + 2$ The graph of $y = \sin x$ is shifted vertically upward 2 units.
- **94. a.** $y = \cos x 4$ The graph of $y = \cos x$ is shifted vertically downward 4 units.
 - **b.** $y = \cos(x 4)$ The graph of $y = \cos x$ is shifted 4 units to the right.

Write About It

95. Is $f(x) = \sin x$ one-to-one? Explain why or why not.

No. If f were one-to-one, then f(a) = f(b) would imply that a = b. However, as a counterexample, $f\left(\frac{\pi}{4}\right) = f\left(\frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$ but $\frac{\pi}{4} \neq \frac{3\pi}{4}$.

96. Is $f(x) = \cos x$ one-to-one? Explain why or why not.

No. If f were one-to-one, then f(a) = f(b) would imply that a = b. However, as a counterexample, $f\left(\frac{\pi}{3}\right) = f\left(\frac{5\pi}{3}\right) = \frac{1}{2}$ but $\frac{\pi}{3} \neq \frac{5\pi}{3}$.

97. If f and g are both periodic functions with period P, is (f + g)(x) also periodic? Explain why or why not.

Yes. (f+g)(x) = f(x) + g(x) = f(x+P) + g(x+P) = (f+g)(x+P) for all x in the domain of f and g.

Expanding Your Skills

98. Explain why a function that is increasing on its entire domain cannot be periodic.

If a function f is increasing over its entire domain, then for any interval [a, b] in the domain of f, f(a) < f(b). However, if f were periodic, then there would exist a positive number P such that f(a + P) = f(a). This contradicts the fact that f is increasing.

For Exercises 99–100, find the average rate of change on the given interval. Give the exact value and an approximation to 4 decimal places. Verify that your results are reasonable by comparing the results to the slopes of the lines given in the graph.

99.
$$f(x) = \sin x$$

a.
$$\left[0, \frac{\pi}{6}\right] \quad \frac{3}{\pi} \approx 0.9549$$

a.
$$\left[0, \frac{\pi}{6}\right] \quad \frac{3}{\pi} \approx 0.9549$$
b. $\left[\frac{\pi}{6}, \frac{\pi}{3}\right] \quad \frac{3(\sqrt{3} - 1)}{\pi} \approx 0.6991$
c. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right] \quad \frac{3(2 - \sqrt{3})}{\pi} \approx 0.2559$

c.
$$\left[\frac{\pi}{3}, \frac{\pi}{2}\right] \quad \frac{3(2-\sqrt{3})}{\pi} \approx 0.2559$$

100.
$$f(x) = \cos x$$

a.
$$\left[0, \frac{\pi}{6}\right] \frac{3(\sqrt{3}-2)}{\pi} \approx -0.2559$$

b. $\left[\frac{\pi}{6}, \frac{\pi}{3}\right] \frac{3(1-\sqrt{3})}{\pi} \approx -0.6991$
c. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right] -\frac{3}{\pi} \approx -0.9549$

b.
$$\left[\frac{\pi}{6}, \frac{\pi}{3}\right] \quad \frac{3(1-\sqrt{3})}{\pi} \approx -0.6991$$

c.
$$\left[\frac{\pi}{3}, \frac{\pi}{2}\right] - \frac{3}{\pi} \approx -0.9549$$

101.
$$\sin x = x - 2$$
 One solution

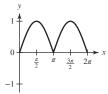
102.
$$\cos x = -x$$
 One solution

103.
$$\sin 2x = -2$$
 No solution

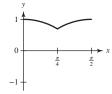
104.
$$2 \sin 2x = -2$$
 Infinitely many solutions

For Exercises 105–106, graph the piecewise-defined function.

105.
$$g(x) = \begin{cases} \sin x \text{ for } 0 \le x \le \pi \\ -\sin x \text{ for } \pi < x \le 2\pi \end{cases}$$



106.
$$f(x) = \begin{cases} \cos x \text{ for } 0 \le x \le \frac{\pi}{4} \\ \sin x \text{ for } \frac{\pi}{4} < x \le \frac{\pi}{2} \end{cases}$$

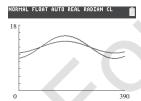


Technology Connections

107. Functions a and m approximate the duration of daylight, respectively, for Albany, New York, and Miami, Florida, for a given year for day t. The value t=1 represents January 1, t=2 represents February 1, and so on.

$$a(t) = 12 + 3.1 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$
 $m(t) = 12 + 1.6 \sin\left[\frac{2\pi}{365}(t - 80)\right]$

a. Graph the two functions with a graphing utility and comment on the difference between the two graphs.



b. Both functions have a constant term of 12. What does this represent graphically and in the context of this problem?

The constant term of 12 is the vertical shift of the parent sine function. It represents the "equilibrium" value for duration of sunlight from which the amplitude is measured.

The factors 3.1 and 1.6, represent the amplitude of the functions a and m, respectively. The amplitude represents the greatest amount of deviation above and below the line y = 12.

d. What is the period of each function?

The period of each function is 365 days.

e. What does the horizontal shift of 80 units represent in the context of this problem?

The horizontal shift represents the number of days from January 1 to the point where the duration of light and darkness is the same (12 hr each). This is the approximate number of days to the March equinox (first day of spring).

f. Use the Intersect feature to approximate the points of intersection. (80, 12) and (262.5, 12)

g. Interpret the meaning of the points of intersection.

The points of intersection represent the points where the duration of light and darkness is the same. They represent the March and September equinoxes where the durations of light and darkness are each 12 hr.

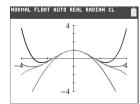
For Exercises 108–109, we demonstrate that trigonometric functions can be approximated by polynomial functions over a given interval in the domain.

Graph functions f, g, h, and k on the viewing window $-4 \le x \le 4$, $-4 \le y \le 4$. Then use a Table feature on a graphing utility to evaluate each function for the given values of x. How do functions g, h, and k compare to function f for x values farther from 0? [Hint: For a given natural number n, the value n!, read as "n factorial," is defined as n! = n (n – 1)(n – 2) \cdots 1. For example, 3! = n 2 · 1 = 6.]

108.

| Function | x = 0.1 | <i>x</i> = 0.5 | x = 1.0 | <i>x</i> = 1.5 |
|---|---------|----------------|---------|----------------|
| $f(x) = \cos x$ | | | | |
| $g(x) = 1 - \frac{x^2}{2!}$ | | | | |
| $h(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ | | | | |
| $k(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$ | | | | |

Each polynomial function gives a better approximation for values of x close to zero than for values farther from zero. Furthermore, the greater the degree of the polynomial, the better the approximation of $\cos x$ for each value of x.

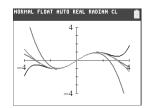


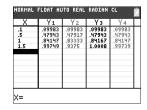
| X | Υı | Y 2 | Υз | Y4 |
|-----|--------|------|--------|--------|
| .1 | .995 | .995 | .995 | .995 |
| .5 | .87758 | .875 | .8776 | .87758 |
| 1.5 | .5403 | .5 | .54167 | .54028 |
| 1.5 | .07074 | 125 | .08594 | .07012 |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |
| | | | | |

109.

| Function | x = 0.1 | <i>x</i> = 0.5 | x = 1.0 | x = 1.5 |
|---|---------|----------------|---------|---------|
| $f(x) = \sin x$ | | | | |
| $g(x) = x - \frac{x^3}{3!}$ | | | | |
| $h(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ | | | | |
| $k(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ | | | | |

Each polynomial function gives a better approximation for values of x close to zero than for values farther from zero. Furthermore, the greater the degree of the polynomial, the better the approximation of $\sin x$ for each value of x.



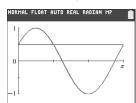


For Exercises 110–111, use a graph to solve the equation on the given interval.

110.
$$\cos(2x - \frac{\pi}{3}) = 0.5$$
 on $[0, \pi]$

Viewing window:
$$\left[0, \pi, \frac{\pi}{3}\right]$$
 by $\left[-1, 1, \frac{1}{2}\right]$

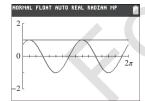
$$x = 0, x = \frac{\pi}{3}, x = \pi$$



111.
$$\sin(2x + \frac{\pi}{4}) = 1$$
 on $[0, 2\pi]$

Viewing window:
$$\left[0, 2\pi, \frac{\pi}{8}\right]$$
 by $\left[-2, 2, 1\right]$

$$x = \frac{\pi}{8}, x = \frac{9\pi}{8}$$

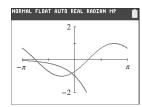


For Exercises 112–113, use a graph to solve the equation on the given interval. Round the answer to 2 decimal places.

112.
$$\sin(x - \frac{\pi}{4}) = -e^x \text{ on } [-\pi, \pi]$$

Viewing window: $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $\left[-2, 2, 1\right]$

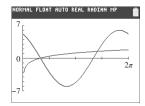
$$x \approx -2.25, x \approx -0.19$$



113.
$$6\cos\left(x + \frac{\pi}{6}\right) = \ln x$$
 on $[0, 2\pi]$

Viewing window: $\left[0, 2\pi, \frac{\pi}{2}\right]$ by $\left[-7, 7, 1\right]$

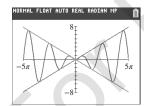
$$x \approx 1.04, x \approx 4.44$$



Variable Amplitude: The graphs of functions such as $f(x) = a\cos x$ and $g(x) = a\sin x$ are curves with a *constant* amplitude, a. If we have a nonconstant coefficient a(x), we will have a *variable* amplitude with the graph of sine or cosine bounded by the graphs of y = a(x) and y = -a(x). Use this information for Exercises 114–115.

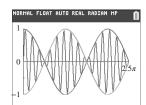
114. Graph $f(x) = 0.5x \sin x$ on the viewing window $[-5\pi, 5\pi, \pi]$ by [-8, 8, 1] and give the equations of the bounding functions.

$$y = 0.5x$$
, $y = -0.5x$



115. Graph $g(x) = \cos x \sin 12x$ on the viewing window $[0, 2.5\pi, 0.25\pi]$ by [-1, 1, 0.5] and give the equations of the bounding functions.

$$y = \cos x, y = -\cos x$$



Lesson 4-6 Graphs of Other Trigonometric Functions

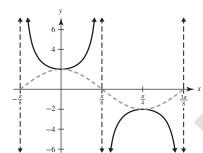
Teaching Tips

Each fully worked example is paired with end-of-lesson exercises that mirror the example. These end-of-lesson exercises are excellent problems to work through together in class. Teacher Notes on specific issues are included where applicable.

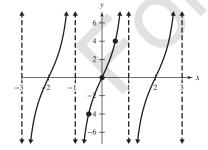
| Example | Exercise | Teacher Notes |
|---------|----------|--|
| 1 | 22 | Emphasize to students that the reciprocal relationships among the six trigonometric functions are crucial when determining the equations of the vertical asymptotes. |
| 2 | 32 | Consider using the graphs of the secant, cosecant, tangent, and cotangent functions to review the notation used for unbounded behavior. For example, as $x \to 0+$, $\csc x \to \infty$. |
| 3 | 42 | |
| 4 | 58 | |

Lesson 4-6 Apply the Skills

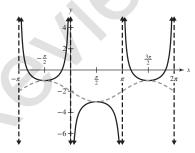
1. Graph $y = 2 \sec 4x$.



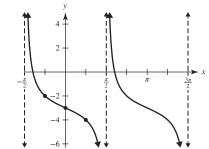
3. Graph $y = 4 \tan \frac{\pi}{2} x$.



2. Graph $y = \csc(x + \pi) - 2$.



4. Graph $y = \cot(x + \frac{\pi}{2}) - 3$.



Lesson 4-6 Practice

Prerequisite Review

R.1. Write the domain of the function in interval notation.

 $f(x) = \frac{x^2 - 16}{x - 4} \quad (-\infty, 4) \cup (4, \infty)$

R.2. Refer to the graph of the function to complete the statement.

a. As $x \to -2^-$, $f(x) \to \underline{\hspace{1cm} -\infty}$.

b. As $x \to -2^+$, $f(x) \to _{---}$

R.3. Determine the vertical asymptotes of the graph of the function.

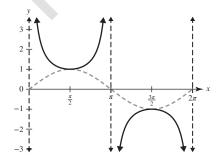
 $f(x) = \frac{6}{x - 5}$ x = 5

Concept Connections

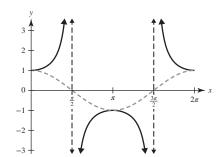
- **1.** At values of x for which $\sin x = 0$, the graph of $y = \csc x$ will have a <u>vertical</u> asymptote. This occurs for $x = \underline{n\pi}$ for all integers n.
- **3.** The relative maxima on the graph of $y = \sin x$ correspond to the <u>relative</u> minima on the graph of $y = \csc x$.
- **4.** The graph of $y = \csc x$ is symmetric with respect to the <u>origin</u>. The graph of $y = \sec x$ is symmetric with respect to the <u>y</u>-axis.
- **5.** If a function is an odd function, then each point (x, y) in Quadrant I will have a corresponding point $(\underline{-x}, \underline{-y})$ in Quadrant $\underline{|||}$.
- **6.** The range of $y = \tan x$ and $y = \cot x$ is all real numbers or $(-\infty, \infty)$ or R.
- **7.** The graphs of both $y = \tan x$ and $y = \cot x$ are symmetric with respect to the origin
- **8.** For the functions $y = A \tan(Bx C)$ and $y = A \cot(Bx C)$ with B > 0, the vertical scaling factor is $A = A \cot(Bx C)$, the period is $A = B = A \cot(Bx C)$, and the phase shift is $A = B = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$ with $A = A \cot(Bx C)$ and $A = A \cot(Bx C)$

Learn: Graph the Secant and Cosecant Functions

9. Sketch the graph of $y = \csc x$ from memory. Use the graph of $y = \sin x$ for reference.



10. Sketch the graph of $y = \sec x$ from memory. Use the graph of $y = \cos x$ for reference.

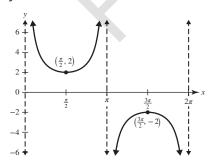


For Exercises 11–16, identify the statements among a-h that follow directly from the given condition about x.

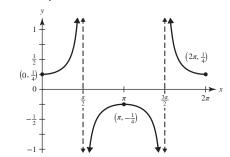
- **a.** csc x is undefined.
- **b.** sec *x* is undefined.
- **c.** The graph of $y = \sec x$ has a relative maximum at x.
- **d.** The graph of $y = \csc x$ has a relative minimum at x.
- **e.** The graph of $y = \sec x$ has a vertical asymptote.
- **f.** The graph of $y = \csc x$ has a vertical asymptote.
- **g.** The graph of $y = \csc x$ has a relative maximum at x.
- **h.** The graph of $y = \sec x$ has a relative minimum at x.
- **11.** $\sin x = 0$ a, f
- **12.** $\cos x = 0$ b, e
- **13.** The graph of $y = \cos x$ has a relative maximum at x.
- **14.** The graph of $y = \sin x$ has a relative minimum at x. g
- **15.** The graph of $y = \cos x$ has a relative minimum at x.
- **16.** The graph of $y = \sin x$ has a relative maximum at x.

For Exercises 17–32, graph one period of the function. (See Examples 1–2, pp. 554–555)

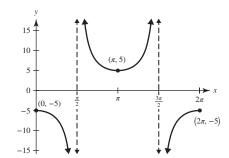
17. $y = 2 \csc x$



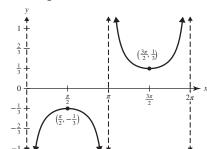
18. $y = \frac{1}{4} \sec x$



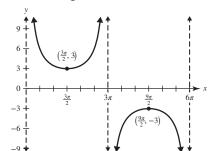
19.
$$y = -5 \sec x$$



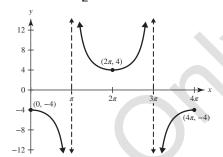
20.
$$y = -\frac{1}{3}\csc x$$



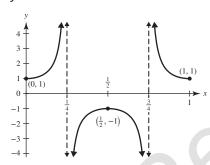
21.
$$y = 3 \csc \frac{x}{3}$$



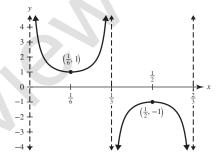
22.
$$y = -4 \sec \frac{x}{2}$$



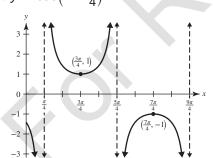
23.
$$y = \sec 2\pi x$$



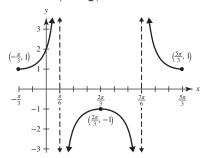
24.
$$y = \csc 3\pi x$$



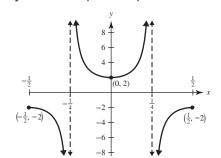
25.
$$y = \csc(x - \frac{\pi}{4})$$



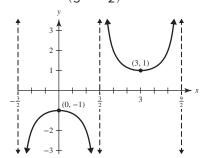
26.
$$y = \sec(x + \frac{\pi}{3})$$



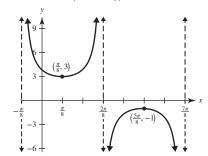
27.
$$y = -2\sec(2\pi x + \pi)$$



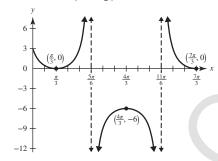
28.
$$y = -\csc(\frac{\pi}{3}x + \frac{\pi}{2})$$



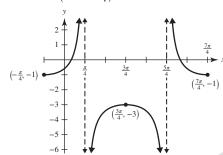
29.
$$y = 2\csc(2x + \frac{\pi}{4}) + 1$$



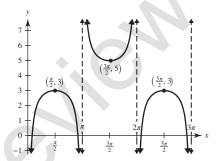
30.
$$y = 3\sec\left(x - \frac{\pi}{3}\right) - 3$$



31.
$$y = \sec\left(-x - \frac{\pi}{4}\right) - 2$$



32.
$$y = -\csc(-x + \pi) + 4$$



For Exercises 33-34, write the range of the function in interval notation.

33. a.
$$y = -4\csc(2x - \pi) + 7$$
 $(-\infty, 3] \cup [11, \infty)$

b.
$$y = 2 \csc(\frac{\pi}{3}x) - 10 \quad (-\infty, -12] \cup [-8, \infty)$$

34. a.
$$y = -3\sec(5\pi x) - 1$$
 $(-\infty, -4] \cup [2, \infty)$

b.
$$y = 5 \sec(3x - \frac{\pi}{2}) - 2 \quad (-\infty, -7] \cup [3, \infty)$$

35. Write a function of the form
$$y = A \sec Bx$$
 for the given graph. $y = 2 \sec 3x$

36. Write a function of the form $y = \csc(Bx - C)$ for the given graph.

$$y = \csc\left(2x - \frac{\pi}{4}\right)$$

- 37. A plane flying at an altitude of 5 mi travels on a path directly over a radar tower.
 - **a.** Express the distance $d(\theta)$ (in miles) between the plane and the tower as a function of the angle θ in standard position from the tower to the plane. $d(\theta) = 5 \csc \theta$

- **b.** If $d(\theta) = 5$, what is the measure of the angle and where is the plane located relative to the tower? $\theta = 90^{\circ}$ and the plane is directly overhead.
- **c.** Can the value of θ be π ? Explain your answer in terms of the function d. No. π is not in the domain of d.

38. The distance d(x) (in feet) between an observer 30 ft from a straight highway and a police car traveling down the highway is given by $d(x) = 30 \sec x$, where x is the angle (in degrees) between the observer and the police car.

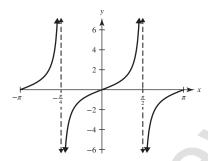
a. Use a calculator to evaluate d(x) for the given values of x. Round to the nearest foot.

| X | 45 | 60 | 70 | 80 | 89 |
|------|----|----|----|-----|------|
| d(x) | 42 | 60 | 88 | 173 | 1719 |

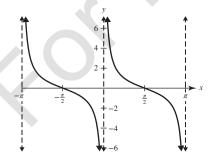
b. Try experimenting with values of x closer to 90°. What happens as $x \to 90^\circ$? The distance approaches infinity.

Learn: Graph the Tangent and Cotangent Functions

39. a. Graph $y = \tan x$ on the interval $[-\pi, \pi]$.



- **b.** How many periods of the tangent function are shown on the interval $[-\pi, \pi]$?
- **40. a.** Graph $y = \cot x$ on the interval $[-\pi, \pi]$.

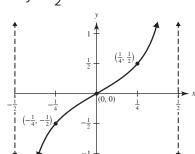


b. How many periods of the cotangent function are shown on the interval $[-\pi, \pi]$?

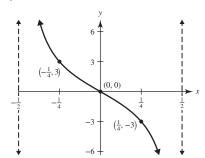
2

For Exercises 41–42, graph one complete period of the function. Identify the x-intercept and evaluate the function for x values midway between the x-intercept and the asymptotes. (See Example 3, p. 559)

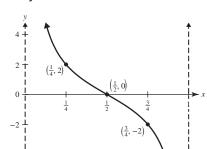
41. a.
$$y = \frac{1}{2} \tan \pi x$$



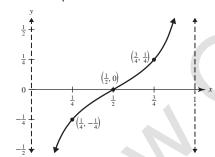
b.
$$y = -3 \tan \pi x$$



42. a.
$$y = 2 \cot \pi x$$

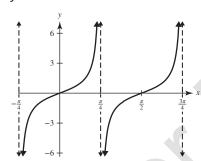


b.
$$y = -\frac{1}{4}\cot \pi x$$

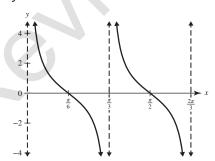


For Exercises 43–58, graph the function. (See Examples 3–4, pp. 559–560)

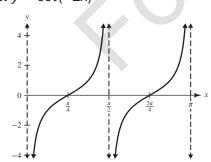
43.
$$y = \tan 2x$$



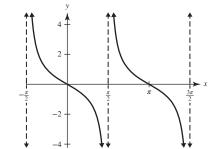
44.
$$y = \cot 3x$$

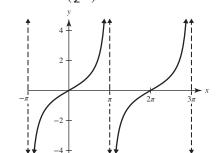


45.
$$y = \cot(-2x)$$

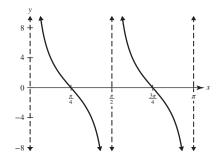


46.
$$y = \tan(-x)$$

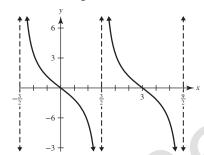




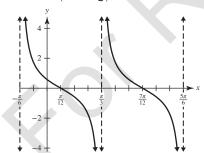
49. $y = 4 \cot 2x$



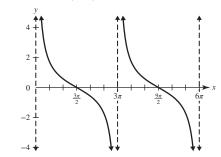
51. $y = -2 \tan \frac{\pi}{3} x$



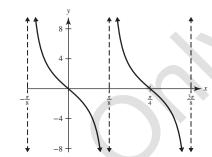
53. $y = \cot(2x + \frac{\pi}{3})$



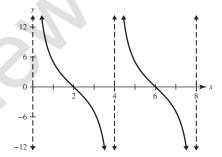
48. $y = \cot(\frac{1}{3}x)$



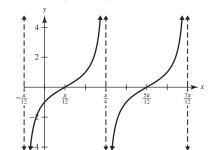
50. $y = -3 \tan 4x$



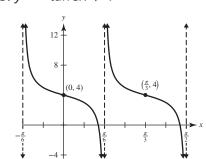
52. $y = 5 \cot \frac{\pi}{4} x$



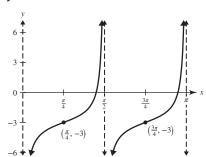
54. $y = \tan(3x - \frac{\pi}{4})$



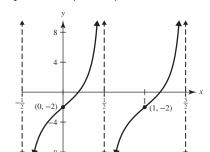
55.
$$y = -\tan 3x + 4$$



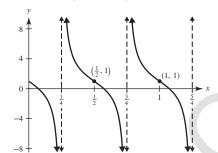
56.
$$y = -\cot 2x - 3$$



57.
$$y = 3 \tan(\pi x + \pi) - 2$$



58. $y = 2\cot\left(2\pi x - \frac{\pi}{2}\right) + 1$



59. Write a function of the form $y = \tan(Bx - C)$ for the given graph.

$$y = \tan\left(x - \frac{\pi}{4}\right)$$

60. Write a function of the form $y = \cot(Bx)$ for the given graph. $y = \cot(4\pi x)$

Mixed Exercises

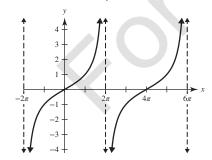
For Exercises 61–64, given y = f(x) and y = g(x),

a. Find $(f \circ g)(x)$ and graph the resulting function.

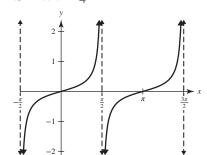
b. Find $(g \circ f)(x)$ and graph the resulting function.

61. $f(x) = \tan x$ and $g(x) = \frac{x}{4}$

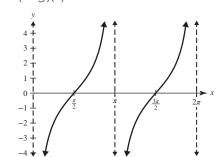
a.
$$(f \circ g)(x) = \tan \frac{x}{4}$$



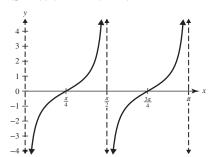
b.
$$(g \circ f)(x) = \frac{1}{4} \tan x$$



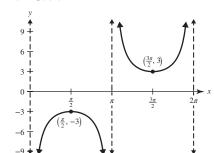
- **62.** f(x) = -2x and $g(x) = \cot x$
 - **a.** $(f \circ g)(x) = -2 \cot x$



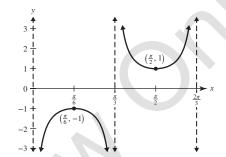
b. $(g \circ f)(x) = \cot(-2x)$



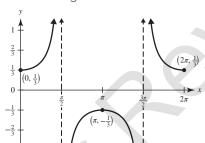
- **63.** f(x) = -3x and $g(x) = \csc x$
 - **a.** $(f \circ g)(x) = -3 \csc x$



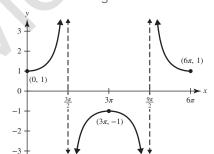
b. $(g \circ f)(x) = \csc(-3x)$



- **64.** $f(x) = \frac{x}{3}$ and $g(x) = \sec x$
 - **a.** $(f \circ g)(x) = \frac{1}{3} \sec x$



b. $(g \circ f)(x) = \sec \frac{x}{3}$



- For Exercises 65–68, complete the statements for the function provided.
- **65.** $f(x) = \tan x$

a. As
$$x \to -\frac{\pi^-}{2}$$
, $f(x) \to \underline{\hspace{1cm}} \infty$

- **b.** As $x \to -\frac{\pi}{2}^+$, $f(x) \to -\infty$
- **66.** $f(x) = \cot x$

a. As
$$x \to 0^-$$
, $f(x) \to \underline{\qquad -\infty}$

b. As $x \to 0^+$, $f(x) \to$ ______

a. As
$$x \to 0^-$$
, $f(x) \to \underline{\qquad -\infty}$

b. As
$$x \to 0^+$$
, $f(x) \to _-$

68.
$$f(x) = \sec x$$

a. As
$$x \to \frac{\pi}{2}$$
, $f(x) \to \underline{\hspace{1cm}} \infty$

b. As
$$x \to \frac{\pi^+}{2}$$
, $f(x) \to \underline{\hspace{1cm} -\infty}$

Write About It

69. Explain how to find two consecutive vertical asymptotes of $y = A \tan (Bx - C)$ for B > 0.

Solve the inequality $-\frac{\pi}{2} < Bx - C < \frac{\pi}{2}$. The endpoints of the resulting interval give the location of two consecutive vertical asymptotes: $x = \frac{1}{B} \left(C - \frac{\pi}{2} \right)$ and $x = \frac{1}{B} \left(C + \frac{\pi}{2} \right)$.

70. Explain how to find two consecutive vertical asymptotes of $y = A \cot(Bx - C)$ for B > 0

Solve the inequality $0 < Bx - C < \pi$. The endpoints of the resulting interval give the location of two consecutive vertical asymptotes: $x = \frac{C}{B}$ and $x = \frac{C + \pi}{B}$.

71. Explain how to graph $y = A \sec(Bx - C) + D$.

First graph the related reciprocal function, $y = A\cos(Bx - C)$, without the vertical shift. Then graph $y = A\sec(Bx - C)$ by first graphing the vertical asymptotes. The vertical asymptotes occur where $y = A\cos(Bx - C) = 0$.

Then plot the relative maximum and minimum points for $y = A\sec(Bx - C)$. These correspond to the relative minimum and maximum points of $y = A\cos(Bx - C)$. Sketch the general shape of the "parent" function $y = \sec x$ between the asymptotes.

Then shift the graph of $y = A \sec(Bx - C)$ vertically D units.

72. Explain how to graph $y = A\csc(Bx - C) + D$.

First graph the related reciprocal function, $y = A\sin(Bx - C)$, without the vertical shift. Then graph $y = A\csc(Bx - C)$ by first graphing the vertical asymptotes. The vertical asymptotes occur where $y = A\sin(Bx - C) = 0$.

Expanding Your Skills

For Exercises 73–76, solve each equation for x on the interval $[0, 2\pi)$.

73.
$$\tan x = 1 \quad \left\{ \frac{\pi}{4}, \frac{5\pi}{4} \right\}$$

74.
$$\sec x = -1$$
 { π }

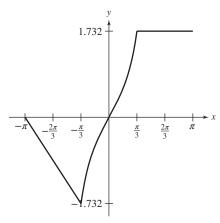
75.
$$\csc x = 1 \quad \left\{ \frac{\pi}{2} \right\}$$

76.
$$\cot x = -1 \quad \left\{ \frac{3\pi}{4}, \frac{7\pi}{4} \right\}$$

For Exercises 77–78, graph the piecewise-defined function.

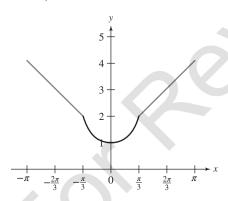
77.
$$f(x) = \begin{cases} -\frac{3\sqrt{3}}{2\pi}x - \frac{3\sqrt{3}}{2} & \text{for } x \le -\frac{\pi}{3} \\ \tan x & \text{for } -\frac{\pi}{3} < x < \frac{\pi}{3} \\ \sqrt{3} & \text{for } x \ge \frac{\pi}{3} \end{cases}$$

Hint: Graph the function on the interval $[-\pi, \pi]$. Use increments of $\frac{\pi}{3}$ on the *x*-axis and increments of $\sqrt{3}$ on the *y*-axis.



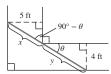
78.
$$g(x) = \begin{cases} |x| + 1 & \text{for } x \le -\frac{\pi}{3} \\ \sec x & \text{for } -\frac{\pi}{3} < x < \frac{\pi}{3} \\ |x| + 1 & \text{for } x \ge \frac{\pi}{3} \end{cases}$$

Hint: Graph the function on the interval $[-\pi, \pi]$.



79. Show that the maximum length L (in feet) of a beam that can fit around the corner shown in the figure is $L = 5 \sec \theta + 4 \csc \theta$.

Let *x* and *y* represent the hypotenuse of the upper and lower right triangles, respectively.



 $\sin(90^{\circ} - \theta) = \frac{5}{x} = \cos\theta \text{ and } \sin\theta = \frac{4}{y}.$

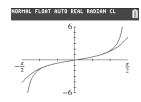
Therefore, $x = \frac{5}{\cos \theta} = 5 \sec \theta$ and $y = \frac{4}{\sin \theta} = 4 \csc \theta$.

 $L = x + y = 5 \sec \theta + 4 \csc \theta$

Technology Connections

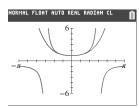
80. Graph the functions $y = \tan x$ and $y = x + \frac{x^3}{3} + \frac{2x^5}{15}$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. How do the functions compare for values of x taken close to 0?

The function values are close for *x* taken near 0.

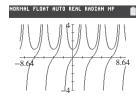


81. Graph the functions $y = \sec x$ and $y = 1 + \frac{x^2}{2} + \frac{5x^4}{24}$ on the interval $[-\pi, \pi]$. How do the functions compare for values of x taken close to 0?

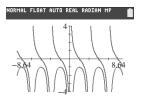
The function values are close for x taken near 0.



- **82.** Given $f(x) = x^2$, $g(x) = \tan x$, and $h(x) = \sec x$,
 - **a.** Find $(f \circ h)(x)$. $(f \circ h)(x) = \sec^2 x$
 - **b.** Graph g(x) and $(f \circ h)(x)$ together using the ZTRIG window. The relationship between the two graphs will be studied in calculus. For a given value of x in the domain of $g(x) = \tan x$, $y = \sec^2 x$ gives the slope of a line tangent to g at x.



- **83.** Given $r(x) = -x^2$, $s(x) = \cot x$, and $t(x) = \csc x$,
 - **a.** Find $(r \circ t)(x)$ $(r \circ t)(x) = -\csc^2 x$
 - **b.** Graph s(x) and $(r \circ t)(x)$ together on the ZTRIG window. The relationship between the two graphs will be studied in calculus. For a given value of x in the domain of $s(x) = \cot x$, $y = -\csc^2 x$ gives the slope of a line tangent to s at x.



Problem Recognition Exercises

Comparing Graphical Characteristics of Trigonometric Functions

For Exercises 1–16, identify which functions shown here (f, g, h), and so on) have the given characteristics.

$$f(x) = \sin\left(\frac{\pi}{2}x\right) + 3$$

$$g(x) = -3\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$$

$$h(x) = 3\sin\left(-\frac{1}{2}x - \frac{\pi}{4}\right)$$

$$k(x) = -3\sec(2x + \pi)$$

$$m(x) = 2\csc\left(2x - \frac{\pi}{2}\right) - 3$$

$$n(x) = 3\tan\left(x - \frac{\pi}{2}\right)$$

$$p(x) = -2\cot\left(\frac{1}{2}x + \pi\right)$$

$$t(x) = -3 + 2\cos x$$

- **1.** Has an amplitude of 3 g, h
- **2.** Has an amplitude of 2 t
- **3.** Has no amplitude k, m, n, p
- **4.** Has a period of 2π ρ , t
- **5.** Has a period of 4π g, h
- **6.** Has a period of π k, m, n
- **7.** Has a vertical shift upward from the parent graph f
- **8.** Has a vertical shift downward from the parent graph m, t
- **9.** Has no asymptotes f, g, h, t
- **10.** Has no *x*-intercepts f, k, t
- **11.** Has no *y*-intercept n, p
- **12.** Has a range of all real numbers n, p
- **13.** Has domain of all real numbers f, g, h, t
- **14.** Has a phase shift of $-\frac{\pi}{2}$ h, k
- **15.** Has a phase shift of $\frac{\pi}{2}$ g, n
- **16.** Has no phase shift f, t

Lesson 4-7 Inverse Trigonometric **Functions**

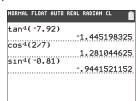
Teaching Tips

Each fully worked example is paired with end-of-lesson exercises that mirror the example. These end-of-lesson exercises are excellent problems to work through together in class. Teacher Notes on specific issues are included where applicable.

| Example | Exercise | Teacher Notes |
|---------|------------|---|
| | | Remind students of the difference between an inverse and an inverse function. The graph of $x = \sin y$ does not define y as a function of x . |
| | | $ \begin{array}{cccccccccccccccccccccccccccccccccccc$ |
| | | restricted domain |
| 1 | 8, 10, 12 | Perhaps the most common student mistake when evaluating an inverse trigonometric function is failing to recognize the range of the function. Emphasize this point again and again. |
| 2 | 18, 22, 24 | |
| 3 | 34 | |
| 4 | 38, 40, 44 | |
| 5 | 48, 54 | |
| 6 | 68 | |
| 7 | 62 | |
| 8 | 72 | |
| 9 | 80 | |
| 10 | 96, 100 | |

Lesson 4-7 Apply the Skills

- **1.** Find the exact values or state that the expression is undefined.
 - **a.** $\sin^{-1}\frac{\sqrt{2}}{2}$ $\frac{\pi}{4}$
 - **b.** arcsin (-1) $-\frac{\pi}{2}$
 - c. $\sin^{-1}(-3)$ Undefined
- 2. Find the exact values.
 - **a.** $\cos^{-1}(-1)$ π
 - **b.** $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right) \frac{\pi}{6}$
 - **c.** arctan 0 0
- **3.** Use a calculator to approximate the function values in both radians and degrees.
 - **a.** $tan^{-1}(-7.92)$
 - **b.** $\arccos \frac{2}{7}$
 - **c.** $\sin^{-1}(-0.81)$



In radians:

| tan4(-7.92 |) |
|-------------|--------------|
| | -82.80376457 |
| cos¹(2/7) | 73.3984504 |
| sin-1(-0.81 | |
| | -54.09593142 |

- In degrees:
- **4.** Use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions.
 - **a.** $\sin \theta = -\frac{3}{7}$ and $180^{\circ} \le \theta \le 270^{\circ}$ 205.4°
 - **b.** $\tan \theta = -\frac{8}{3} \text{ and } \frac{\pi}{2} < \theta < \pi$ 1.9296
- 5. Find the exact values.
 - **a.** $\cos[\cos^{-1}(-1)]$ -1
 - **b.** $\cos^{-1}(\cos\frac{4\pi}{3}) = \frac{2\pi}{3}$
- **6.** Find the exact value of $\sin \left[\tan^{-1} \left(\frac{12}{5} \right) \right]$. $\frac{12}{13}$
- **7.** Find the exact value of $\cos \left[\sin^{-1} \left(-\frac{2}{11} \right) \right]$. $\frac{\sqrt{117}}{11}$
- **8.** Write the expression $\tan\left(\sin^{-1}\frac{x}{\sqrt{x^2+9}}\right)$ as an algebraic expression for x > 0. $\frac{x}{3}$

- **9.** For the construction of a house, a 16-ft by 6-ft wooden frame is made. Find the angle that the diagonal beam makes with the base of the frame. Round to the nearest tenth of a degree. 20.6°
- 10. Approximate each expression in radians, rounded to 4 decimal places.

a. $sec^{-1}(-4)$ 1.8235

b. $\cot^{-1}\left(-\frac{5}{12}\right)$ 1.9656

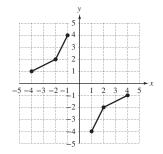
Lesson 4-7 Practice

Prerequisite Review

- **R.1.** Determine if the relation defines y as a one-to-one function of x. Yes
- **R.2.** Determine if the relation defines y as a one-to-one function of x. No
- **R.3.** A one-to-one function is given. Write an equation for the inverse function.

$$g(x) = \frac{3-x}{4}$$
 $g^{-1}(x) = 3-4x$

R.4. The graph of a function is given. Graph the inverse function.



Concept Connections

- **1.** A function must be <u>one-to-one</u> on its entire domain to have an inverse function
- **2.** If $\left(\frac{2\pi}{3}, -\frac{1}{2}\right)$ is on the graph of $y = \cos x$, what is the related point on $y = \cos^{-1} x$? $\left(-\frac{1}{2}, \frac{2\pi}{3}\right)$
- **3.** The graph of $y = \tan^{-1} x$ has two <u>horizontal</u> (horizontal/vertical) asymptotes represented by the equations $y = -\frac{\pi}{2}$ and $y = \frac{\pi}{2}$.
- **4.** The domain of $y = \arctan x$ is $\frac{R}{-\frac{\pi}{2}}$. The output is a real number (or angle in radians) between $\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- **5.** In interval notation, the domain of $y = \cos^{-1} x$ is $\underline{[-1, 1]}$. The output is a real number (or angle in radians) between $\underline{0}$ and $\underline{\pi}$, inclusive
- **6.** In interval notation, the domain of $y = \sin^{-1} x$ is $\frac{[-1, 1]}{-\frac{\pi}{2}}$. The output is a real number (or angle in radians) between $\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive.

Learn: Evaluate the Inverse Sine Function

For Exercises 7–12, find the exact value or state that the expression is undefined. (See Example 1, p. 568)

7.
$$\arcsin \frac{\sqrt{2}}{2} \quad \frac{\pi}{4}$$

8.
$$\sin^{-1}\left(-\frac{1}{2}\right) - \frac{\pi}{6}$$
 9. $\sin^{-1}\pi$ Undefined

9.
$$\sin^{-1} \pi$$
 Undefined

10.
$$\sin^{-1}\frac{3}{2}$$
 Undefined **11.** $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ $-\frac{\pi}{4}$ **12.** $\arcsin\frac{\sqrt{3}}{2}$ $\frac{\pi}{3}$

11.
$$\arcsin\left(-\frac{\sqrt{2}}{2}\right) \quad -\frac{\pi}{4}$$

12. arcsin
$$\frac{\sqrt{3}}{2}$$

For Exercises 13–16, find the exact value.

13.
$$\sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{2} \quad \frac{\pi}{2}$$

14.
$$\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}(-1)$$
 $\frac{3\pi}{4}$

13.
$$\sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{1}{2} \quad \frac{\pi}{2}$$
14. $\sin^{-1}\frac{\sqrt{2}}{2} - \sin^{-1}(-1) \quad \frac{3\pi}{4}$
15. $2\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right) - \frac{\pi}{3} \quad -\frac{5\pi}{6}$
16. $\frac{\pi}{2} + 3\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \quad -\frac{\pi}{2}$

16.
$$\frac{\pi}{2} + 3\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \frac{\pi}{2}$$

Learn: Evaluate the Inverse Cosine and Tangent Functions

For Exercises 17-28, find the exact value or state that the expression is undefined. (See Example 2, p. 571)

17.
$$\arccos\left(-\frac{\sqrt{2}}{2}\right) \quad \frac{3\pi}{4}$$
 18. $\tan^{-1}(-\sqrt{3}) \quad -\frac{\pi}{3}$ **19.** $\cos^{-1} 0 \quad \frac{\pi}{2}$

18.
$$tan^{-1}(-\sqrt{3})$$

19.
$$\cos^{-1} 0$$
 $\frac{\pi}{2}$

22.
$$\arccos \frac{4}{3}$$
 Undefined

23. arctan
$$\frac{\sqrt{3}}{3}$$

24.
$$\cos^{-1} \frac{\sqrt{2}}{2}$$
 $\frac{\pi}{4}$

25.
$$tan^{-1} 1 \frac{\pi}{4}$$

20.
$$\tan^{-1}0$$
 0 21. $\cos^{-1}(-2)$ Undefined 22. $\arccos \frac{4}{3}$ Undefined 23. $\arctan \frac{\sqrt{3}}{3} \frac{\pi}{6}$ 24. $\cos^{-1} \frac{\sqrt{2}}{2} \frac{\pi}{4}$ 25. $\tan^{-1}1 \frac{\pi}{4}$ 26. $\arctan \left(-\frac{\sqrt{3}}{3}\right) -\frac{\pi}{6}$ 27. $\arccos \frac{\sqrt{3}}{2} \frac{\pi}{6}$ 28. $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) \frac{5\pi}{6}$

27. arccos
$$\frac{\sqrt{3}}{2}$$
 $\frac{\pi}{6}$

28.
$$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$$

29.
$$\tan^{-1}(-1) + \tan^{-1}\sqrt{3}$$
 $\frac{\pi}{12}$

30.
$$\cos^{-1}\frac{\sqrt{2}}{2} + \cos^{-1}\frac{1}{2} \quad \frac{7\pi}{12}$$

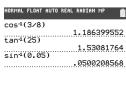
31.
$$2\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \tan^{-1}\frac{\sqrt{3}}{3}$$

For Exercises 29–32, find the exact value.
29.
$$\tan^{-1}(-1) + \tan^{-1}\sqrt{3}$$
 $\frac{\pi}{12}$ 30. $\cos^{-1}\frac{\sqrt{2}}{2} + \cos^{-1}\frac{1}{2}$ $\frac{7\pi}{12}$ 31. $2\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) - \tan^{-1}\frac{\sqrt{3}}{3}$ $\frac{3\pi}{2}$ 32. $3\tan^{-1}1 + \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ $\frac{3\pi}{2}$

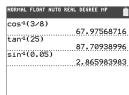
Learn: Approximate Inverse Trigonometric Functions on a Calculator

For Exercises 33-36, use a calculator to approximate the function values in both radians and degrees. (See Example 3, p. 572)

33. a.
$$\cos^{-1}\frac{3}{8}$$



In radians:



In degrees:

- **b.** arccos 0.17
- **c.** $\arctan \frac{7}{4}$

| NORMAL FLOAT AUTO | REAL RADIAN MP |
|--------------------------|----------------|
| sin ⁻¹ (0.93) | |
| cos-1(0.17) | 1.19441284 |
| cos*(0.1/) | 1.39996665 |
| tan•(7/4) | |
| | 1.05165021 |

In degrees:

1.39996665 tan-1(7/4) 1.05165021 In radians:

- **35. a.** tan⁻¹(-28)
 - **b.** $\arccos \frac{\sqrt{3}}{5}$
 - **c.** $\sin^{-1}(-0.14)$

| tan-1(-28) | |
|-----------------------|--------------|
| | -1.535097214 |
| cos-1(\(\bar{3}\/5)\) | |
| sin-1(-0.14) | 1.21705472 |
| 5111-(0.14) | 1404614147 |

In radians: ___

NORMAL FLOAT AUTO REAL DEGREE MP

tan²(-28) -87.95459151

cos²(√3/5) 69.73209894

sin²(-0.14) -8.047846247

ORMAL FLOAT AUTO REAL DEGREE MP

68.43481498

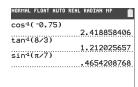
80.21218094

sin-1(0.93)

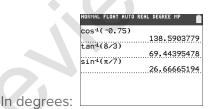
cos-1(0.17)

In degrees:

- **36. a.** $\cos^{-1}(-0.75)$
 - **b.** $tan^{-1} \frac{8}{3}$
 - **c.** $\arcsin \frac{\pi}{7}$



In radians:



For Exercises 37–46, use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle θ subject to the given conditions. (See Example 4, p. 573)

- **37.** $\cos \theta = -\frac{5}{6}$ and $180^{\circ} < \theta < 270^{\circ}$ 213.6°
- **38.** $\sin \theta = -\frac{4}{5}$ and $180^{\circ} < \theta < 270^{\circ}$ 233.1°
- **39.** $\tan \theta = -\frac{12}{5}$ and $90^{\circ} < \theta < 180^{\circ}$ 112.6°
- **40.** $\cos \theta = -\frac{2}{13}$ and $180^{\circ} < \theta < 270^{\circ}$ 261.2°
- **41.** $\sin \theta = \frac{12}{19}$ and $90^{\circ} < \theta < 180^{\circ}$ 140.8°
- **42.** $\tan \theta = \frac{7}{15}$ and $180^{\circ} < \theta < 270^{\circ}$ 205.0°
- **43.** $\cos \theta = -\frac{5}{8}$ and $\pi < \theta < \frac{3\pi}{2}$ 4.0373
- **44.** $\tan \theta = -\frac{9}{5}$ and $\frac{\pi}{2} < \theta < \pi$ 2.0779

45.
$$\sin \theta = -\frac{17}{20}$$
 and $\pi < \theta < \frac{3\pi}{2}$ 4.1576

46.
$$\cos \theta = \frac{1}{17}$$
 and $\frac{3\pi}{2} < \theta < 2\pi$ 4.7712

Learn: Compose Trigonometric Functions and Inverse Trigonometric Functions

For Exercises 47-58, find the exact values. (See Example 5, p. 575)

47.
$$\sin\left(\sin^{-1}\frac{\sqrt{2}}{2}\right) \quad \frac{\sqrt{2}}{2}$$

48.
$$\arcsin\left(\sin\frac{5\pi}{3}\right) - \frac{\pi}{3}$$

49.
$$\sin^{-1} \left(\sin \frac{5\pi}{4} \right) - \frac{\pi}{4}$$

50.
$$\sin \left[\sin^{-1} \left(-\frac{1}{2} \right) \right] -\frac{1}{2}$$

51.
$$\cos(\cos^{-1}\frac{2}{3})$$
 $\frac{2}{3}$

52.
$$\arccos\left(\cos\frac{11\pi}{6}\right) \quad \frac{\pi}{6}$$

53.
$$\cos^{-1} \left(\cos \frac{4\pi}{3} \right) \quad \frac{2\pi}{3}$$

54.
$$\cos \left[\cos^{-1}\left(-\frac{1}{2}\right)\right] - \frac{1}{2}$$

55.
$$\tan^{-1} \left(\tan \frac{2\pi}{3} \right) - \frac{\pi}{3}$$

57. tan [arctan
$$(-\pi)$$
] $-\pi$

58.
$$\tan^{-1} \left[\tan \left(-\frac{\pi}{6} \right) \right] -\frac{\pi}{6}$$

For Exercises 59–70, find the exact values. (See Examples 6–7, pp. 575–576)

59.
$$\cos\left(\tan^{-1}\frac{\sqrt{3}}{3}\right) \frac{\sqrt{3}}{2}$$

60.
$$\sin(\cos^{-1}\frac{1}{2})$$
 $\frac{\sqrt{3}}{2}$

61.
$$\tan \left[\sin^{-1} \left(-\frac{2}{3} \right) \right] -\frac{2\sqrt{5}}{5}$$

62.
$$\sin\left[\cos^{-1}\left(-\frac{2}{3}\right)\right] = \frac{\sqrt{5}}{3}$$

63.
$$\sin(\cos^{-1}\frac{3}{4})$$
 $\frac{\sqrt{7}}{4}$

64.
$$\sin(\tan^{-1}\frac{4}{3}) = \frac{4}{5}$$

65.
$$\sin[\tan^{-1}(-1)] - \frac{\sqrt{2}}{2}$$

66.
$$\cos[\tan^{-1}(-1)] \frac{\sqrt{2}}{2}$$

67.
$$\cos \left[\sin^{-1} \left(-\frac{2}{7} \right) \right] = \frac{3\sqrt{5}}{7}$$

68.
$$\cos\left[\tan^{-1}\left(-\frac{5}{12}\right)\right] \frac{12}{13}$$

69.
$$\tan \left[\cos^{-1}\left(-\frac{5}{6}\right)\right] - \frac{\sqrt{11}}{5}$$

70.
$$\tan \left[\sin^{-1} \left(-\frac{\sqrt{5}}{3} \right) \right] - \frac{\sqrt{5}}{2}$$

For Exercises 71–76, write the given expression as an algebraic expression. It is not necessary to rationalize the denominator. (See Example 8, p. 577)

71.
$$\cos\left(\sin^{-1}\frac{x}{\sqrt{25+x^2}}\right)$$
 for $x > 0$. $\frac{5}{\sqrt{25+x^2}}$

72.
$$\cot\left(\cos^{-1}\frac{\sqrt{x^2-1}}{x}\right)$$
 for $x > 1$. $\sqrt{x^2-1}$

73.
$$\sin(\tan^{-1}x)$$
 for $x > 0$. $\frac{x}{\sqrt{x^2 + 1}}$

74. tan (sin⁻¹ x) for
$$|x| < 1$$
. $\frac{x}{\sqrt{1-x^2}}$

75.
$$\tan(\cos^{-1}\frac{3}{x})$$
 for $x > 3$. $\frac{\sqrt{x^2 - 9}}{3}$

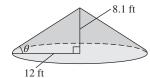
76.
$$\sin\left(\cos^{-1}\frac{\sqrt{x^2-25}}{x}\right)$$
 for $x > 5$. $\frac{5}{x}$

Learn: Apply Inverse Trigonometric Functions

- **77.** To meet the requirements of the Americans with Disabilities Act (ADA), a wheelchair ramp must have a slope of 1:12 or less. That is, for every 1 in. of "rise," there must be at least 12 in. of "run." (See Example 9, p. 578)
 - **a.** If a wheelchair ramp is constructed with the maximum slope, what angle does the ramp make with the ground? Round to the nearest tenth of a degree. 4.8°
 - **b.** If the ramp is 22 ft long, how much elevation does the ramp provide? Round to the nearest tenth of a foot. 1.8 ft
- **78.** A student measures the length of the shadow of the Washington Monument to be 620 ft. If the Washington Monument is 555 ft tall, approximate the angle of elevation of the Sun to the nearest tenth of a degree. 41.8°
- **79.** A balloon advertising an open house is stabilized by two cables of lengths 20 ft and 40 ft tethered to the ground. If the perpendicular distance from the balloon to the ground is $10\sqrt{3}$ ft, what is the degree measure of the angle each cable makes with the ground? Round to the nearest tenth of a degree if necessary.

The short cable makes an angle of 60° with the ground, and the long cable makes an angle of 25.7° with the ground.

- **80.** A group of campers hikes down a steep path. One member of the group has an altimeter to measure altitude. If the path is 1250 yd and the amount of altitude lost is 480 yd, what is the angle of incline? Round to the nearest tenth of a degree. 22.6°
- **81.** Navajo Tube Hill, a snow tubing hill in Utah, is 550 ft long and has a 75-ft vertical drop. Find the angle of incline of the hill. Round to the nearest tenth of a degree. 7.8°
- **82.** A ski run on Giant Steps Mountain in Utah is 1475 m long. The difference in altitude from the beginning to the end of the run is 350 m. Find the angle of the ski run. Round to the nearest tenth of a degree. 13.7°
- **83.** When granular material such as sand or gravel is poured onto a horizontal surface it forms a right circular cone. The angle that the surface of the cone makes with the horizontal is called the **angle of repose**. The angle of repose depends on a number of variables such as the shape of the particles and the amount of friction between them. "Stickier" particles have a greater angle of repose, and "slippery" particles have a smaller angle of repose. Find the angle of repose for the pile of dry sand. Round to the nearest degree. 34°



84. For the construction of a bookcase, a 5-ft by 3-ft wooden frame is made with a cross-brace in the back for stability. Find the angle that the diagonal brace makes with the base of the frame. Round to the nearest tenth of a degree. 59.0°

Learn: Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

85. Show that $\sec^{-1} x = \cos^{-1} \frac{1}{x}$ for $x \ge 1$.

Let
$$\theta = \sec^{-1} x$$
 for $x \ge 1$.

Then,
$$\sec \theta = x$$

$$\frac{1}{\cos\theta} = x$$

$$\cos \theta = \frac{1}{x}$$

$$\theta = \cos^{-1}\frac{1}{x}$$

Hence,
$$\sec^{-1} x = \cos^{-1} \frac{1}{x}$$
.

86. Show that $\csc^{-1} x = \sin^{-1} \frac{1}{x}$ for $x \ge 1$.

Let
$$\theta = \csc^{-1} x$$
 for $x \ge 1$.

Then,
$$\csc \theta = x$$

$$\frac{1}{\sin \theta} = x$$

$$\sin \theta = \frac{1}{X}$$

$$\theta = \sin^{-1}\frac{1}{X}$$

Hence,
$$\csc^{-1} x = \sin^{-1} \frac{1}{x}$$
.

87. Show that $\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$ for $x \ge 1$.

Let
$$\alpha = \sec^{-1} x$$
 for $x \ge 1$, which implies that $\sec \alpha = x$ and

$$\csc\left(\frac{\pi}{2} - \alpha\right) = x$$

$$\frac{\pi}{2} - \alpha = \csc^{-1} X$$

Substituting
$$\alpha = \sec^{-1} x$$
 yields

$$\frac{\pi}{2} - \sec^{-1} x = \csc^{-1} x$$

$$\frac{\pi}{2} = \sec^{-1}x + \csc^{-1}x$$

88. Complete the table, giving the domain and range in interval notation for each inverse function.

| Inverse Function | Domain | Range |
|-------------------|----------------------------------|--|
| $y = \sin^{-1} x$ | [-1, 1] | $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ |
| $y = \csc^{-1} x$ | $(-\infty, -1] \cup [1, \infty)$ | $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ |
| $y = \cos^{-1} x$ | [-1, 1] | [0, π] |
| $y = \sec^{-1} x$ | $(-\infty, -1] \cup [1, \infty)$ | $\left[0,\frac{\pi}{2}\right)\cup\left(\frac{\pi}{2},\pi\right]$ |
| $y = \tan^{-1} x$ | $(-\infty, \infty)$ | $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ |
| $y = \cot^{-1} x$ | $(-\infty, \infty)$ | (0, π) |

For Exercises 89–94, find the exact values.

89.
$$\sec^{-1} \frac{2\sqrt{3}}{3} = \frac{\pi}{6}$$

90.
$$\sec^{-1}(-\sqrt{2})$$
 $\frac{3\pi}{4}$

91.
$$csc^{-1}(-1)$$
 $-\frac{\pi}{2}$

92.
$$csc^{-1}$$
 (2) $\frac{\pi}{6}$

93.
$$\cot^{-1}\sqrt{3}$$
 $\frac{\pi}{6}$

94. cot⁻¹ (1)
$$\frac{\pi}{\Delta}$$

For Exercises 95–100, use a calculator to approximate each expression in radians, rounded to 4 decimal places. (See Example 10, p. 580)

95.
$$\sec^{-1}\frac{7}{4}$$
 0.9626

96.
$$\csc^{-1} \frac{6}{5}$$
 0.9851

99.
$$\cot^{-1}\left(-\frac{8}{15}\right)$$
 2.0608

100.
$$\cot^{-1}\left(-\frac{24}{7}\right)$$
 2.8578

Mixed Exercises

For Exercises 101–104, find the exact value if possible. Otherwise find an approximation to 4 decimal places or state that the expression is undefined.

101. a.
$$\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

b.
$$\sin^{-1}\frac{\pi}{4}$$
 0.9033

c.
$$\sin^{-1} \frac{\sqrt{2}}{2}$$

102. a.
$$\cos \frac{2\pi}{3}$$
 $-\frac{1}{2}$

b.
$$\cos^{-1}\frac{2\pi}{3}$$
 Undefined

c.
$$\cos^{-1}\left(-\frac{1}{2}\right) \quad \frac{2\pi}{3}$$

103. a.
$$\cos^{-1}\left(-\frac{\pi}{6}\right)$$
 2.1219

b.
$$\cos(-\frac{\pi}{6})$$
 $\frac{\sqrt{3}}{2}$

101. a.
$$\sin \frac{\pi}{4} \frac{\sqrt{2}}{2}$$
 b. $\sin^{-1}\frac{\pi}{4} = 0.9033$ **c.** $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ **102. a.** $\cos \frac{2\pi}{3} = -\frac{1}{2}$ **b.** $\cos^{-1}\frac{2\pi}{3} = 0.9033$ **c.** $\sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ **103. a.** $\cos^{-1}\left(-\frac{\pi}{6}\right) = 2.1219$ **b.** $\cos\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ **c.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$ **104. a.** $\sin^{-1}\frac{7\pi}{6}$ Undefined **b.** $\sin\frac{7\pi}{6} = -\frac{1}{2}$ **c.** $\sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

104. a.
$$\sin^{-1} \frac{7\pi}{6}$$
 Undefined

b.
$$\sin \frac{7\pi}{6} - \frac{1}{3}$$

c.
$$\sin^{-1} \left(-\frac{1}{2} \right) - \frac{\pi}{6}$$

For Exercises 105–108, find the inverse function and its domain and range.

105.
$$f(x) = 3\sin x + 2$$
 for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$f^{-1}(x) = \sin^{-1}\left(\frac{x-2}{3}\right);$$

Domain: $\{x | -1 \le x \le 5\}$;

Range:
$$\left\{ y \middle| -\frac{\pi}{2} \le y \le \frac{\pi}{2} \right\}$$

$$g^{-1}(x) = \cos^{-1}\left(\frac{x+4}{6}\right);$$

Domain: $\{x | -10 \le x \le 2\}$;

Range: $\{y \mid 0 \le y \le \pi\}$

107. $h(x) = \frac{\pi}{4} + \tan x \text{ for } -\frac{\pi}{2} < x < \frac{\pi}{2}$

$$h^{-1}(x) = \tan^{-1}(x - \frac{\pi}{4});$$

Domain: \mathbb{R} ;

Range:
$$\left\{ y \middle| -\frac{\pi}{2} < y < \frac{\pi}{2} \right\}$$

108. $k(x) = \pi + \sin x \text{ for } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$$k^{-1}(x) = \sin^{-1}(x - \pi);$$

Domain:

$$\{x \mid -1 + \pi \le x \le 1 + \pi\};$$

Range:
$$\left\{ y \middle| -\frac{\pi}{2} \le y \le \frac{\pi}{2} \right\}$$

- **109.** A video camera located at ground level follows the liftoff of an Atlas V rocket from the Kennedy Space Center. Suppose that the camera is 1000 m from the launch pad.
 - **a.** Write the angle of elevation θ from the camera to the rocket as a function of the rocket's height, h. $\theta = \tan^{-1} \frac{h}{1000}$
 - **b.** Without the use of a calculator, will the angle of elevation be less than 45° or greater than 45° when the rocket is 400 m high?

$$\theta$$
 is less than 45° because the ratio $\frac{400}{1000}$ is less than 1.

- **c.** Use a calculator to find θ to the nearest tenth of a degree when the rocket's height is 400 m, 1500 m, and 3000 m. 21.8°, 56.3°, 71.6°
- **110.** The effective focal length f of a camera is the distance required for the lens to converge light to a single focal point. The angle of view α of a camera describes the angular range (either horizontally, vertically, or diagonally) that is imaged by a camera.
 - **a.** Show that $\alpha = 2 \arctan \frac{d}{2f}$ where d is the dimension of the image sensor or film.

$$\tan\frac{\alpha}{2} = \frac{\frac{1}{2}d}{f}$$

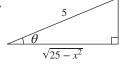
$$\frac{\alpha}{2} = \arctan \frac{d}{2f}$$

$$\alpha = 2 \arctan \frac{d}{2f}$$

b. A typical 35-mm camera has image dimensions of 24 mm (vertically) by 36 mm (horizontally). If the focal length is 50 mm, find the vertical and horizontal viewing angles. Round to the nearest tenth of a degree.

Vertical viewing angle: 27.0°; Horizontal viewing angle: 39.6°

For Exercises 111–114, use the relationship given in the right triangle and the inverse sine, cosine, and tangent functions to write θ as a function of xin three different ways. It is not necessary to rationalize the denominator.



$$\theta = \cos^{-1} \frac{\sqrt{25 - x^2}}{5}$$
 or $\theta = \sin^{-1} \frac{x}{5}$ or $\theta = \tan^{-1} \frac{x}{\sqrt{25 - x^2}}$

112.

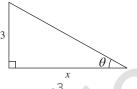


$$\theta = \tan^{-1} \frac{\sqrt{x^2 - 4}}{2}$$
 or $\theta = \sin^{-1} \frac{\sqrt{x^2 - 4}}{x}$ or $\theta = \cos^{-1} \frac{2}{x}$



$$\theta = \sin^{-1}\frac{x}{6}$$
 or $\theta = \cos^{-1}\frac{\sqrt{36-x^2}}{6}$ or $\theta = \tan^{-1}\frac{x}{\sqrt{36-x^2}}$

114.



$$\theta = \tan^{-1}\frac{3}{x}$$
 or $\theta = \cos^{-1}\frac{x}{\sqrt{x^2 + 9}}$ or $\theta = \sin^{-1}\frac{3}{\sqrt{x^2 + 9}}$

For Exercises 115–120, find the exact solution to each equation.

115.
$$-2\sin^{-1}x - \pi = 0$$
 {-1}

116.
$$3\cos^{-1}x - \pi = 0$$
 $\left\{\frac{1}{2}\right\}$

117.
$$6\cos^{-1}x - 3\pi = 0$$
 {0}

118.
$$4 \sin^{-1} x + \pi = 0$$
 $\left\{ -\frac{\sqrt{2}}{2} \right\}$
120. $6 \tan^{-1} 2x = 2\pi$ $\left\{ \frac{\sqrt{3}}{2} \right\}$

119.
$$4 \tan^{-1} 2x = \pi$$
 $\left\{ \frac{1}{2} \right\}$

120.
$$6 \tan^{-1} 2x = 2\pi$$
 $\left\{ \frac{\sqrt[3]{3}}{2} \right\}$

Write About It

121. Explain the difference between the reciprocal of a function and the inverse of a function.

Given y = f(x), the reciprocal $y = [f(x)]^{-1}$ is the quotient of 1 and f(x), that is $y = \frac{1}{f(x)}$. The inverse of the function, denoted by $y = f^{-1}(x)$, is the function that reverses the operations performed by the original function. Explain the difference between the reciprocal of a function and the inverse of a function.

- **122.** Explain the flaw in the logic: $\cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$. Therefore, $\cos^{-1}\frac{\sqrt{2}}{2} = -\frac{\pi}{4}$. By the definition of the inverse cosine function, the range (output) is a number between 0 and π , inclusive.
- **123.** In terms of angles, explain what is meant when we find $\sin^{-1}\left(-\frac{1}{2}\right)$. We want to find an angle such that the sine of the angle is $-\frac{1}{2}$. However, the angle must be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.
- **124.** Why must the domains of the sine, cosine, and tangent functions be restricted in order to define their inverse functions?

The domains must be restricted because the sine, cosine, and tangent functions are not one-to-one on their entire domains.

Expanding Your Skills

- **125.** In calculus, we can show that the area below the graph of $f(x) = \frac{1}{1+x^2}$, above the *x*-axis, and between the lines x = a and x = b for a < b, is given by $\tan^{-1}b \tan^{-1}a$
 - **a.** Find the area under the curve between x = 0 and x = 1. $\frac{\pi}{4}$ square units
 - **b.** Evaluate f(0) and f(1). $f(0) = 1, f(1) = \frac{1}{2}$
 - **c.** Find the area of the trapezoid defined by the points (0, 0), (1, 0), [0, f(0)], and [1, f(1)] to confirm that your answer from part (a) is reasonable.

$$A \text{trapezoid} = \frac{1}{2}(b_1 + b_2)h$$

$$= \frac{1}{2}[f(0) + f(1)](1 - 0)$$

$$=\frac{1}{2}\left(1+\frac{1}{2}\right)(1)=\frac{3}{4};$$

The answer from part (a) is $\frac{\pi}{4} \approx 0.785$, and the approximated value is $\frac{3}{4} = 0.75$. These are very close.

126. In calculus, we can show that the area below the graph of $f(x) = \frac{1}{\sqrt{1-x^2}}$, above the *x*-axis, and between the lines x = a and x = b for a < b, is given by

$$\sin^{-1}b - \sin^{-1}a$$

- **a.** Find the area under the curve between x = 0 and x = 0.5. $\frac{\pi}{6}$ square units
- **b.** Evaluate f(0) and f(0.5). f(0) = 1, $f(0.5) = \frac{2\sqrt{3}}{3}$

c. Find the area of the trapezoid defined by the points (0, 0), (1, 0), [0, f(0)], and [0.5, f(0.5)] to confirm that your answer from part (a) is reasonable.

$$A_{\text{trapezoid}} = \frac{1}{2} (b_1 + b_2) h$$

$$= \frac{1}{2} [f(0) + f(0.5)](0.5 - 0)$$

$$= \frac{1}{2} \left(1 + \frac{2\sqrt{3}}{3} \right) (0.5) = \frac{3 + 2\sqrt{3}}{12};$$

The answer from part (a) is $\frac{\pi}{6}\approx 0.524$, and the approximated value is $\frac{3+2\sqrt{3}}{12}\approx 0.539$. These are very close.

- **127.** The vertical viewing angle θ to a movie screen is the angle formed from the bottom of the screen to a viewer's eye to the top of the screen. Suppose that the viewer is sitting x horizontal feet from an IMAX screen 53 ft high and that the bottom of the screen is 10 vertical feet above the viewer's eye level. Let α be the angle of elevation to the bottom of the screen.
 - **a.** Write an expression for tan α . $\tan \alpha = \frac{10}{x}$
 - **b.** Write an expression for tan $(\alpha + \theta)$. $\tan(\theta + \alpha) = \frac{63}{X}$
 - **c.** Using the relationships found in parts (a) and (b), show that $\theta = \tan^{-1} \frac{63}{X} \tan^{-1} \frac{10}{X}$.

$$\tan(\theta + \alpha) = \frac{63}{X}$$

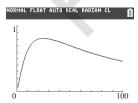
$$\theta + \alpha = \tan^{-1}\frac{63}{X}$$

$$\theta = \tan^{-1}\frac{63}{X} - \alpha$$

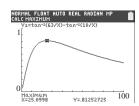
$$\theta = \tan^{-1}\frac{63}{X} - \tan^{-1}\frac{10}{X}$$

Technology Connections

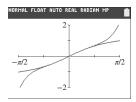
- **128.** Refer to the movie screen and observer in Exercise 127. The vertical viewing angle is given by $\theta = \tan^{-1}\frac{63}{x} \tan^{-1}\frac{10}{x}$.
 - **a.** Find the vertical viewing angle (in radians) for an observer sitting 15, 25, and 35 ft away. Round to 2 decimal places. 0.75, 0.81, 0.79
 - **b.** Graph $y = \tan^{-1} \frac{63}{x} \tan^{-1} \frac{10}{x}$ on a graphing utility on the window [0, 100, 10] by [0, 1, 0.1].



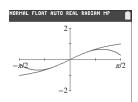
c. Use the Maximum feature on the graphing utility to estimate the distance that an observer should sit from the screen to produce the maximum viewing angle. Round to the nearest tenth of a foot. 25.1 ft



129. a. Graph the functions $y = \tan^{-1} x$ and $y = x - \frac{x^3}{3} + \frac{x^5}{5}$ on the window $\left[-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4} \right]$ by [-2, 2, 1].



b. Graph the functions $y = \tan^{-1} x$ and $y = \frac{x}{1 + \frac{x^2}{3 - x^2}}$ on the window $\left[-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4} \right]$ by [-2, 2, 1].



c. How do the functions in parts (a) and (b) compare for values of *x* taken close to 0? The function values are close for *x* taken near 0.

The function values are close for *x* taken near 0.

Chapter 4 Review

Chapter 4 Test

- **1.** Convert 15.36° to DMS (degree-minute-second) form. Round to the nearest second if necessary. 15° 21′ 36″
- 2. Convert 130.3° to radians. Round to 2 decimal places. 2.27
- **3.** Find the exact length of the arc intercepted by a central angle of 27° on a circle of radius 5 ft. $\frac{3\pi}{4}$ ft
- **4.** A skateboard designed for rough surfaces has wheels with diameter 60 mm. If the wheels turn at 2200 rpm, what is the linear speed in mm per minute? $132,000\pi$ mm/min
- **5.** A pulley is 20 in. in diameter. Through how many degrees should the pulley rotate to lift a load 3 ft? Round to the nearest degree. 206°
- **6.** A circle has a radius of 9 yd. Find the area of a sector with a central angle of 120° . $27\pi \text{ yd}^2$
- **7.** For an acute angle θ , if $\sin \theta = \frac{5}{6}$, evaluate $\cos \theta$ and $\tan \theta$. $\cos \theta = \frac{\sqrt{11}}{6}$, $\tan \theta = \frac{5\sqrt{11}}{11}$
- **8.** Evaluate $\tan \frac{\pi}{6} \cot \frac{\pi}{6}$ without the use of a calculator. $-\frac{2\sqrt{3}}{3}$
- **9.** Given $\sin \theta = \frac{5}{13}$, use a Pythagorean identity to find $\cos \theta$. $\cos \theta = \frac{12}{13}$
- **10.** Given sec $75^{\circ} = \sqrt{2} + \sqrt{6}$, find a cofunction of another angle with the same function value. csc 15°
- **11.** Use a calculator to approximate $\sin \frac{2\pi}{11}$ to 4 decimal places. 0.5406
- **12.** A flagpole casts a shadow of 20 ft when the angle of elevation of the Sun is 40°. How tall is the flagpole? Round to the nearest foot. 17 ft
- **13.** A newly planted tree is anchored by a covered wire running from the top of the tree to a post in the ground 5 ft from the base of the tree. If the angle between the wire and the top of the tree is 20°, what is the length of the wire? Round to the nearest foot. 15 ft

For Exercises 14–21, evaluate the function or state that the function is undefined at the given value.

14.
$$\sin\left(-\frac{3\pi}{4}\right) - \frac{\sqrt{2}}{2}$$

15.
$$\tan 930^{\circ} \frac{\sqrt{3}}{3}$$

16.
$$\sec \frac{11\pi}{6} = \frac{2\sqrt{3}}{3}$$

18.
$$\cot 20\pi$$
 Undefined

19.
$$\cos 690^{\circ} \frac{\sqrt{3}}{2}$$

20.
$$\tan \frac{7\pi}{3} = \sqrt{3}$$

22. Given
$$\sin \theta = \frac{5}{8}$$
 and $\cos \theta < 0$, find $\tan \theta$. $-\frac{5\sqrt{39}}{39}$

- **23.** Given $\sec \theta = \frac{4}{3}$ and $\sin \theta < 0$, find $\csc \theta$. $-\frac{4\sqrt{7}}{7}$
- **24.** Given $\tan \theta = -\frac{4}{3}$ and θ is in Quadrant IV, find $\sec \theta$. $\frac{5}{3}$

For Exercises 25–28, select the trigonometric function, $f(t) = \sin t$, $g(t) = \cos t$, $h(t) = \tan t$ or $r(t) = \cot t$, with the given properties.

- **25.** The function is odd, with period 2π , and domain of all real numbers. $f(t) = \sin t$
- **26.** The function is odd, with period π , and domain of all real numbers excluding odd multiples of $\frac{\pi}{2}$. $h(t) = \tan t$
- **27.** The function is even, with period 2π , and domain of all real numbers.

$$g(t) = \cos t$$

- **28.** The function is odd, with period π , and domain of all real numbers excluding multiples of π . $r(t) = \cot t$
- **29.** Use the even-odd and periodic properties of the trigonometric functions to simplify $\cos(-\theta 2\pi) \sin(\theta 2\pi) \cdot \cot(\theta + \pi) = 0$
- **30.** Suppose that a rectangle is bounded by the *x*-axis and the graph of $y = \sin x$ on the interval $[0, \pi]$.
 - **a.** Write a function that represents the area A(x) of the rectangle for $0 < x < \frac{\pi}{2}$.

$$A(x) = (\pi - 2x) \sin x$$

b. Determine the area of the rectangle for $x = \frac{\pi}{6}$ and $x = \frac{\pi}{4}$.

$$A\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$$
 square units and $A\left(\frac{\pi}{6}\right) = \frac{\sqrt{2}\pi}{4}$ square units

For Exercises 31–32, determine the amplitude, period, phase shift, and vertical shift for each function.

31.
$$y = \frac{3}{4} \sin(2\pi x - \frac{\pi}{6})$$

$$y = \frac{3}{4}\sin(2\pi x - \frac{1}{6})$$

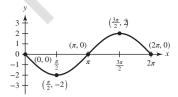
Amplitude = $\frac{3}{4}$, Period = 1, Phase shift = $\frac{1}{12}$, Vertical shift = 0

32.
$$y = -5\cos(5x + \pi) + 7$$

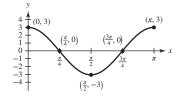
Amplitude = 5, Period =
$$\frac{2\pi}{5}$$
, Phase shift = $-\frac{\pi}{5}$, Vertical shift = 7

For Exercises 33–36, graph one period of the function.

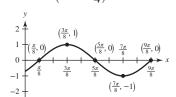
33.
$$y = -2\sin x$$



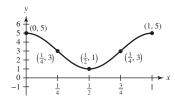
34.
$$y = 3\cos 2x$$



35.
$$y = \sin(2x - \frac{\pi}{4})$$



36.
$$y = 3 - 2\cos(2\pi x - \pi)$$



37. Write a function of the form $f(x) = A\cos(Bx)$ for the given graph.

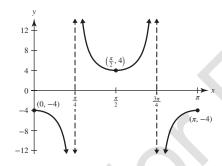
$$f(x) = -3\cos 3x$$

- 38. Identify each statement as true or false. If a statement is false, explain why.
 - **a.** The relative maxima of the graph of $y = \sin x$ correspond to the relative minima of the graph of $y = \csc x$. True
 - **b.** The period of $y = \tan 2x$ is π . False; The period is $\frac{\pi}{2}$.
 - **c.** The amplitude of $y = 2 \cot x$ is 2.
 - False; The cotangent function has no amplitude because it is unbounded.
 - **d.** The period of $y = \sec 2x$ is π . True
 - **e.** The vertical asymptotes of the graph of $y = \cot x$ occur where the graph of $y = \cos x$ has x-intercepts.

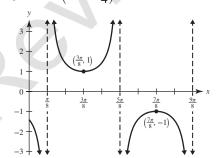
False; The asymptotes occur where the graph of $y = \sin x$ has x-intercepts.

For Exercises 39-40, graph one period of the function.

39.
$$y = -4 \sec 2x$$

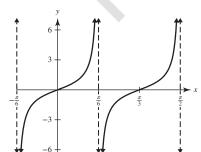


40.
$$y = \csc(2x - \frac{\pi}{4})$$

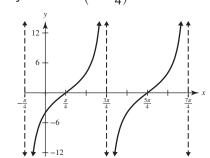


For Exercises 41–42, graph two periods of the function.

41.
$$y = \tan 3x$$



42.
$$y = -4 \cot \left(x + \frac{\pi}{4}\right)$$



43. Simplify each expression.

a.
$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\frac{\sqrt{2}}{2} - \frac{\pi}{12}$$

- **b.** cos[arctan $(-\sqrt{3})$ $\frac{1}{2}$
- **44.** Use a calculator to approximate the degree measure (to 1 decimal place) of the angle θ subject to the given conditions. 158.0°

$$\sin \theta = \frac{3}{8}$$
 and $90^{\circ} \le \theta \le 180^{\circ}$

- **45.** Find the exact value of $\sin^{-1}\left(\sin\frac{11\pi}{6}\right)$. $-\frac{\pi}{6}$
- **46.** Find the exact value of $\tan(\cos^{-1}\frac{7}{8})$. $\frac{\sqrt{15}}{7}$
- **47.** Find the exact value of $\cos \left[\arcsin\left(-\frac{5}{6}\right)\right]$. $\frac{\sqrt{11}}{6}$
- **48.** Write the expression $\tan \left(\cos^{-1} \frac{x}{\sqrt{x^2 + 25}}\right)$ for $x \neq 0$ as an algebraic expression. $\frac{5}{x}$
- **49.** A radar station tracks a plane flying at a constant altitude of 6 mi on a path directly over the station. Let θ be the angle of elevation from the radar station to the plane.
 - **a.** Write θ as a function of the plane's ground distance x > 0 from the station. $\theta = \tan^{-1} \frac{6}{x}$
 - **b.** Without the use of a calculator, will the angle of elevation be less than 45° or greater than 45° when the plane's ground distance is 3.2 mi away? θ is greater than 45° because the ratio $\frac{6}{3.2}$ is greater than 1.
 - **c.** Use a calculator to find θ to the nearest degree for x = 3.2, 1.6, and 0.5 mi. 62° , 75° , 85°

Cumulative Review Exercises

For Exercises 1–9, solve the equations and inequalities. Write the solution set to inequalities in interval notation.

1.
$$|3x - 5| + 3 \ge 6$$
 $\left(-\infty, \frac{2}{3}\right] \cup \left[\frac{8}{3}, \infty\right)$

2.
$$(x-2)^2(x+5) < 0 \quad (-\infty, -5)$$

3.
$$\frac{x+2}{x-3} \ge 2$$
 (3, 8]

4.
$$3x^4 - x^2 - 10 = 0$$
 $\left\{ \pm \sqrt{2}, \pm \frac{\sqrt{15}}{3}i \right\}$

5.
$$\sqrt{x+7} = 4 - \sqrt{x-1}$$
 {2}

6.
$$2x^4 + 5x^3 - 29x^2 - 17x + 15 = 0$$
 $\left\{ -5, -1, \frac{1}{2}, 3 \right\}$

7.
$$e^{3x} = 2 \quad \left\{ \frac{\ln 2}{3} \right\}$$

8.
$$\log(3x+2) - \log x = 3 \quad \left\{\frac{2}{997}\right\}$$

9.
$$\ln(x-1) + \ln x = \ln 2$$
 {2}

10. Given
$$x^2 + y^2 - 4x + 6y + 9 = 0$$

a. Write the equation of the circle in standard form.
$$(x-2)^2 + (y+3)^2 = 4$$

b. Identify the center and radius. Center:
$$(2, -3)$$
; Radius: 2

11. Given
$$y = -x^2 + 2x - 4$$

a. Identify the vertex.
$$(1, -3)$$

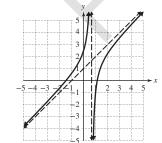
Domain:
$$(-\infty, \infty)$$
; Range: $(-\infty, -3]$

12. Given
$$f(x) = x^4 - 2x^3 - 4x^2 + 8x$$
,

a. Find the x-intercepts of the graph of f.
$$(-2, 0)$$
, $(0, 0)$, $(2, 0)$

Up to the left and up to the right

13. a. Graph
$$y = \frac{3x^2 + x - 5}{3x - 2}$$



b. Identify the asymptotes.

Vertical asymptote: $x = \frac{2}{3}$, Slant asymptote: y = x + 1

- **14.** Given $y = -2 \sin(4x \pi) 6$, identify the
 - **a.** Domain and range in interval notation. Domain: $(-\infty, \infty)$; Range: [-8, -4]
 - **b.** Amplitude. 2
 - **c.** Period. $\frac{\pi}{2}$
 - **d.** Phase shift. $\frac{\pi}{4}$
 - **e.** Vertical shift. −6
- **15.** Given $f(x) = \tan x$ and $g(x) = \frac{x}{2}$, evaluate
 - **a.** $(f \circ g) \left(\frac{\pi}{2}\right)$ 1
 - **b.** $(g \circ f)(\pi) = 0$
- **16.** Evaluate $\cos(-450^{\circ})$. 0
- **17.** Evaluate $\tan \left[\arcsin\left(-\frac{3}{8}\right)\right]$. $-\frac{3\sqrt{55}}{55}$
- **18.** Write the logarithm as the sum or difference of logarithms. Simplify as much as possible. $\log\left(\frac{x^2y}{100z}\right) = 2\log x + \log y \log z 2$
- **19.** Divide. Write the answer in standard form, a + bi. $\frac{2 8i}{3 5i}$ $\frac{23}{17} \frac{7}{17}i$
- **20.** Suppose that y varies inversely as x and directly as z. If y is 12 when x is 8 and z is 3, find the constant of variation k. k = 32