## First Edition

# Precalculus 

## High School Edition

Student Edition Sample Chapter


First Edition Precalculus

High School Edition

## Mc Graw Hil

Julie Miller

Daytona State College

Donna Gerken
Miami Dade College

## mheonline.com/honorselectives

## Mc <br> Graw <br> Hill

## PRECALCULUS, HIGH SCHOOL 1st EDITION

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Online-Only: Chapter 12 Preview of CalculusThe topics in Chapter 12 are outside the scope of most high school precalculuscourses. The chapter is available online as part of the student eBook.

## Preface

## About the Authors: Julie Miller and Donna Gerken

Julie Miller is from Daytona State College, where she taught developmental and upper-level mathematics courses for 20 years. Prior to her work at DSC, she worked as a software engineer for General Electric in the area of flight and radar simulation. Julie earned a bachelor of science in applied mathematics from Union College in Schenectady, New York, and a master of science in mathematics from the University of Florida. In addition to this textbook, she has authored textbooks in developmental mathematics, trigonometry, and precalculus, as well as several short works of fiction and nonfiction for young readers.
"My father was a medical researcher, and I got hooked on math and science when I was young and would visit his laboratory. I remember doing simple calculations with him and using graph paper to plot data points for his experiments. He would then tell me what the peaks and features in the graph meant in the context of his experiment. I think that applications and hands-on experience made math come alive for me, and l'd like to see math come alive for my students."

Donna Gerken was a professor at Miami Dade College where she taught developmental courses, honors classes, and upper-level mathematics classes for decades. Throughout her career she has been actively involved with many projects at Miami Dade, including those on computer learning, curriculum design, and the use of technology in the classroom. Donna's bachelor of science in mathematics and master of science in mathematics are both from the University of Miami.

## Dedications

In honor of Gus, Winston, Sine, Cosine, George, Pebbles, Lucy, Princess, Amber, Lady, Brownie, Aero, Chloe, Stanly, and Oliver for your past and present inspiration.

- Julie Miller \& Donna Gerken


## Acknowledgments

Paramount to the development of Precalculus was the invaluable feedback provided by instructors from around the country who reviewed the manuscript. In particular, we want to thank Alina Coronel for her amazing work in bringing our content into ALEKS, and Jennifer Blue for her steadfast scrutiny of our manuscript.

A special thanks to all of the event attendees who helped shape Precalculus. Focus groups and symposia were conducted to ensure the text was meeting the needs of students.

## Preface

## A Letter from the Author

Precalculus serves as a gateway course for many students interested in many disciplines. For some students, it is the entrance into the higher mathematics needed for careers in science, technology, engineering, and mathematics. For others, precalculus serves as the primary resource needed to understand the complexities of a modern world awash with statistics, investment strategies, financial planning, and even medical decisions. With the broad scope of this foundational course in mind, we worked to make the textbook, the digital tools, and the supplements as clear, relevant, and accessible as possible.

As part of our revision plan, we modernized content with new data sets and contemporary topics that are germane to all of today's students. Inside, you will find data and information relating to healthcare costs and ethics, income tax rates, the development of modern computers, extreme weather, and the work of modern scientists and inventors. We have also placed an emphasis on inclusion and diversity so that all students feel connected to the content in this book and can envision themselves in the many fields and situations in life that embrace mathematics.

In the text, we have expanded our exercise sets with an emphasis on mixed exercises that require multiple tools to complete their solutions. Likewise, we've added additional "challenge" exercises to promote more critical thinking and less rote repetition. Throughout this revision we were especially cognizant of the struggles for both teachers and students in adapting traditional content to the digital style of teaching that emerged during the pandemic.

We are excited for you, the student, to join an amazing journey in mathematics and hope it serves you well in your future.

Best to all,
Julie Miller
Donna Gerken

## Preface

## Key Features for Students

Clear, Precise Writing Precalculus is written for the diverse group of students taking this course. Julie Miller has written this text to use simple and accessible language. Through her friendly and engaging writing style, students are able to understand the material easily.

Fully Worked Examples The examples in the textbook are stepped-out in detail with thorough annotations at the right explaining each step. Following each example is a similar Apply the Skills exercise to engage students by practicing what they have just learned.

Modeling and Applications One of the most important tools to motivate our students is to make the mathematics they learn meaningful in their lives. The textbook is filled with robust applications and numerous opportunities for mathematical modeling for those instructors looking to incorporate these features into their course.

Building Confidence and Skills Throughout the text, students are encouraged to revisit key steps and tools while problem-solving. Specific, point-of-use call outs build student confidence in their abilities to solve complex problems.

- Insights boxes offer additional tips or observations on a concept or procedure
- Check Your Work boxes fend off common mistakes

Diverse Exercise Sets The exercises at the end of each lesson are graded, varied, and carefully organized to maximize student learning:

- Prerequisite Review Exercises begin the section-level exercises and ensure that students have the foundational skills to complete the homework sets successfully.
- Concept Connections prompt students to review the vocabulary and key concepts presented in the section.
- Core Exercises are presented next and are grouped by objective. These exercises are linked to examples in the text and direct students to similar problems whose solutions have been stepped-out in detail.
- Mixed Exercises do not refer to specific examples so that students can dip into their mathematical toolkit and decide on the best technique to use.
- Write About It exercises are designed to emphasize mathematical language by asking students to explain important concepts.
- Technology Connections require the use of a graphing utility and are found at the end of exercise sets. They can be easily skipped for those who do not encourage the use of calculators.
- Expanding Your Skills Exercises challenge and broaden students' understanding of the material.
- Point of Interest boxes feature interesting topics in mathematics from a diverse and inclusive set of contributors. These essays promote critical thinking, discussion, and research and often include follow-up questions and exercises.

End-of-Chapter Materials The textbook has the following end-of-chapter materials for students to review before test time:

- A summary with important equations, vocabulary, and key concepts. Detailed Chapter Summaries are available with the online resources.
- A Chapter Test reviewing the learning objectives of the full chapter.
- Cumulative Review Exercises that synthesize the learning objectives of the current chapter as well as all preceding chapters.


## Key Features for Teachers

The accompanying teacher manual has been authored specifically for the high school classroom. This manual helps teachers to build inclusive, engaging lectures and classroom experiences, alongside providing key information about answers and pacing.

Pairing Examples and Exercises saves teachers time and helps to build lectures and presentations. Each fully worked example is paired with end-of-lesson Practice Exercises that mirror the example.

Guided Lecture Notes are keyed to the objectives in each section of the text. The notes step through the material with a series of questions and exercises that can be used in conjunction with lecture.

Classroom Activities, including online and in-class exercises, build collaboration and confidence in students. A diversity of exercises are available, including:

- Wolfram Alpha activities promote active learning in the classroom by using a powerful online resource.
- A Group Activity is available for each chapter to promote classroom discussion and collaboration.
- The Problem Recognition Exercises are available as worksheets for students to work on as individuals or in groups to help them determine appropriate methods of solution for related problem types.


## Guided Tour - Chapter and Lesson Openers

## Exclusively for High School Precalculus

Precalculus is crafted specifically for the diverse group of students who will take this course as part of their high school math pathway. This text is organized around clear objectives, with abundant examples, and opportunities for students to develop mastery. Examples and concepts are connected to a wide range of applied topics certain to appeal to students with diverse backgrounds and interests, including business, chemistry, health, space travel, education, sports, travel, and more.

## Stunning Visuals, Intuitive Layout

Making the study of advanced mathematics accessible and approachable for all students starts with its considerate, student-friendly design. Each chapter opens with a bright visual related to the topic of the Launch Activity. The chapter Outline provides a quick reference to what is covered in each chapter. The chapter openers are designed to be inviting, intriguing, and easy to navigate.


## Clear, Precise Narrative

At the core of Precalculus is the accessible narrative that invites students to engage with and grasp the content through clear, student-friendly instruction. The narrative is augmented with point-of-use features and elements designed to enhance student interest and understanding.


The Launch Activity helps students to develop their comfort and confidence with the mathematical practices including modeling, asking questions, and using mathematical reasoning.

Lessons open with What Will You Learn? sections, which summarize the main learning objectives of the lesson. They also connect the content of the lesson to the wider world, emphasizing the connection between precalculus and students' everyday lives.

The Go online icon indicates that the lesson is extended online. Students can work independently or in groups to answer the questions in the launch activity in their digital course.

## Lesson 1-1

## The Rectangular Coordinate System and Graphing Utilities

## What Will You Learn?

After completing this lesson, you should be able to:

- Plot Points on a Rectangular Coordinate System
- Use the Distance and Midpoint Formulas
- Graph Equations by Plotting Points
- Identify $x$ - and $y$-intercepts
- Graph Equations Using a Graphing Uovily

Websites, newspapers, sporting events, and the workploce all tolize graphs and tables to present dava. Therefore, his important to leam how to create and interpret meaningful grophs. Understanding how points are located resitive to a fixed origin is important for many graphing applications. For example. computer

## Guided Tour - Lesson Features

## Guided Learning

Each chapter in Precalculus is divided into lessons, which follow a predictable pattern. This structure and repetition helps students build their foundational understanding of the concepts.

## Learn: Use the Distance and Midpoint Formulas

Recall that the distance between two points $A$ and $B$ on a number line can be represented by $|A-B|$ or $|B-A|$. Now we want to find the distance between two points in a coordinate plane. For example, consider the points $(1,5)$ and $(4,9)$. The distance $d$ between the points is labeled in Figure 1-3. The dashed horizontal and vertical line segments form a right triangle with hypotenuse $d$.


Figure 1-3

Chapter Lessons are divided into Learn sections, which focus on a particular topic or skill. Each Learn section starts with a Learn statement, and models a fully guided example for students to follow.

The horizontal distance between the points is $\mid 4$
The vertical distance between the points is $\mid 9-$ : Applying the Pythagorean theorem, we have

$$
\begin{aligned}
& d^{2}=\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2} \\
& d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
\end{aligned}
$$

We can drop the absolute value bars because $|a|^{2}=(a)^{2}$ for all real numbers $a$. Likewise
$\left|x_{2}-x_{1}\right|^{2}=\left(x_{2}-x_{1}\right)^{2}$ and $\left|y_{2}-y_{1}\right|^{2}=\left(y_{2}-y_{1}\right)^{2}$.

Important formulas and rules are highlighted in easy-to-find boxes for accessible student reference.

## Distance Formula

The distance between points $\left(x_{1}, y_{1}\right)$ and $\left(x_{2}, y_{2}\right)$ is given by

$$
d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}
$$

Example 1: Finding the Distance Between Two Points
Find the distance between the points $(-5,1)$ and $(7,-3)$. Give the exact distance and an approximation to 2 decimal places.

## Solution:

```
(-5, 1) and (7,-3)
(x,y) and ( }\mp@subsup{x}{2}{},\mp@subsup{y}{2}{}
d=\sqrt{}{[7-(-5)\mp@subsup{]}{}{2}+(-3-1)}
    = \sqrt{}{(12\mp@subsup{)}{}{2}+(-4\mp@subsup{)}{}{2}}
    = \sqrt{}{160}
    =4\sqrt{}{10}\approx12.65 The exact distance is 4\sqrt{}{10}\mathrm{ units.}
    This is approximately }12.65\mathrm{ units.
```

Label the points. Note that the choice for $\left(x_{1}, y_{7}\right)$ and $\left(x_{2}, y_{2}\right)$ will not affect the outcome.

Apply the distance formula. $d=\sqrt{\left(x_{2}-x_{1}\right)^{2}+\left(y_{2}-y_{1}\right)^{2}}$
Simplify the radical.

The exact distance is $4 \sqrt{10}$ units.
This is approximately 12.65 units.

## Apply the Skills

1. Find the distance between the points $(-1,4)$ and $(3,-6)$. Give the exact distance and an approximation to 2 decimal places.

Examples are fully worked out with step-bystep explanations, illustrations, and detailed solutions. Students build confidence in their own problem-solving ability, and are then immediately able to apply the skills they've learned in the Apply the Skills exercise.

## Meaningful Practice

In each lesson, students build toward independent problem solving. Scaffolded practice and timely reminders help to reinforce the learning while building their confidence. Several boxed reminders pop up that prompt students to double-check the assumptions they are making while problem-solving.

| Solution: |  |  |
| :---: | :---: | :---: |
| a. $9(-2)=2(-2)+1$ | Substitute -2 for $x$. | The function values represent the ordered pairs $(-2,-3)$. $(-1,-1),(0,1),(1,3)$, and (2,5). The line through the points represents all ordered pairs defined by this function. <br> This is the graph of the function. |
| $=-3$ | $g(-2)=-3$ |  |
| b. $g(-1)=2(-1)+1$ | Substitute -1 for $\boldsymbol{x}$. |  |
| = -1 | $g(-1)=-1$ |  |
| c. $g(0)=2(0)+1$ | Substitute 0 for $x$. |  |
| $=1$ | $g(0)=1$ |  |
| d. $g(1)=2(0)+1$ | Substitute 1 for $x$. |  |
| $=3$ | $g(0)=3$ |  |
| e. $g(2)=2(2)+1$ | Substitute 2 for $x$. |  |
| $=5$ | $g(2)=5$ |  |

5. Evaluate the function defined by $h(x)=4 x-3$ for the given values of $x$.
a. $h(-3)$
b. $h(-1)$
c. $h(0)$
d. $h(1)$
e. $h(3)$

## Good Practices

The name of a function can be represented by any letter or symbol. However, lowercase letters such as $f, g, h$, and so on are often used.

Each example is followed by Apply the Skills exercises, which give students an immediate opportunity to practice the concepts they just learned in the example. Good Practices and Check Your Work boxes encourage students to slow down and refresh their problem-solving skills.

## Apply the Skills

1. Graph the line represented by each equation.
a. $4 x+2 y=2$
b. $y=1$
c. $-3 x=12$

## Check Your Work

The graph of a linear equation is a line. Therefore, a minimum of two points is needed to graph the line. A third point can be used to verify that the line is graphed correctly. The points must all line up.

## Insights

Since $\left(x_{2}-x_{1}\right)^{2}=\left(x_{1}-x_{2}\right)^{2}$ and $\left(y_{2}-y_{1}\right)^{2}=\left(y_{1}-y_{2}\right)^{2}$, the distance formula can also be expressed as

$$
d=\sqrt{\left(x_{1}-x_{2}\right)^{2}+\left(y_{1}-y_{2}\right)^{2}} .
$$

## Good Practices

For more accuracy in the graph, plot one or two points near the vertex. Then use the symmetry of the curve to find additional points on the graph.
For example, the points $(1,-2)$ and $(0,-5)$ are on the left branch of the parabola. The corresponding points to the right of the axis of symmotry are $(3,-2)$ and $(4,-5)$.

## Check Your Work

By clearing fractions, the result of Example 1 can be checked by multiplication.

$$
\begin{aligned}
\text { Dividend } & =(\text { Divisor)Quotient }+ \text { Remainder } \\
6 x^{3}-5 x^{2}-3 & \doteq(3 x+2)\left(2 x^{2}-3 x+2\right)+(-7) \\
& \doteq 6 x^{3}-5 x^{2}+4+(-7) \\
& \doteq 6 x^{3}-5 x^{2}-3
\end{aligned}
$$

Insights provide additional context or explanation, often offering an alternate formula or way of thinking about a problem.

Good Practices highlight proper notation and equation structure, to ensure consistency in problem solving.

Check Your Work fends off common pitfalls by highlighting areas where mistakes are commonly made.

## Guided Tour - Lesson Features

## Real-World Connections

One of the most important tools to motivate students is to make the mathematics they learn meaningful to their lives. Numerous robust applications connect the abstract concepts to tangible events and circumstances for students.

## TECHNOLOGY CONNECTIONS

Setting a Square Viewing Window and Graphing a Circle
A graphing calculator expects an equation with the $y$ variable isolated. Therefore, to graph an equation of a circle such as $(x+5)^{2}+(y-3)^{2}=9$, from Example 3, we first solve for $y$.
$(x+5)^{2}+(y-3)^{2}=9$
$(v-3)^{2}-9-(x+5)^{2}$


Point of Interest features interesting topics from diverse perspectives and a variety of applications. Perfect for partner activities or class discussions, these features provide students with the opportunity to see the relevance of mathematics to their everyday life.

Technology Connections clearly explain how to solve different types of problems with the use of a graphing calculator. Students will apply this at the end of each lesson.

```
Foint of Interest
Why 60%
One hour may be subdivied irto t0 minkes, und each minute mar be furter dvided vto
60 seconks. The 360 degees in a cricle myy be sinmaty subdvided by diving each degre
ito 60 mintet and each minute isto 60 secondt (These muller mesusemerts of angle tre
```



```
*)
A number whtem baved on 40 is calid rewpepingh,
```





```
2, 15, 20.30, mod bo
135. Purhuss plad habks de terd, bat why not use 100 minutes in an hour? Why not convide
    N0, degrees toc a croel infiesting) in D93. durng Hye Frenct Alvolion a new systen,
```




```
    en fiench Almationoy Time and the timepieces created to teep time in this turkich
```



## Support for All Learners

Differentiated support addresses a range of abilities and proficiency levels, including students who need support with prerequisite skills and those who are ready for more challenging content.


For accelerated learners, the online-only Chapter 12 offers a preview of calculus.

The printable, interactive Corequisite Workbook is included in the digital resources. This workbook provides extra math practice on prerequisite content for each lesson in the textbook.

Chapter 12 (online only): Preview of Calculus
Lesson 12.4 Inteoduction to Limis Theough Tables and Graphs
tesson 12-2 Algetraic Properties of Limits
Probiem Recognition Exercises: Limits and Continuity Lesson 12.3 The Tangert Line Problemc introduction to Derivatives
Lesson 12-4 Limits at Infinity and Limits of Sepuences
Lesson 12.5 Area Under a Cunve

## Guided Tour - Lesson Review

## Abundant and Varied Lesson Assessments

Each lesson ends with a graded, varied, and carefully organized set of Practice Exercises. Several strands of problems are present to review the full breadth of knowledge covered in each lesson.


Prerequisite Review problems open each set of Practice Exercises.

Concept Connections ask students to review the vocabulary and key concepts presented in the section.

Each Learn title is reviewed independently with several practice exercises.

Mixed Exercises review topics across multiple learning objectives within a lesson.

## Mixed Exercises

For Exercises 99-102, write an equation of the line from the graph. Write the answer in slope-intercept form.



Write About It exercises emphasize mathematical language, giving students an opportunity to answer questions and describe key concepts in their own words.

Expanding Your Skills challenges and broadens students' understanding of the material.

Technology Connections are designed for use with students' graphing calculators.
102.

## Write About It

107. Explain how you can determine from a linear equation $A x+B y=C(A$ and $B$ not both zero) whether the line is slanted, horizontal, or vertical.
108. Explain how you can determine from a linear equation $A x+B y=C(A$ and $B$ not both zero) whether the line passes through the origin.
109. What is the benefit of writing an equation of a line in slope-intercept form?
110. Explain how the average rate of change of a function $f$ on the interval $\left[x_{T} x_{2}\right]$ is related to slope.

Expanding Your Skills
111. Determine the area in the second quadrant enclosed by the equation $y=2 x+4$ and the $x$ - and $y$-axes.
112. Determine the area enclosed by the equations.

$$
y=x+6
$$

$y=-2 x+6$ $y=0$
113. Determine the area enclosed by the
116. Use the results from Exercise 115 to determine the slope and $y$-intercept for the graphs of the lines.
a. $5 x-9 y=6$
b. $0.052 x-0.013 y=0.39$

Technology Connections
For Exercises 117-120, solve the equation in part (a) and verify the solution on a graphing calculator. Then use the graph to find the solution set to the inequalities in parts (b) and (c). Write the solution sets to the inequalities in interval notation. (See Example 9)
117. a. $3.1-2.2(t+1)=6.3+1.4 t$
b. $31-2.2(t+1)>6.3+1.4 t$
c. $31-2.2(t+1)<6.3+1.4 t$
118. อ. $-11.2-4.6(c-3)+1.8 c=0.4(c+2)$
b. $-11.2-4.6(c-3)+1.8 c>0.4(c+2)$
c. $-11.2-4.6(c-3)+1.8 c<0.4(c+2)$
119. a. $|2 x-3.8|-4.6=72$
b. $|2 x-3.8|-4.6 \geq 7.2$
c. $|2 x-3.8|-4.6 \leq 7.2$

## Guided Tour - Chapter Review

## Comprehensive Chapter Review

Each chapter ends with a Chapter Review including a summary with important equations, vocabulary, and key concepts. These elements review the full extent of the content covered in the chapter.


The Chapter Review is organized into Key Concepts tables. Each table summarizes the most important vocabulary words and formulas from each lesson.

Lesson 14 Linear Equations in Two Variables and Linear Functions

```
Kgy Concepts Meserence
Let A,B, and C represent real numbers where, A and & are not both ip S0
zerc.A linear equation in the variabies }x\mathrm{ and }y\mathrm{ is an equation that
can be wrisen as Ax+By=C
The slope of a line passing trough the divingt ponts (x,y) and p. p. 53
```



```
Gven a line with slope it and y intercept (0. ol, the slepe-intercept p.55
form of the line is given by y =rtx+b
|f is defined on me intervel \x, x, , then the average rate of change
of fon the intevad [x, x,] is the sispe of the secart line containing.
```



```
Thex-coordinetes ef the poins of insersection between the grohs of p. 59
y=N(0) and y=g(x) are the solutions to the equation f(x)=g(0)
```

Each key formula or vocabulary word is tagged with a page reference, so that students can easily navigate back to the initial introduction of that concept for review.


Detailed Chapter Summaries, available with the digital resources, are excellent study tools.

## Robust, Versioned Assessments

The Chapter Test assesses students on all the learning objectives covered in the chapter and can be used for self-study and review. Additional versioned Lesson Quizzes and Chapter Tests are available as part of the digital resources, alongside an expansive, editable question bank.

## Chapter 1 Test

1. The endpoints of a diameter of a circle are $(-2,3)$ and $(8,-5)$.
a. Determine the center of the circle.
b. Determine the radius of the circle.
c. Write an equation of the circle in standard form.
2. Given $x=|y|-4$,
a. Determine the $x$ - and $y$-intercepts of the graph of the equation.
b. Does the equation define $y$ as a function of $x$ ?
3. Given $x^{2}+y^{2}+14 x-10 y+70=0$,
a. Write the equation of the circle in standard form.
b. Identify the center and radius.

For Exercises 4-5, determine if the relation defines $y$ as a function of $x$.
4.

5.

6. Given $f(x)=-2 x^{2}+7 x-3$, find
a. $f(-1)$.
b. $f(x+h)$
c. The difference quotient: $\frac{f(x+h)-f(x)}{h}$.
d. The $x$-intercepts of the graph of $f$.
e. The $y$-intercept of the graph of $f$.
f. The average rate of change of $f$ on the interval $[1,3]$.
7. Use the graph of $y=f(x)$ to estimate
a. $f(0)$.
b. $f(-4)$.
c. The values of $x$ for which $f(x)=2$.

d. The interval(s) over which $f$ is increasing.
e. The interval(s) over which $f$ is decreasing.
f. Determine the location and value of any relative minima.
g. Determine the location and value of any relative maxima.
h. The domain.
i. The range.
j. Whether $f$ is even, odd, or neither.

For Exercises 8-9, write the domain in interval notation.
8. $f(w)=\frac{2 w}{3 w+7}$
9. $f(c)=\sqrt{4-c}$
10. Given $3 x=-4 y+8$,
a. Identify the slope.
b. Identify the $y$-intercept.
c. Graph the line.
d. What is the slope of a line perpendicular to this line?
e. What is the slope of a line parallel to this line?
11. Write an equation of the line passing through the point $(-2,6)$ and perpendicular to the line defined by $x+3 y=4$.
12. Use the graph to solve the equation and inequalities. Write the solutions to the inequalities in interval notation.

a. $2 x+8=-\frac{1}{2} x+3$
b. $2 x+8<-\frac{1}{2} x+3$
c. $2 x+8 \geq-\frac{1}{2} x+3$

# Best in Class Digital Resources 

## Precalculus is enriched with multimedia content that enhance the teaching and learning experience both inside and outside of the classroom.

Developed with the world's leading subject matter experts and organized by chapter level, the resources provide students with multiple opportunities to contextualize and apply their understanding. Teachers can save time, customize lessons, monitor student progress, and make data-driven decisions in the classroom with the flexible, easy-tonavigate instructional tools.

## Student Assignments

Resources are organized at the chapter level. To enhance the core content, teachers can add assignments, activities, and instructional aids to any lesson. The chapter landing page gives students access to:

- assigned activities;
- customizable, auto-graded assessments;
- an interactive eBook;
- data sets, data projects,
 and classroom activities;
- and an interactive and printable Corequisite Workbook for additional math skills practice.


## Mobile Ready

Access to course content on-the-go is easier and more efficient than ever before with the McGraw Hill K12 Portal App.

## Teacher Resources

Teachers have access to the interactive eBook, plus a wealth of customizable chapter resources and powerful gradebook tools. Resources include:

- a solutions manual with answers to the end-of-chapter questions in the student edition;
- actionable reporting features that track student progress with data-driven insights;
- customizable PowerPoint presentations, visual aids, and additional ideas for lecture enrichment;
- and customizable assignments and quiz banks that are automatically graded and populate easy-to-read reports.



## Adaptive Learning with ALEKS

Available with the digital subscription, as an add-on. ALEKS uses adaptive questioning to quickly and accurately determine exactly what math topics a student knows and doesn't know and instructs each student on the topics they are most ready to learn. With ALEKS, teachers can:

- assess a student's proficiency and knowledge with an "Initial Knowledge Check,"

- track which topics have been mastered,
- identify areas that need more study,
- and build student confidence with preparatory modules, video tutorials, and practice questions.


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## Chapter 4

Trigonometric Functions

## Outline

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Launch Activity: Make Use of Structure
Revolutions of the Wheel
Your robotics team is preparing to compete in a local competition. Once the course for the competition is complete your team can view the layout and measure distances of the path and angles of the turns prior to the event. All coding for this course must be completed before the competition begins. For this competition, the standard radius of each wheel is 3.5 cm .

One teammate from your robotics team measured the first straight segment of the competition path and found the length to be 4 feet 3 inches long. Coding for the robot's movement does not use distance as input but rather revolutions of the wheel.

Talk About It

1. What do you notice?
2. What questions can you ask?

You will work with your partner to answer these questions: How many revolutions will the wheel of the robot make in the first straight segment of the course? How many degrees will the wheel travel for this section of the course?

Analyze the Problem

1. What assumptions are you making? Why are you making these assumptions?
2. What constraints, if any, need to be considered? How might they affect the solution?
3. What type of model best represents the situation?
4. What variables will you use? What do they represent and how are they related?

G® GO ONLINE to answer these questions

## Angles and Their Measure

## What Will You Learn?

After completing this lesson, you should be able to:

- Find Degree Measure
- Find Radian Measure
- Determine Coterminal Angles
- Compute Arc Length of a Sector of a Circle
- Compute Linear and Angular Speed
- Compute the Area of a Sector of a Circle


## Learn: Find Degree Measure

The Tour de France is the most famous bicycle race in the world, spanning 3500 km $(2175 \mathrm{mi})$ in 21 days through France and two mountain ranges. The speed of a racer depends on a number of variables, including the gear ratio and the cadence. The gear ratio determines the number of times the rear wheel turns with each pedal stroke, and the cadence is the number of revolutions of the pedals per minute. In this lesson, we study angles and their measure, and the relationship between angular and linear speeds to study such applications as the speed of a bicycle.


A ray is a part of a line that consists of an endpoint and all points on the line to one side of the endpoint. In Figure 4-1, ray $\overrightarrow{P Q}$ is named by using the endpoint $P$ and another point $Q$ on the ray. Notice that the rays $\overrightarrow{P Q}$ and $\overrightarrow{Q P}$ are different because the initial points are different and the rays extend in opposite directions.


Figure 4-1

## Good Practices

A ray has only one endpoint, which is always written first when naming the ray.

An angle is formed by rotating a ray about its endpoint. The starting position of the ray is called the initial side of the angle, and the final position of the ray is called the terminal side. The common endpoint is called the vertex of the angle, and the vertex is often named by a capital letter such as $A$ (Figure 4-2).

Angle $A$ in Figure 4-2 can be denoted by $\angle A$ (read as "angle $A$ ") or by $\angle B A C$, where $B$ is a point on the initial side of the angle, $A$ represents the vertex, and $C$ is a point on the terminal side. Alternatively, Greek letters such as $\theta$ (theta), $\alpha$ (alpha), $\beta$ (beta), and $\gamma$ (gamma) are often used to denote angles.

Insights When denoting an angle such as $\angle B A C$, the vertex is the middle letter.

An angle is in standard position if its vertex is at the origin in the $x y$-plane and its initial side is the positive $x$-axis. In Figure 4-3, angle $\alpha$ is drawn in standard position.


Figure 4-2


Figure 4-3

The measure of an angle quantifies the direction and amount of rotation from the initial side to the terminal side. The measure of an angle is positive if the rotation is counterclockwise, and the measure is negative if the rotation is clockwise. One unit with which to measure an angle is the degree. One full rotation of a ray about its endpoint is 360 degrees, denoted $360^{\circ}$. Therefore, $1^{\circ}$ is $\frac{1}{360}$ of a full rotation.
Figure $4-4$ shows a variety of angles and their measures.





Figure 4-4

## Insights

The hands of a clock each resemble a ray. When the minute hand moves from 12:00 to $12: 15$, the rotation is $-90^{\circ}$.


If the measure of an angle $\theta$ is $30^{\circ}$, we denote the measure of $\theta$ as $m(\theta)=30^{\circ}$ or simply $\theta=30^{\circ}$. We may also refer to $\theta$ as a $30^{\circ}$ angle, rather than using the more formal, but cumbersome language "an angle whose measure is $30^{\circ}$." We also introduce several key terms associated with the measure of an angle (Figure 4-5).

A right angle is a $90^{\circ}$ angle and a straight angle is a $180^{\circ}$ angle. An acute angle is one with measure strictly between $0^{\circ}$ and $90^{\circ}$, while an obtuse angle measures strictly bewteen $90^{\circ}$ and $180^{\circ}$.


If the sum of the measures of two angles is $90^{\circ}$, we say that the angles are complementary (for example, the complement of a $20^{\circ}$ angle is a $70^{\circ}$ angle and vice versa). If the sum of the measures of two angles is $180^{\circ}$, we say that the angles are supplementary (for example, the supplement of a $20^{\circ}$ angle is a $160^{\circ}$ angle and vice versa).

A degree can be divided into 60 equal parts called minutes (min or '), and each minute is divided into 60 equal parts called seconds (sec or "). Note that the terms minutes and seconds when used in the context of angular measure are sometimes called arcminutes and arcseconds to avoid confusion with measurements of time.

- 1 min $=\left(\frac{1}{60}\right)^{\circ}$ or $1^{\prime}=\left(\frac{1}{60}\right)^{\circ}$
- $1 \mathrm{sec}=\left(\frac{1}{60}\right)^{\prime}=\left(\frac{1}{3600}\right)^{\circ}$ or $1^{\prime \prime}=\left(\frac{1}{60}\right)^{\prime}=\left(\frac{1}{3600}\right)^{\circ}$

For example, 74 degrees, 42 minutes, 15 seconds is denoted $74^{\circ} 42^{\prime} 15^{\prime \prime}$.

Example 1: Converting from Degrees, Minutes, Seconds (DMS) to Degree Decimal Form
Convert $74^{\circ} 42^{\prime} 15^{\prime \prime}$ to decimal degrees. Round to 4 decimal places.

## Solution:

Convert the minute and second portions of the angle to degrees. Choose the conversion factors so that the original units (minutes and seconds) "cancel," leaving the measurement in degrees.

$$
\begin{aligned}
74^{\circ} 42^{\prime} 15^{\prime \prime} & =74^{\circ}+(42 \mathrm{~min}) \cdot\left(\frac{1^{\circ}}{60 \mathrm{~min}}\right)+(15 \sec ) \cdot\left(\frac{1^{\circ}}{3600 \sec }\right) \\
& =74^{\circ}+0.7^{\circ}+0.0041 \overline{6}^{\circ} \\
& \approx 74.7042^{\circ}
\end{aligned}
$$

## Apply the Skills

1. Convert $131^{\circ} 12^{\prime} 33^{\prime \prime}$ to decimal degrees. Round to 4 decimal places.

## Example 2: Converting from Decimal Degrees to Degrees, Minutes, Seconds (DMS) Form

Convert $159.26^{\circ}$ to degree, minute, second form.

## Solution:

$159.26^{\circ}=159^{\circ}+0.26^{\circ}$
$=159^{\circ}+0.26^{\circ} \cdot\left(\frac{60^{\prime}}{1^{\circ}}\right)$
$=159^{\circ}+15.6^{\prime}$
$=159^{\circ}+15^{\prime}+0.6^{\prime}$
$=159^{\circ}+15^{\prime}+0.6^{\prime} \cdot\left(\frac{60^{\prime \prime}}{1^{\prime}}\right)$

$$
\begin{aligned}
& =159^{\circ}+15^{\prime}+36^{\prime \prime} \\
& =159^{\circ} 15^{\prime} 36^{\prime \prime}
\end{aligned}
$$

Write the decimal as a whole number part plus a fractional part. The fractional part of $1^{\circ}$ needs to be converted from degrees to minutes and seconds.
Use the conversion factor $60^{\prime}=1^{\circ}$ to convert to minutes.

Write $15.6^{\prime}$ as a whole number part plus a fractional part.

Now convert the fractional part of 1 minute to seconds. Use the conversion factor $60^{\prime \prime}=1^{\prime}$ to convert to seconds.
路

Apply the Skills
2. Convert $26.48^{\circ}$ to degree, minute, second form.

## Learn: Find Radian Measure

Degree measure is used extensively in many applications of engineering, surveying, and navigation. Another type of angular measure that is better suited for applications in trigonometry and calculus is radian measure. To begin, we define a central angle as an angle with the vertex at the center of a circle.

## Definition of One Radian

A central angle that intercepts an arc on the circle with length equal to the radius of the circle has a measure of 1 radian (Figure 4-6).

Note: One radian may be denoted as 1 radian, 1 rad, or simply 1 . That is, radian measure carries no units.

When two lines or rays cross a circle, the part of the circle between the intersection points is called the intercepted arc and is often denoted by s.


Figure 4-6
Any central angle can be measured in radians by dividing the length $s$ of the intercepted arc by the radius $r$. For example, in Figure 4-7, the length of the red arc is $2 r$ (twice the radius).


Figure 4-7
Therefore, the measure of angle $\theta$ is given by

$$
\theta=\frac{s}{r}=\frac{2 r}{r}=2(2 \text { radians })
$$

## Definition of Radian Measure of an Angle

The radian measure of a central angle $\theta$ subtended by an arc of length $s$ on a circle of radius $r$ is given by $\theta=\frac{S}{r}$.

You may have an intuitive feel for angles measured in degrees (for example, $90^{\circ}$ is one-quarter of a full rotation). However, radian measure is unfamiliar. From Figure $4-6$, notice that an angle of 1 radian ( 1 rad ) is approximately $57.3^{\circ}$. If we divide the circumference of a circle into arcs of length $r$ (Figure 4-8), we see that there are just over 6 rad in one full rotation. In fact, we can show that there are exactly $2 \pi$ rad in one full rotation.


Figure 4-8

## Insights

Radian measure carries no units because it is measured as a ratio of two lengths with the same units (the units associated with $s / r$ "cancel"). So, $2 \pi \mathrm{rad}$ is simply written as $2 \pi$. It is universally understood that the measure is in radians. Sometimes the notation "rad" is included for emphasis, but is not necessary.

Recall that $\pi$ is defined as the ratio of the circumference of a circle to its diameter $d$. Therefore, the circumference $C$ is given by $C=\pi d$ or equivalently $C=2 \pi r$, where $r$ is the radius of the circle. The circumference is the arc length of a full circle, and dividing this by the radius gives the number of radians in one revolution.


The angular measure of one full rotation is $2 \pi$ ( $2 \pi \mathrm{rad}$ ). Therefore, we have the following relationships.


The statement $\pi=180^{\circ}$ gives us a conversion factor to convert between degree measure and angular measure.

## Converting Between Degree and Radian Measure

- To convert from degrees to radians, multiply the degree measure by $\frac{\pi}{180^{\circ}}$.
- To convert from radians to degrees, multiply the radian measure by $\frac{180^{\circ}}{\pi}$.


## Good Practices

Some angles are used frequently in the study of trigonometry. Their degree measures and equivalent radian measures are worth memorizing.

$$
30^{\circ}=\frac{\pi}{6} \quad 45^{\circ}=\frac{\pi}{4} \quad 60^{\circ}=\frac{\pi}{3} \quad 90^{\circ}=\frac{\pi}{2}
$$

## Example 3: Converting from Degrees to Radians

Convert from degrees to radians.
a. $210^{\circ}$
b. $-135^{\circ}$

## Solution:

a. $210^{\circ} \cdot\left(\frac{\pi \mathrm{rad}}{180^{\circ}}\right)=\frac{210 \pi}{18 \sigma} \mathrm{rad}=\frac{7 \pi}{6} \mathrm{rad}=\frac{7 \pi}{6}$

The units of degrees "cancel" in the numerator and denominator in the first step, leaving units of radians.
b. $-135^{\circ} \cdot\left(\frac{\pi}{180^{\circ}}\right)=-\frac{13^{3} 5 \pi}{18 \sigma}=-\frac{3 \pi}{4}$

The units of "rad" are implied in the numerator of the conversion factor $\frac{\pi}{180^{\circ}}$.

## Apply the Skills

3. Convert from degrees to radians.
a. $300^{\circ}$
b. $-70^{\circ}$

## Example 4: Converting from Radians to Degrees

Convert from radians to degrees.
a. $\frac{\pi}{12}$
b. $-\frac{4 \pi}{3}$

## Solution:

a. $\frac{\pi}{12} \operatorname{rad} \cdot\left(\frac{180^{\circ}}{\pi \mathrm{rad}}\right)=\left(\frac{180^{5} \pi}{122 \pi}\right)^{\circ}=15^{\circ}$
b. $-\frac{4 \pi}{3} \cdot\left(\frac{180^{\circ}}{\pi}\right)=\left(-\frac{7220 \pi}{3_{1}^{24} \pi}\right)^{\circ}=-240^{\circ}$

The units of rad "cancel" in the numerator and denominator in the first step, leaving units of degrees.

The units of "rad" are implied in the denominator of the conversion factor $\frac{180^{\circ}}{\pi}$.

## Insights

To help determine which conversion factor to use, remember that you want the original units to "cancel," leaving the new unit of measurement in the numerator.

## Apply the Skills

4. Convert from radians to degrees.
a. $\frac{\pi}{18}$
b. $-\frac{7 \pi}{4}$

## Learn: Determine Coterminal Angles

Two angles in standard position with the same initial side and same terminal side are called coterminal angles. Figure 4-9 illustrates three angles in standard position that are coterminal to $30^{\circ}$. Notice that each angle is $30^{\circ}$ plus or minus some number of full revolutions clockwise or counterclockwise.

$$
30^{\circ}+(1)\left(360^{\circ}\right)=390^{\circ}
$$


$30^{\circ}+(-1)\left(360^{\circ}\right)=-330^{\circ}$


Figure 4-9 Coterminal Angles

## Point of Interest

Aerial snowboarding is a winter sport in which competitors perform aerial tricks after launching from the sides of a halfpipe. The degree of difficulty of a move is measured in part by the number of full rotations that a competitor completes through the air. Ulrik Badertscher of Norway was the first to rotate through the the air an amazing 4.5 full rotations, an angle of $1620^{\circ}$.

## Example 5: Finding Coterminal Angles between $0^{\circ}$ and $360^{\circ}$

Find an angle coterminal to $\theta$ between $0^{\circ}$ and $360^{\circ}$.
a. $\theta=960^{\circ}$
b. $\theta=-225^{\circ}$

## Solution:

a. $\theta=960^{\circ}$ is more than $360^{\circ}$. Therefore, we will subtract some multiple of $360^{\circ}$ to get an angle coterminal to $\theta$ between $0^{\circ}$ and $360^{\circ}$.

- One revolution measures $360^{\circ}$.
- Two revolutions measure $720^{\circ}$.
- Three revolutions measure $1080^{\circ}$.
$360^{\circ}(2)$ is subtracted from $960^{\circ}$
because $\theta$ is between 2 and 3
revolutions.


Therefore,
$960^{\circ}-\left(360^{\circ}\right)(2)=960^{\circ}-720^{\circ}=240^{\circ}$

## Insights

The number of revolutions contained in $960^{\circ}$ can be found by dividing $960^{\circ}$ by $360^{\circ}$.

$\theta=960^{\circ}$ is two full revolutions plus $240^{\circ}$.
b. An angle of $-225^{\circ}$ is less than one revolution. Therefore, we will add $360^{\circ}$ (1) to obtain a positive angle between $0^{\circ}$ and $360^{\circ}$ and coterminal to $-225^{\circ}$.
$-225^{\circ}+360^{\circ}=135^{\circ}$


## Apply the Skills

5. Find an angle coterminal to $\theta$ between $0^{\circ}$ and $360^{\circ}$.
a. $\theta=1230^{\circ}$
b. $\theta=-315^{\circ}$

Two angles in standard position are coterminal if their measures differ by a multiple of $360^{\circ}$ or $2 \pi \mathrm{rad}$. The angles shown in Figure 4-10 are coterminal to an angle of $\frac{\pi}{6}$ rad (or $30^{\circ}$ ).




Figure 4-10 Coterminal Angles

## Example 6: Finding Coterminal Angles between 0 and $2 \pi$

Find an angle coterminal to $\theta$ on the interval $[0,2 \pi$ ).
a. $\theta=-\frac{5 \pi}{6}$
b. $\theta=\frac{13 \pi}{2}$

## Solution:

a. One revolution is $2 \pi$ rad or equivalently $\frac{12 \pi}{6}$.

Therefore, adding any multiple of $\frac{12 \pi}{6}$ to $-\frac{5 \pi}{6}$ results in an angle coterminal to $-\frac{5 \pi}{6}$.

$$
-\frac{5 \pi}{6}+\frac{12 \pi}{6}=\frac{7 \pi}{6}
$$


b. One revolution is $2 \pi \mathrm{rad}$, or equivalently $\frac{4 \pi}{2}$.

We can divide $\frac{13 \pi}{2} \div \frac{4 \pi}{2}=\frac{13 \pi}{2} \cdot \frac{2}{4 \pi}=\frac{13}{4}=3 \frac{1}{4}$.
Therefore, $\frac{13 \pi}{2}$ is three full revolutions plus $\frac{1}{4}$ of a revolution.
Subtracting three revolutions results in an angle coterminal to $\frac{13 \pi}{2}$.



## Apply the Skills

6. Find an angle coterminal to $\theta$ on the interval $[0,2 \pi)$.
a. $\theta=-\frac{\pi}{8}$
b. $\quad \theta=\frac{19 \pi}{4}$

## Learn: Compute Arc Length of a Sector of a Circle

By definition, the measure of a central angle $\theta$ in radians equals the length $s$ of the intercepted arc divided by the radius $r$, that is, $\theta=\frac{s}{r}$. Solving this relationship for $s$ gives $s=r \theta$, which enables us to compute arc length if the measure of the central angle and radius are known.

## Arc Length

Given a circle of radius $r$, the length $s$ of an arc intercepted by a central angle $\theta$ (in radians) is given by

$$
s=r \theta
$$



## Example 7: Determining Arc Length

Find the length of the arc made by an angle of $105^{\circ}$ on a circle of radius 15 cm . Give the exact arc length and round to the nearest tenth of a centimeter.

## Solution:

$$
\begin{array}{ll}
\theta=105^{\circ} \cdot \frac{\pi}{180^{\circ}}=\frac{7 \pi}{12} & \text { Convert } \theta \text { to radians } \\
s=r \theta=(15 \mathrm{~cm}) \cdot \frac{7 \pi}{12}=8.75 \pi \mathrm{~cm} & \text { Apply the arc length formula. }
\end{array}
$$

The arc is $8.75 \pi \approx 27.5 \mathrm{~cm}$.

## Apply the Skills

7. Find the length of the arc made by an angle of $220^{\circ}$ on a circle of radius 9 in .

The Earth is approximately spherical, and the most common way to locate points on the surface is by using latitude and longitude. These coordinates are measured in degrees and represent central angles measured from the center of the Earth.

The equator is an imaginary circle around the Earth equidistant between the north and south poles. Latitude is the angular measure of a central angle measuring north ( N ) or south ( S ) from the equator. There are $90^{\circ}$ of latitude measured north from the equator
 and $90^{\circ}$ of latitude measured south from the equator. The equator has a latitude of $0^{\circ}$, the north pole has a latitude of $90^{\circ} \mathrm{N}$, and the south pole has a latitude of $90^{\circ} \mathrm{S}$ (Figure 4-11).


Figure 4-11
Lines of longitude, called meridians, are circles that pass through both poles and run perpendicular to the equator. By international agreement, $0^{\circ}$ longitude is taken to be the meridian line through Greenwich, England. This is called the prime meridian. So, longitude is the angular measure of a central angle east ( E ) or west (W) of the prime meridian. The Earth is divided into $360^{\circ}$ of longitude. There are $180^{\circ}$ east (E) of the prime meridian and $180^{\circ}$ west (W) of the prime meridian.

For example, New York City is located at $40.7^{\circ} \mathrm{N}, 74.0^{\circ} \mathrm{W}$. This means that New York City is located $40.7^{\circ}$ north of the equator and $74.0^{\circ}$ west of the prime meridian.

## Example 8: Determining the Distance Between Cities

Seattle, Washington, is located at $47.6^{\circ} \mathrm{N}, 122.4^{\circ} \mathrm{W}$, and San Francisco, California, is located at $37.8^{\circ} \mathrm{N}, 122.3^{\circ} \mathrm{W}$. Since the longitudes are nearly the same, the cities are roughly due north-south of each other. Using the difference in latitude, approximate the distance between the cities assuming that the radius of the earth is 3960 mi . Round to the nearest mile.


## Solution:

$$
\begin{array}{ll}
47.6^{\circ}-37.8^{\circ}=9.8^{\circ} & \text { Calculate the difference in latitude. } \\
\theta=9.8^{\circ} \cdot \frac{\pi}{180^{\circ}} \approx 0.17104 & \text { Convert } \theta \text { to radians. } \\
s=r \theta & \text { Apply the arc length formula. } \\
s=(3960)(0.17104) \approx 677 \mathrm{mi} &
\end{array}
$$

## Check Your Work

When applying the formula $s=r \theta$, always use radian measure for $\theta$.

## Apply the Skills

8. Lincoln, Nebraska, is located at $40.8^{\circ} \mathrm{N}, 96.7^{\circ} \mathrm{W}$ and Dallas, Texas, is located at $32.8^{\circ} \mathrm{N}, 96.7^{\circ} \mathrm{W}$. Since the longitudes are the same, the cities are north-south of each other. Using the difference in latitudes, approximate the distance between the cities assuming that the radius of the Earth is 3960 mi. Round to the nearest mile.

## Learn: Compute Linear and Angular Speed

Consider a ceiling fan rotating at a constant rate. For a small increment of time, suppose that point $A$ on the tip of a blade travels a distance $s_{1}$. In the same amount of time, point $B$ will travel a shorter distance $s_{2}$ (Figure 4-12). Therefore, point $A$ on the tip has a greater linear speed $v$ than point $B$. However, each point has the same angular speed $\omega$ (omega) because they move through the same angle for a given unit of time.


Figure 4-12

## Angular and Linear Speed

If a point on a circle of radius $r$ moves through an angle of $\theta$ radians in time $t$, the angular and linear speeds of the point are

$$
\begin{aligned}
& \text { angular speed: } \omega=\frac{\theta}{t} \\
& \text { linear speed: } v=\frac{s}{t} \text { or } \quad v=\frac{r \theta}{t} \text { or } \quad v=r \omega
\end{aligned}
$$

## Insights

In an application, if a rate is given in degrees, radians, or revolutions per unit time, it is an angular speed. If a rate is given in linear units (such as feet) per unit time, it is a linear speed.

## Example 9: Finding Linear and Angular Speed

A ceiling fan rotates at 90 rpm (revolutions per minute). For a point at the tip of a 2-ft blade,
a. Find the angular speed.
b. Find the linear speed. Round to the nearest whole unit.

## Solution:

a. For each revolution of the blade, the point moves through an angle of $2 \pi$ radians.
$\omega=\frac{\theta}{t}=\frac{90 \mathrm{rev}}{\mathrm{min}} \cdot \frac{2 \pi \mathrm{rad}}{\mathrm{ret}}=180 \pi \mathrm{rad} / \mathrm{min}=180 \pi / \mathrm{min}$
b. $v=r \omega$
$v=(2 \mathrm{ft})\left(\frac{180 \pi}{\mathrm{~min}}\right)=360 \pi \mathrm{ft} / \mathrm{min} \approx 1131 \mathrm{ft} / \mathrm{min}$

## Insights

Converting the result to miles per hour, we have

$$
\frac{1131 \mathrm{ft}}{\min } \cdot \frac{1 \mathrm{mi}}{5280 \mathrm{ft}} \cdot \frac{60 \mathrm{~min}}{1 \mathrm{hr}} \approx 12.9 \mathrm{mph} .
$$

## Apply the Skills

9. A bicycle wheel rotates at 2 revolutions per second.
a. Find the angular speed.
b. How fast does the bicycle travel (in $\mathrm{ft} / \mathrm{sec}$ ) if the wheel is 2.2 ft in diameter? Round to the nearest tenth.

## Point of Interest

The Earth rotates at a constant angular speed of $360^{\circ}$ in 24 hr or equivalently $15^{\circ}$ in 1 hr . Therefore, each time zone is approximately $15^{\circ}$ of longitude in width (with local variations) and is 1 hr earlier than the zone immediately to the east.

## Learn: Compute the Area of a Sector of a Circle

A sector of a circle is a "pie-shaped" wedge of a circle bounded by the sides of a central angle and the intercepted arc (Figure 4-13).

The area of a sector of a circle is proportional to the measure of the central angle. The expression $\frac{\theta}{2 \pi}$ is the fractional amount of a full rotation represented by angle $\theta$ (in radians). So the area of a sector formed by $\theta$ is



Figure 4-13

## Area of a Sector

The area $A$ of a sector of a circle of radius $r$ with central angle $\theta$ (in radians) is given by

$$
A=\frac{1}{2} r^{2} \theta
$$

## Example 10: Determining the Area of a Sector

A crop sprinkler rotates through an angle of $150^{\circ}$ and sprays water a distance of 90 ft . Find the amount of area watered. Round to the nearest whole unit.


## Solution:

To use the formula $A=\frac{1}{2} r^{2} \theta$, we need to convert $\theta$ to radians.
$150^{\circ} \cdot \frac{\pi}{180^{\circ}}=\frac{5 \pi}{6}$
$A=\frac{1}{2} r^{2} \theta=\frac{1}{2}(90 \mathrm{ft})^{2}\left(\frac{5 \pi}{6}\right)=3375 \pi \mathrm{ft}^{2} \approx 10,603 \mathrm{ft}^{2}$

## Apply the Skills

10. A sprinkler rotates through an angle of $120^{\circ}$ and sprays water a distance of 30 ft . Find the amount of area watered. Round to the nearest whole unit.

## Practice 4-1

## Practice Exercises

## Prerequisite Review

R.1. Solve for the specified variable.
$p=h z$ for $z$
R.2. If a plane travels 280 mph for 3.5 hr , find the distance traveled.

For Exercises R.3-R.5, convert the unit of time.
R.3. $210 \mathrm{~min}=$ $\qquad$ hr
R.4. $120 \mathrm{sec}=$ $\qquad$ min
R.5. $7200 \mathrm{sec}=$ $\qquad$ hr
R.6. Find the circumference of a circle with a radius of $2 \frac{1}{2} \mathrm{~m}$.
a. Give the exact answer in terms of $\pi$.
b. Approximate the answer by using 3.14 for $\pi$. Round to 1 decimal place.
R.7. Determine the area of a circle with a diameter of 40 ft . Use 3.14 for $\pi$. Round to the nearest whole unit.

## Concept Connections

1. The measure of an angle is (positive/ negative) $\qquad$ if its rotation from the initial side to the terminal side is clockwise. If the rotation is counterclockwise, then the measure is $\qquad$ .
2. An angle with its vertex at the origin of an $x y$-coordinate plane and with initial side on the positive $x$-axis is in $\qquad$ position.
3. Two common units used to measure angles are $\qquad$ and $\qquad$ .
4. One degree is what fractional amount of a full rotation?
5. An angle that measures $360^{\circ}$ has a measure of $\qquad$ radians.
6. Two angles are called $\qquad$ if the sum of their measures is $90^{\circ}$. Two angles are called $\qquad$ if the sum of their measures is $180^{\circ}$.
7. An angle with measure $\qquad$ or $\qquad$ radians is a right angle. A straight angle has a measure of $\qquad$ or $\qquad$ radians.
8. $A(n)$ $\qquad$ angle has a measure between $0^{\circ}$ and $90^{\circ}$, whereas $a(n)$ $\qquad$ angle has a measure between $90^{\circ}$ and $180^{\circ}$.
9. One degree is equally divided into 60 parts called $\qquad$ -
10. One minute is equally divided into 60 parts called $\qquad$ .
11. $1^{\circ}=$ $\qquad$ $=$ $\qquad$
12. An angle with its vertex at the center of a circle is called $a(n)$ $\qquad$ angle.
13. A central angle of a circle that intercepts an arc equal in length to the radius of the circle has a measure of $\qquad$ .
14. Which angle has a greater measure, $2^{\circ}$ or 2 radians?
15. To convert from radians to degrees, multiply by $\qquad$ . To convert from degrees to radians, multiply by $\qquad$ .
16. Two angles are $\qquad$ if they have the same initial side and same terminal side.
17. The measure of all angles coterminal to $\frac{7 \pi}{4}$ differ from $\frac{7 \pi}{4}$ by a multiple of $\qquad$ -
18. The measure of all angles coterminal to $112^{\circ}$ differ from $112^{\circ}$ by a multiple of $\qquad$ .
19. The length $s$ of an arc made by angle $\theta$ on a circle of radius $r$ is given by the formula $\qquad$ , where $\theta$ is measured in $\qquad$ .
20. To locate points on the surface of the Earth that are north or south of the equator, we measure the $\qquad$ of the point. To locate points that are east or west of the prime meridian, we measure the of the point.
21. The relationship $v=\frac{\text { arc length }}{\text { time }}$ represents the $\qquad$ speed of a point traveling in a circular path.
22. The symbol $\omega$ is typically used to denote
$\qquad$ speed and represents the number of radians per unit time that an object rotates.
23. A wedge of a circle, similar in shape to a slice of pie, is called a(n) $\qquad$ of the circle.
24. The area $A$ of a sector of a circle of radius $r$ with central angle $\theta$ is given by the formula
$\qquad$ , where $\theta$ is measured in $\qquad$ —.

## Learn: Find Degree Measure

For Exercises 25-26, sketch the angles in standard position.
25.a. $60^{\circ}$
b. $225^{\circ}$
c. $-210^{\circ}$
d. $-86^{\circ}$
26.a. $30^{\circ}$
b. $120^{\circ}$
c. $-135^{\circ}$
d. $-73^{\circ}$

For Exercises 27-30, convert the given angle to decimal degrees. Round to 4 decimal places. (See Example 1, p. 449)
27. $17^{\circ} 34^{\prime}$
28. $215^{\circ} 47^{\prime}$
29. $54^{\circ} 36^{\prime} 55^{\prime \prime}$
30. $23^{\circ} 42^{\prime} 48^{\prime \prime}$

For Exercises 31-34, convert the given angle to DMS (degree-minute-second) form. Round to the nearest second if necessary.
(See Example 2, p. 449)
31. $46.418^{\circ}$
32. $82.074^{\circ}$
33. $-84.64^{\circ}$
34. $-61.46^{\circ}$

## Learn: Find Radian Measure

35. Use the conversion factor $\pi=180^{\circ}$ along with the symmetry of the circle to complete the degree or radian measure of the missing angles.

36. Insert the appropriate symbol <, >, or $=$ in the blank.
a. $\frac{5 \pi}{6} \square 120^{\circ}$
b. $-\frac{4 \pi}{3} \square-270^{\circ}$

For Exercises 37-40, convert from degrees to radians. Give the answers in exact form in terms of $\pi$. (See Example 3, p. 452)
37. $75^{\circ}$
38. $240^{\circ}$
39. $-210^{\circ}$
40. $-195^{\circ}$

For Exercises 41-44, convert from degrees to radians. Round to $\mathbf{4}$ decimal places.
41. $-64.6^{\circ}$
42. $-312.4^{\circ}$
43. $12^{\circ} 6^{\prime} 36^{\prime \prime}$
44. $108^{\circ} 42^{\prime} 9^{\prime \prime}$

For Exercises 45-56, convert from radians to decimal degrees. Round to 1 decimal place if necessary. (See Example 4, p. 453)
45. $\frac{\pi}{4}$
46. $\frac{11 \pi}{6}$
47. $-\frac{5 \pi}{3}$
48. $-\frac{7 \pi}{6}$
49. $\frac{5 \pi}{18}$
50. $\frac{7 \pi}{9}$
51. $-\frac{2 \pi}{5}$
52. $-\frac{3 \pi}{8}$
53. 2.7
54. 5.3
55. $\frac{9 \pi}{2}$
56. $7 \pi$

## Learn: Determine Coterminal Angles

For Exercises 57-64, find a positive angle and a negative angle that is coterminal to the given angle. (See Examples 5-6, pp. 454-455)
57. $57^{\circ}$
58. $313^{\circ}$
59. $-105^{\circ}$
60. $-12^{\circ}$
61. $\frac{5 \pi}{6}$
62. $\frac{3 \pi}{4}$
63. $-\frac{3 \pi}{2}$
64. $-\pi$

For Exercises 65-70, find an angle between $0^{\circ}$ and $360^{\circ}$ or between 0 and $2 \pi$ that is coterminal to the given angle.
(See Examples 5-6, pp. 454-455)
65. $1521^{\circ}$
66. $-603^{\circ}$
67. $\frac{17 \pi}{4}$
68. $\frac{11 \pi}{3}$
69. $-\frac{7 \pi}{18}$
70. $-\frac{5 \pi}{9}$

## Learn: Compute Arc Length of a Sector of a Circle

For Exercises 71-74, find the exact length of the arc intercepted by a central angle $\theta$ on a circle of radius $r$. Then round to the nearest tenth of a unit. (See Example 7, p. 456)
71. $\theta=\frac{\pi}{3}, r=12 \mathrm{~cm}$
72. $\theta=\frac{5 \pi}{6}, r=4 m$
73. $\theta=135^{\circ}, r=10 \mathrm{in}$.
74. $\theta=315^{\circ}, r=2 \mathrm{yd}$
75. A 6 -ft pendulum swings through an angle of $40^{\circ} 36^{\prime}$. What is the length of the arc that the tip of the pendulum travels? Round to the nearest hundredth of a foot.
76. A gear with a $1.2-\mathrm{cm}$ radius moves through an angle of $220^{\circ} 15^{\prime}$. What distance does a point on the edge of the gear move? Round to the nearest tenth of a centimeter.
77. Which location would best fit the coordinates $12^{\circ} \mathrm{S}, 77^{\circ} \mathrm{W}$ ?
a. Paris, France
b. Lima, Peru
c. Miami, Florida
d. Moscow, Russia
78. a. What is the geographical relationship between two points that have the same latitude?
b. What is the geographical relationship between two points that have the same longitude?

For Exercises 79-82, assume that the Earth is approximately spherical with radius 3960 mi . Approximate the distances to the nearest mile. (See Example 8, p. 457)
79. Barrow, Alaska ( $71.3^{\circ} \mathrm{N}, 156.8^{\circ} \mathrm{W}$ ), and Kailua, Hawaii ( $19.7^{\circ} \mathrm{N}, 156.1^{\circ} \mathrm{W}$ ), have approximately the same longitude, which means that they are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.
80. Rochester, New York ( $43.2^{\circ} \mathrm{N}, 77.6^{\circ} \mathrm{W}$ ), and Richmond, Virginia ( $37.5^{\circ} \mathrm{N}, 77.5^{\circ} \mathrm{W}$ ), have approximately the same longitude, which means that they are roughly due northsouth of each other. Use the difference in latitude to approximate the distance between the cities.
81. Raleigh, North Carolina $\left(35.8^{\circ} \mathrm{N}, 78.6^{\circ} \mathrm{W}\right)$, is located north of the equator, and Quito, Ecuador ( $0.3^{\circ} \mathrm{S}, 78.6^{\circ} \mathrm{W}$ ), is located south of the equator. The longitudes are the same, indicating that the cities are due north-south of each other. Use the difference in latitude to approximate the distance between the cities.
82. Trenton, New Jersey ( $40.2^{\circ} \mathrm{N}, 74.8^{\circ} \mathrm{W}$ ), is located north of the equator, and Ayacucho, Peru $\left(13.2^{\circ} \mathrm{S}, 74.2^{\circ} \mathrm{W}\right)$, is located south of the equator. The longitudes are nearly the same, indicating that the cities are roughly due north-south of each other. Use the difference in latitude to approximate the distance between the cities.
83. A pulley is 16 cm in diameter.
a. Find the distance the load will rise if the pulley is rotated $1350^{\circ}$. Find the exact distance in terms of $\pi$ and then round to the nearest centimeter.
b. Through how many degrees should the pulley rotate to lift the load 100 cm ? Round to the nearest degree.
84. A pulley is 1.2 ft . in diameter.
a. Find the distance the load will rise if the pulley is rotated $630^{\circ}$. Find the exact distance in terms of $\pi$ and then round to the nearest tenth of a foot.
b. Through how many degrees should the pulley rotate to lift the load 24 ft ? Round to the nearest degree.
85. A hoist is used to lift a pallet of bricks. The drum on the hoist is 15 in . in diameter. How many degrees should the drum be rotated to lift the pallet a distance of 6 ft ? Round to the nearest degree.

86. A winch on a sailboat is 8 in . in diameter and is used to pull in the "sheets" (ropes used to control the corners of a sail). To the nearest degree, how far should the winch be turned to pull in 2 ft of rope?

Before the widespread introduction of electronic devices to measure distances, surveyors used a subtense bar to measure a distance $x$ that is not directly measurable. A subtense bar is a bar of known length $h$ with marks or "targets" at either end. The surveyor measures the angle $\theta$ formed by the location of the surveyor's scope and the top and bottom of the bar (this is the angle subtended by the bar). Since the angle and height of the bar are known, right triangle trigonometry can be used to find the horizontal distance. Alternatively, if the distance from the surveyor to the bar is large, then the distance can be approximated by the radius $r$ of the arc $s$ intercepted by the bar. Use this information for Exercises 87-88.

87. A surveyor uses a subtense bar to find the distance across a river. If the angle of sight between the bottom and top marks on a $2-\mathrm{m}$ bar is $57^{\prime} 18^{\prime \prime}$, approximate the distance across the river between the surveyor and the bar. Round to the nearest meter.
88. A surveyor uses a subtense bar to find the distance across a canyon. If the angle of sight between the bottom and top marks on a $2-\mathrm{m}$ bar is $24^{\prime} 33^{\prime \prime}$, approximate the distance across the river between the surveyor and the bar. Round to the nearest meter.

## Learn: Compute Linear and Angular Speed

89. A circular paddle wheel of radius 3 ft is lowered into a flowing river. The current causes the wheel to rotate at a speed of 12 rpm . To 1 decimal place,
a. What is the angular speed? (See Example 9, p. 459)
b. Find the speed of the current in $\mathrm{ft} / \mathrm{min}$.
c. Find the speed of the current in mph.
90. An energy-efficient hard drive has a $2.5-\mathrm{in}$. diameter and spins at 4200 rpm .
a. What is the angular speed?
b. How fast in in./min does a point on the edge of the hard drive spin? Give the exact speed and the speed rounded to the nearest in./min.
91. A $7 \frac{1}{4}$ in.-diameter circular saw has 24 teeth and spins at 5800 rpm .
a. What is the angular speed?
b. What is the linear speed of one of the "teeth" on the outer edge of the blade? Round to the nearest inch per minute.
92. On a weed-cutting device, a thick nylon line rotates on a spindle at 3000 rpm .
a. Determine the angular speed.
b. Determine the linear speed (to the nearest inch per minute) of a point on the tip of the line if the line is 5 in .
93. A truck has $2.5-\mathrm{ft}$ tires (in diameter).
a. What distance will the truck travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of a foot.
b. How far will the truck travel with 10,000 rotations of the wheels? Give the exact distance and an approximation to the nearest foot.
c. If the wheels turn at 672 rpm , what is the angular speed?
d. If the wheels turn at 672 rpm , what is the linear speed in feet per minute? Give the exact distance and an approximation to the nearest whole unit.
e. If the wheels turn at 672 rpm , what is the linear speed in miles per hour? Round to the nearest mile per hour. (Hint: $1 \mathrm{mi}=5280 \mathrm{ft}$ and $1 \mathrm{hr}=60 \mathrm{~min}$.)
94. A bicycle has $25-\mathrm{in}$. wheels (in diameter).
a. What distance will the bicycle travel with one rotation of the wheels? Give the exact distance and an approximation to the nearest tenth of an inch.
b. How far will the bicycle travel with 200 rotations of the wheels? Give the exact distance and approximations to the nearest inch and nearest foot.
c. If the wheels turn at 80 rpm , what is the angular speed?
d. If the wheels turn at 80 rpm , what is the linear speed in inches per minute? Give the exact speed and an approximation to the nearest inch per minute.
e. If the wheels turn at 80 rpm , what is the linear speed in miles per hour? Round to the nearest mile per hour. (Hint: $1 \mathrm{ft}=12 \mathrm{in}$., $1 \mathrm{mi}=5280 \mathrm{ft}$, and $1 \mathrm{hr}=60 \mathrm{~min}$.)

## Learn: Compute the Area of a Sector of a Circle

For Exercises 95-98, find the exact area of the sector. Then round the result to the nearest tenth of a unit. (See Example 10, p. 460)
95.

96.

97.

98.

99. A slice of a circular pizza 12 in . in diameter is cut into a wedge with a $45^{\circ}$ angle. Find the area and round to the nearest tenth of a square inch.
100. A circular cheesecake 9 in . in diameter is cut into a slice with a $20^{\circ}$ angle. Find the area and round to the nearest tenth of a square inch.
101. The back wiper blade on an SUV extends 3 in. from the pivot point to a distance of 17 in . from the pivot point. If the blade rotates through an angle of $175^{\circ}$, how much area does it cover? Round to the nearest square inch.
102. A robotic arm rotates through an angle of $160^{\circ}$. It sprays paint between a distance of 0.5 ft and 3 ft from the pivot point. Determine the amount of area that the arm makes. Round to the nearest square foot.

## Mixed Exercises

For Exercises 103-108, find the (a) complement and (b) supplement of the given angle.
103. $16.21^{\circ}$ 104. $49.87^{\circ}$
105. $18^{\circ} 13^{\prime} 37^{\prime \prime}$
107. $9^{\circ} 42^{\prime \prime} 7^{\prime \prime}$ 106. $22^{\circ} 9^{\prime} 54^{\prime \prime}$
108. $82^{\circ} 15^{\prime} 3^{\prime \prime}$
109. The second hand of a clock moves from 12:10 to 12:30.
a. How many degrees does it move during this time?
b. How many radians does it move during this time?
c. If the second hand is 10 in . in length, determine the exact distance that the tip of the second hand travels during this time.
d. Determine the exact angular speed of the second hand in radians per second.
e. What is the exact linear speed (in inches per second) of the tip of the second hand?
f. What is the amount of area that the second hand sweeps out during this time? Give the exact area in terms of $\pi$ and then approximate to the nearest square inch.
110. The minute hand of a clock moves from 12:10 to 12:15.
a. How many degrees does it move during this time?
b. How many radians does it move during this time?
c. If the minute hand is 9 in . in length, determine the exact distance that the tip of the minute hand travels during this time.
d. Determine the exact angular speed of the minute hand in radians per minute.
e. What is the exact linear speed (in inches per minute) of the tip of the minute hand?
f. What is the amount of area that the minute hand sweeps out during this time? Give the exact area in terms of $\pi$ and then approximate to the nearest square inch.
111. The Earth's orbit around the Sun is elliptical (oval shaped). However, the elongation is small, and for our discussion here, we take the orbit to be circular with a radius of approximately $93,000,000 \mathrm{mi}$.
a. Find the linear speed (in mph ) of the Earth through its orbit around the Sun. Round to the nearest hundred miles per hour.
b. How far does the Earth travel in its orbit in one day? Round to the nearest thousand miles.
112. The Earth completes one full rotation around its axis (poles) each day.
a. Determine the angular speed (in radians per hour) of the Earth during its rotation around its axis.
b. The Earth is nearly spherical with a radius of approximately 3960 mi. Find the linear speed of a point on the surface of the Earth rounded to the nearest mile per hour.
113. Two gears are calibrated so that the smaller gear drives the larger gear. For each rotation of the smaller gear, how many degrees will the larger gear rotate?

114. Two gears are calibrated so that the larger gear drives the smaller gear. The larger gear has a 6 -in. radius, and the smaller gear has a $1.5-\mathrm{in}$. radius. For each rotation of the larger gear, by how many degrees will the smaller gear rotate?
115. A spinning-disc confocal microscope contains a rotating disk with multiple small holes arranged in a series of nested Archimedean spirals. An intense beam of light is projected through the holes, enabling biomedical researchers to obtain detailed video images of live cells. The spinning disk has a diameter of 55 mm and rotates at a rate of 1800 rpm . At the edge of the disk,

a. Find the angular speed.
b. Find the linear speed. Round to the nearest whole unit.

For Exercises 116-119, approximate the area of the shaded region to 1 decimal place. In the figure, $s$ represents arc length, and $r$ represents the radius of the circle.
116. $s=28 \mathrm{in}$.

117. $s=9 \mathrm{~cm}$

118. $s=18 \mathrm{~m}$

119. $s=4.5$ in.


## Write About It

120. Explain what is meant by 1 radian. Explain what is meant by $1^{\circ}$.
121. For an angle drawn in standard position, explain how to determine in which quadrant the terminal side lies.
122. As the fan rotates (see figure), which point $A$ or $B$ has a greater angular speed? Which point has a greater linear speed? Why?

123. If an angle of a sector is held constant, but the radius is doubled, how will the arc length of the sector and area of the sector be affected?
124. If an angle of a sector is doubled, but the radius is held constant, how will the arc length of the sector and the area of the sector be affected?

## Expanding Your Skills

125. When a person pedals a bicycle, the front sprocket moves a chain that drives the back wheel and propels the bicycle forward. For each rotation of the front sprocket, the chain moves a distance equal to the circumference of the front sprocket. The back sprocket is smaller, so it will simultaneously move through a greater rotation. Furthermore, since the back sprocket is rigidly connected to the back wheel, each rotation of the back sprocket generates a rotation of the wheel.

Suppose that the front sprocket of a bicycle has a 4 -in. radius and the back sprocket has a 2-in. radius.

a. How much chain will move with one rotation of the pedals (front sprocket)?
b. How many times will the back sprocket turn with one rotation of the pedals?
c. How many times will the wheels turn with one rotation of the pedals?
d. If the wheels are 27 in . in diameter, how far will the bicycle travel with one rotation of the pedals?
e. If the bicyclist pedals 80 rpm , what is the linear speed (in ft/min) of the bicycle?
f. If the bicyclist pedals 80 rpm , what is the linear speed (in mph) of the bicycle? (Hint: $1 \mathrm{mi}=5280 \mathrm{ft}$, and $1 \mathrm{hr}=60 \mathrm{~min}$ )
126. In the third century B.C., the Greek astronomer Eratosthenes approximated the Earth's circumference. On the summer solstice at noon in Alexandria, Egypt, Eratosthenes measured the angle $a$ of the Sun relative to a line perpendicular to the ground. At the same time in Syene (now Aswan), located on the Tropic of Cancer, the Sun was directly overhead.

a. If $a=\frac{1}{50}$ of a circle, find the measure of $\alpha$ in degrees. (In Eratosthenes' time, the degree measure had not yet been defined.)
b. If the distance between Alexandria and Syene is 5000 stadia, find the circumference of the Earth measured in stadia.
c. If 10 stadia $\approx 1 \mathrm{mi}$, find Eratosthenes' approximation of the circumference of the Earth in miles (the modern-day approximation at the equator is 24,900 mi).
127. The space shuttle program involved 135 crewed space flights in 30 yr . In addition to supplying and transporting astronauts to the International Space Station, space shuttle missions serviced the Hubble Space Telescope and deployed satellites. For a particular mission, a space shuttle orbited the Earth in 1.5 hr at an altitude of 200 mi .
a. Determine the angular speed (in radians per hour) of the shuttle.
b. Determine the linear speed of the shuttle in miles per hour. Assume that the Earth's radius is 3960 mi . Round to the nearest hundred miles per hour.
128. What is the first time (to the nearest second) after 12:00 midnight for which the minute hand and hour hand of a clock make a $120^{\circ}$ angle?

## Technology Connections

For Exercises 129-130, use the functions in the ANGLE menu on your calculator to
a. Convert from decimal degrees to DMS (degree-minute-second) form and
b. Convert from DMS form to decimal degrees.

On some calculators, the " symbol is accessed by hitting followed by $\pm$.
129. a. $-216.479^{\circ}$
b. $42^{\circ} 13^{\prime} 5.9^{\prime \prime}$
130. a. $-14.908^{\circ}$
b. $71^{\circ} 19^{\prime} 4.7^{\prime \prime}$

For Exercises 131-132, use a calculator to convert from degrees to radians.
131. a. $147^{\circ} 26^{\prime} 9^{\prime \prime}$
b. $-228.459^{\circ}$
132. a. $36^{\circ} 4^{\prime} 47^{\prime \prime}$
b. $-25.716^{\circ}$

For Exercises 133-134, use a calculator to convert from radians to degrees.
133. a. $\frac{4 \pi}{9}$
b. -5.718
134. a. $\frac{11 \pi}{18}$
b. -1.356

## Point of Interest

Why 60?
One hour may be subdivided into 60 minutes, and each minute may be further divided into 60 seconds. The 360 degrees in a circle may be similarly subdivided by dividing each degree into 60 minutes and each minute into 60 seconds. (These smaller measurements of angle are sometimes referred to as arcminutes and arcseconds to avoid confusion with time measurements.) Both the measurements for time and for angles are based on the number 60. A number system based on 60 is called sexagesimal.

The base 60 system used for both time and degree measure is usually attributed to the ancient Sumerians and Babylonians or possibly Egyptians, although it is difficult to be sure of a specific origin. One advantage of a base 60 system is that there are many divisors of $60: 1,2,3,4,5,6,10$, $12,15,20,30$, and 60.
135. Perhaps old habits die hard, but why not use 100 minutes in an hour? Why not consider 100 degrees for a circle? Interestingly, in 1793, during the French Revolution, a new system of time was declared called French Revolutionary Time. Timepieces during that period had a 10 -hour day with 100 minutes in an hour and 100 seconds in a minute. Even today, there are still manufacturers that make watches and clocks that keep decimal time. Write a short report on French Revolutionary Time and the timepieces created to keep time in this fashion.
136. When a circle is divided into 6 parts, what is the measure (in degrees) of each central angle? If we inscribe a regular hexagon in a circle of radius $r$, what is the length of each side of the hexagon?

137. For a circle of radius $r$, the length $s$ of an arc intercepted by a central angle $\theta$ (in radlans) is given by $s=r \theta$. What would be an equivalent formula if $\theta$ is measured in degrees?

## Trigonometric Functions Defined on the Unit Circle

## What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate Trigonometric Functions Using the Unit Circle
- Identify the Domains of the Trigonometric Functions
- Use Fundamental Trigonometric Identities
- Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions
- Approximate Trigonometric Functions on a Calculator


## Learn: Evaluate Trigonometric Functions Using the Unit Circle

The functions we have studied so far have been used to model phenomena such as exponential growth and decay, projectile motion, and profit and cost, to name a few. However, none of the functions in our repertoire represent cyclical behavior such as the orbits of the Moon and planets, the variation in air pressure that produces sound, the back-and-forth oscillations of a stretched spring, and so on. To model these behaviors, we introduce six new functions called trigonometric functions.

Historically, the study of trigonometry arose from the need to study relationships among the angles and sides of a triangle. An alternative approach is to define trigonometric functions as circular functions. To begin, we define the unit circle as the circle of radius 1 unit, centered at the origin. The unit circle consists of all points $(x, y)$ that satisfy the equation $x^{2}+y^{2}=1$. For example, $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is a point on the unit circle because $\left(-\frac{1}{2}\right)^{2}+\left(\frac{\sqrt{3}}{2}\right)^{2}=1$ (Figure 4-14).


Figure 4-14

## Insights

The circumference of the unit circle is $2 \pi r=2 \pi(1)=2 \pi$. So, the value $t=\pi$ represents one-half of a revolution and falls on the point ( $-1,0$ ). Likewise the value $t=\frac{\pi}{2}$ represents one-quarter of a revolution and falls on the point $(0,1)$.

Now suppose we wrap the real number line around the unit circle by placing the point $t=0$ from the number line on the point $(1,0)$ on the circle. Positive real numbers from the number line ( $t>0$ ) wrap onto the unit circle in the counterclockwise direction. Negative numbers from the number line ( $t<0$ ) wrap onto the unit circle in the clockwise direction (Figure 4-15). For a value of $t$ greater than $2 \pi$ (or less than $-2 \pi$ ) more than one revolution around the unit circle is required.


Figure 4-15
The purpose of wrapping the real number line around the unit circle is to associate each real number $t$ with a unique point $(x, y)$ on the unit circle.

For a real number $t$ corresponding to a point $P(x, y)$ on the unit circle, we can use the coordinates of $P$ to define six trigonometric functions of $t$ (Table 4-1). Notice that rather than using letters such as $f, g, h$, and so on, trigonometric functions are given the word names sine, cosine, tangent, cosecant, secant, and cotangent. These functions are abbreviated as "sin," "cos," "tan," "csc," "sec," and "cot," respectively. The value $t$ is the input value or argument of each function.

Let $P(x, y)$ be the point associated with a real number $t$ measured along the circumference of the unit circle from the point ( 1,0 ).

TABLE 4-1 Unit Circle Definitions of the Trigonometric Functions

| Function Name | Definition |
| :---: | :---: |
| sine | $\sin t=y$ |
| cosine | $\cos t=x$ |
| tangent | $\tan t=\frac{y}{x}(x \neq 0)$ |
| cosecant | $\csc t=\frac{1}{y}(y \neq 0)$ |
| secant | $\sec t=\frac{1}{x}(x \neq 0)$ |
| cotangent | $\cot t=\frac{x}{y}(y \neq 0)$ |



## Note:

- If $P$ is on the $y$-axis [either $(0,1)$ or $(0,-1)]$, then $x=0$ and the tangent and secant functions are undefined.
- If $P$ is on the $x$-axis [either $(1,0)$ or $(-1,0)$ ], then $y=0$ and the cotangent and cosecant functions are undefined.


## Example 1: Evaluating Trigonometric Functions

Suppose that the real number $t$ corresponds to the point $P\left(-\frac{2}{3},-\frac{\sqrt{5}}{3}\right)$ on the unit circle. Evaluate the six trigonometric functions of $t$.

## Solution:

$\sin t=y=-\frac{\sqrt{5}}{3} \quad \cos t=x=-\frac{2}{3}$
$\tan t=\frac{y}{x}=\frac{-\frac{\sqrt{5}}{3}}{-\frac{2}{3}}=-\frac{\sqrt{5}}{3} \cdot\left(-\frac{3}{2}\right)=\frac{\sqrt{5}}{2}$
$\csc t=\frac{1}{y}=\frac{1}{-\frac{\sqrt{5}}{3}}=-\frac{3}{\sqrt{5}}=-\frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}=-\frac{3 \sqrt{5}}{5}$
$\sec t=\frac{1}{x}=\frac{1}{-\frac{2}{3}}=-\frac{3}{2}$
$\cot t=\frac{x}{y}=\frac{-\frac{2}{3}}{-\frac{\sqrt{5}}{3}}=-\frac{2}{3} \cdot\left(-\frac{3}{\sqrt{5}}\right)=\frac{2}{\sqrt{5}}=\frac{2}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}}=\frac{2 \sqrt{5}}{5}$


## Insights

In the figure in Example 1, $t$ was arbitrarily taken to be positive as noted by the red arc wrapping counterclockwise. However, we can just as easily use a negative value of $t$, terminating at the same point in Quadrant III.

## Apply the Skills

1. Suppose that the real number $t$ corresponds to the point $P\left(\frac{\sqrt{5}}{5}, \frac{2 \sqrt{5}}{5}\right)$ on the unit circle. Evaluate the six trigonometric functions of $t$.

Let $P(x, y)$ be the point on the unit circle associated with a real number $t \geq 0$. Let $\theta \geq 0$ be the central angle in standard position measured in radians with terminal side through $P$ and arc length $t$ (Figure 4-16). Because the radius of the unit circle is 1 , the arc length formula $s=r \theta$ becomes $t=1 \cdot \theta$ or simply $t=\theta$.


Figure 4-16

A similar argument can be made for $t<0$ and $\theta<0$. The arc length formula is $s=r(-\theta)$, or equivalently $-t=r(-\theta)$, which also implies that $t=\theta$ (Figure 4-17).


Figure 4-17

## Check Your Work

If $\theta$ is negative, then the opposite of $\theta$ is used in the arc length formula to ensure that the length of the arc is positive.

From this discussion, we have the following important result. The real number $t$ taken along the circumference of the unit circle gives the radian measure of the corresponding central angle. That is, $\theta=t$ radians. Furthermore, this one-to-one correspondence between the real number $t$ and the radian measure of the central angle $\theta$ means that the trigonometric functions can also be defined as functions of $\theta$.

Trigonometric Functions of Real Numbers and Angles

If $\theta=t$ radians, then

$$
\begin{aligned}
& \sin t=\sin \theta \\
& \csc t=\csc \theta
\end{aligned}
$$

$$
\cos t=\cos \theta
$$

$$
\tan t=\tan \theta
$$

$$
\sec t=\sec \theta
$$

$$
\cot t=\cot \theta
$$

We now want to determine the values of the trigonometric functions for several "special" values of $t$ corresponding to central angles $\theta$ that are integer multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. Consider the point $P(x, y)$ on the unit circle corresponding to $t=\frac{\pi}{4}$ (Figure 4-18). Point $P$ lies on the line $y=x$, and the coordinates of $P$ can be written as $(x, x)$.


Figure 4-18

Substituting the coordinates of $(x, x)$ into the equation $x^{2}+y^{2}=1$, we have

$$
\begin{aligned}
x^{2}+x^{2} & =1 \\
2 x^{2} & =1 \\
x^{2} & =\frac{1}{2} \quad \quad \quad \begin{array}{l}
\text { Choose } x \text { positive for } \\
\text { a first quadrant point. }
\end{array} \\
x & =\sqrt{\frac{1}{2}} \\
x & =\frac{1}{\sqrt{2}}
\end{aligned}
$$

Rationalizing the denominator, we have $x=\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2}$.
Since $y=x$, point $P$ has coordinates

$$
\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text { or }\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) .
$$

From the symmetry of the circle, the points corresponding to $t=\frac{3 \pi}{4}, t=\frac{5 \pi}{4}$, and $t=\frac{7 \pi}{4}$ have coordinates $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right),\left(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$, and $\left(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right)$, respectively (Figure 4-19).


Figure 4-19
Now consider the point $Q(x, y)$ on the unit circle corresponding to $t=\frac{\pi}{6}$
(Figure 4-20). Dropping a line segment from $Q$ perpendicular to the $x$-axis at point $A$, we construct a right triangle ( $\triangle O A Q$ ). The acute angles in $\triangle O A Q$ are $30^{\circ}$ and $60^{\circ}$, and the hypotenuse of the triangle is 1 unit.


Figure 4-20

Placing two such triangles adjacent to one another on opposite sides of the $x$-axis, we have an equilateral triangle ( $\triangle O B Q$ ) with sides of 1 unit and angles of $60^{\circ}$ (Figure 4-21). From $\triangle O B Q$ we have $2 y=1$. So, $y=\frac{1}{2}$.


Figure 4-21
Since $Q(x, y)$ is a point on the unit circle, we can substitute $y=\frac{1}{2}$ into the equation $x^{2}+y^{2}=1$ and solve for $x$.

$$
\begin{aligned}
x^{2}+y^{2} & =1 \\
x^{2}+\left(\frac{1}{2}\right)^{2} & =1 \\
x^{2}+\frac{1}{4} & =1 \\
x^{2} & =\frac{3}{4} \\
x & =\sqrt{\frac{3}{4}}=\frac{\sqrt{3}}{2}
\end{aligned}
$$

So, $Q$ has coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Using symmetry, we can also find coordinates of the points on the unit circle corresponding to $t=\frac{5 \pi}{6}, t=\frac{7 \pi}{6}$, and $t=\frac{11 \pi}{6}$ (Figure 4-22).


Figure 4-22
Using similar reasoning, we can show that the point $R(x, y)$ corresponding to $t=\frac{\pi}{3}$ has coordinates $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$.

## Insights

The value $t=\frac{\pi}{3}$ corresponds to the point $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ on the unit circle. $\triangle O A R$ is congruent to the triangle shown in Figure 4-20, but is oriented with the $60^{\circ}$ angle in standard position rather than the $30^{\circ}$ angle in standard position. As a result, notice that the $x$ - and $y$-coordinates of points $Q$ and $R$ are reversed.


We summarize our findings in Figure 4-23 for selected values of $t$ and the corresponding central angle $\theta$.


Figure 4-23
Figure 4-23 looks daunting, but using the symmetry of the circle and recognizing several patterns can make the coordinates of these special points easy to determine. First we recommend memorizing the coordinates of the three special points in Quadrant I.

- Note that for $t=\frac{\pi}{4}$, the $x$ - and $y$-coordinates are both $\frac{\sqrt{2}}{2}$, or equivalently $\frac{1}{\sqrt{2}}$.
- For $t=\frac{\pi}{6}$ and $t=\frac{\pi}{3}$, the $x$ - and $y$-coordinates are reversed.
- The value $\frac{\sqrt{3}}{2} \approx 0.866$ is greater than the value $\frac{1}{2}=0.5$. For $t=\frac{\pi}{6}$, the $x$-coordinate is greater than the $y$-coordinate. Therefore, $x$ must be $\frac{\sqrt{3}}{2}$. For $t=\frac{\pi}{3}$, the $x$-coordinate is less than the $y$-coordinate. Therefore, $x$ must be $\frac{1}{2}$.
- For the special points in Quadrants II, III, and IV, use the values $\frac{1}{2}, \frac{\sqrt{3}}{2}$, and $\frac{\sqrt{2}}{2}$ with the appropriate signs attached.


## Example 2: Evaluating Trigonometric Functions

Evaluate the six trigonometric functions of the real number $t$.
a. $t=\frac{5 \pi}{3}$
b. $t=-\frac{5 \pi}{4}$

## Solution:

a. $t=\frac{5 \pi}{3}$ corresponds to the point $(x, y)=\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ on the unit circle.

$$
\begin{aligned}
& \sin \frac{5 \pi}{3}=y=-\frac{\sqrt{3}}{2} \\
& \cos \frac{5 \pi}{3}=x=\frac{1}{2} \\
& \tan \frac{5 \pi}{3}=\frac{y}{x}=\frac{-\frac{\sqrt{3}}{2}}{\frac{1}{2}}=-\frac{\sqrt{3}}{2} \cdot \frac{2}{1}=-\sqrt{3} \\
& \csc \frac{5 \pi}{3}=\frac{1}{y}=\frac{1}{-\frac{\sqrt{3}}{2}}=-\frac{2}{\sqrt{3}}=-\frac{2}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}=-\frac{2 \sqrt{3}}{3} \\
& \sec \frac{5 \pi}{3}=\frac{1}{x}=\frac{1}{\frac{1}{2}}=2 \\
& \cot \frac{5 \pi}{3}=\frac{x}{y}=\frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}}=\frac{1}{2} \cdot\left(-\frac{2}{\sqrt{3}}\right)=-\frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}=-\frac{\sqrt{3}}{3}
\end{aligned}
$$


b. Since the circumference of the unit circle is $2 \pi$, the values $t=-\frac{5 \pi}{4}$ and $t_{1}=-\frac{5 \pi}{4}+2 \pi=\frac{3 \pi}{4}$ have the same location in Quadrant II on the unit circle. Both correspond to the point $(x, y)=\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, or equivalently $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
$\sin \left(-\frac{5 \pi}{4}\right)=y=\frac{\sqrt{2}}{2}$


$$
\cos \left(-\frac{5 \pi}{4}\right)=x=-\frac{\sqrt{2}}{2}
$$

$$
\tan \left(-\frac{5 \pi}{4}\right)=\frac{y}{x}=\frac{\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}}=\frac{\sqrt{2}}{2} \cdot\left(-\frac{2}{\sqrt{2}}\right)=-1
$$

$$
\csc \left(-\frac{5 \pi}{4}\right)=\frac{1}{y}=\frac{1}{\frac{1}{\sqrt{2}}}=\sqrt{2}
$$

$$
\sec \left(-\frac{5 \pi}{4}\right)=\frac{1}{x}=\frac{1}{-\frac{1}{\sqrt{2}}}=-\sqrt{2}
$$

$\cot \left(-\frac{5 \pi}{4}\right)=\frac{x}{y}=\frac{\frac{-\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}}=-\frac{\sqrt{2}}{2} \cdot \frac{2}{\sqrt{2}}=-1$

## Apply the Skills

Evaluate the six trigonometric functions of the real number $t$.
a. $t=\frac{5 \pi}{6}$
b. $t=-\frac{3 \pi}{4}$

## Learn: Identify the Domains of the Trigonometric Functions

From Table 4-1, $\sin t=y$ and $\cos t=x$ have no restrictions on their domain.
However, $\tan t=\frac{y}{x}$ and $\sec t=\frac{1}{x}$ are undefined for all values of $t$ corresponding to the points $(0,1)$ and $(0,-1)$ on the unit circle. These are $t=\frac{\pi}{2}, t=\frac{3 \pi}{2}$, and all other odd multiples of $\frac{\pi}{2}$ (Figure 4-24).

- $\tan t$ and $\sec t$ are undefined for $\ldots-\frac{3 \pi}{2},-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3 \pi}{2}, \frac{5 \pi}{2}, \ldots, \frac{(2 n+1) \pi}{2}$, for all integers $n$.
- Likewise, $\cot t=\frac{x}{y}$ and $\csc t=\frac{1}{y}$ are undefined for all values of $t$ corresponding to the points $(1,0)$ and $(-1,0)$ on the unit circle. These are $t=0, t=\pi, t=2 \pi$, and all other multiples of $\pi$ (Figure 4-24).
- $\cot t$ and $\csc t$ are undefined for $\ldots-3 \pi,-2 \pi,-\pi, 0, \pi, 2 \pi, 3 \pi, \ldots, n \pi$ for all integers $n$.


Figure 4-24

## Insights

For integer values of $n$, the formula $\frac{(2 n+1) \pi}{2}$ generates odd multiples of $\frac{\pi}{2}$.
For example, if $n=6$ :

$$
\frac{[2(6)+1] \pi}{2}=\frac{13 \pi}{2}
$$

The formula $n \pi$ generates multiples of $\pi$.
For example, if $n=6$ :

$$
n x=(6) \pi=6 \pi
$$

## Domains of the Trigonometric Functions

| Function | Domain* | Notes |
| :--- | :--- | :--- |
| $f(t)=\sin t$ | All real numbers | No restrictions |
| $f(t)=\cos t$ | All real numbers | No restrictions |
| $f(t)=\tan t$ | $\left\{t \left\lvert\, t \neq \frac{(2 n+1 \pi)}{2}\right.\right.$ for all integers $\left.n\right\}$ | Exclude odd multiples of $\frac{\pi}{2}$. |
| $f(t)=\cot t$ | $\{t \mid t \neq n \pi$ for all integers $n\}$ | Exclude multiples of $\pi$. |
| $f(t)=\sec t$ | $\left\{t \left\lvert\, t \neq \frac{2 n+1 \pi}{2}\right.\right.$ for all integers $\left.n\right\}$ | Exclude odd multiples of $\frac{\pi}{2}$. |
| $f(t)=\csc t$ | $\{t \mid t \neq n \pi$ for all integers $n\}$ | Exclude multiples of $\pi$. |
| * The range of each trigonometric function will be discussed in Lessons 4.5 and 4.6 when we cover the |  |  |
| graphs of the functions. |  |  |

## Example 3: Evaluate the Trigonometric Functions for Given Values of $t$

Evaluate the six trigonometric functions of the real number $t$.
a. $t=\pi$
b. $t=\frac{5 \pi}{2}$

## Solution:

a. The value $t=\pi$ corresponds to $(-1,0)$ on the unit circle.


$$
\begin{array}{ll}
\sin \pi=y=0 & \csc \pi=\frac{1}{y} \text { is undefined because } \frac{1}{0} \text { is undefined. } \\
\cos \pi=x=-1 & \sec \pi=\frac{1}{x}=\frac{1}{-1}=-1 \\
\tan \pi=\frac{y}{x}=\frac{0}{-1}=0 & \cot \pi=\frac{x}{y} \text { is undefined because } \frac{-1}{0} \text { is undefined. }
\end{array}
$$

b. The circumference of the unit circle is $2 \pi$. Therefore, since $t=\frac{5 \pi}{2}=2 \pi+\frac{\pi}{2}$, the value $t=\frac{5 \pi}{2}$ corresponds to the point $(0,1)$ on the unit circle.


$$
\sin \frac{5 \pi}{2}=y=1
$$

$$
\csc \frac{5 \pi}{2}=\frac{1}{y}=\frac{1}{1}=1
$$

$\cos \frac{5 \pi}{2}=x=0$
$\sec \frac{5 \pi}{2}=\frac{1}{x}$ is undefined because $\frac{1}{0}$ is undefined. $\tan \frac{5 \pi}{2}=\frac{x}{y}$ is undefined because $\frac{1}{0}$ is undefined. $\cot \frac{5 \pi}{2}=\frac{x}{y}=\frac{0}{1}=0$

## Apply the Skills

3. Evaluate the six trigonometric functions of the real number $t$.
a. $t=-2 \pi$
b. $\frac{3 \pi}{2}$

## Insights

Determining the location of a real number $t$ wrapped onto the unit circle is equivalent to finding the point where the terminal side of the central angle $\theta$ intercepts the unit circle. This is true for $\theta$ measured in radians and drawn in standard position. In Example 3(b), the angle $\theta=\frac{5 \pi}{2}$ is coterminal to $\frac{\pi}{2}$, which intercepts the unit circle at $(0,1)$.

## Learn: Use Fundamental Trigonometric Identities

You may have already noticed several relationships among the six trigonometric functions that follow directly from their definitions. For example, for a real number $t$ corresponding to a point $(x, y)$ on the unit circle, $\sin t=y$ and $\csc t=\frac{1}{y}$. So, the sine and cosecant functions are reciprocals for all $t$ in their common domain. Table 4-2 summarizes the reciprocal relationships among the trigonometric functions along with two important quotient properties. These relationships should be committed to memory.

## Check Your Work

It is important to note that the argument must be included when denoting the value of a trigonometric function. For example, we write "tant $=\frac{\sin t \text { " }}{\cos t}$ not "tan $=\frac{\sin }{\cos "}$.

TABLE 4-2 Reciprocal and Quotient Identities

| Trigonometric Functions | Relationship |
| :---: | :--- |
| $\csc t=\frac{1}{\sin t}$ or $\sin t=\frac{1}{\csc t}$ | $\sin t$ and $\csc t$ are reciprocals. |
| $\sec t=\frac{1}{\cos t}$ or $\cos t=\frac{1}{\sec t}$ | $\cos t$ and $\sec t$ are reciprocals. |
| $\cot t=\frac{1}{\tan t}$ or $\tan t=\frac{1}{\cot t}$ | $\tan t$ and $\cot t$ are reciprocals. |
| $\tan t=\frac{\sin t}{\cos t}$ | $\tan t$ is the ratio of $\sin t$ and $\cos t$. |
| $\cot t=\frac{\cos t}{\sin t}$ | $\cot t$ is the ratio of $\cos t$ and $\sin t$. |

## Example 4: Using the Reciprocal and Quotient Identities

Given that $\sin t=\frac{5}{8}$ and $\cos t=\frac{\sqrt{39}}{8}$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of $t$.

## Solution:

Given the values of $\sin t$ and $\cos t$, we can use the quotient identities to find $\tan t$ and $\cot t$.
$\tan t=\frac{\sin t}{\cos t}=\frac{\frac{5}{8}}{\frac{\sqrt{39}}{8}}=\frac{5}{8} \cdot \frac{8}{\sqrt{39}}=\frac{5}{\sqrt{39}}=\frac{5}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}}=\frac{5 \sqrt{39}}{39}$
$\cot t=\frac{\cos t}{\sin t}=\frac{\frac{\sqrt{39}}{8}}{\frac{5}{8}}=\frac{\sqrt{39}}{8} \cdot \frac{8}{5}=\frac{\sqrt{39}}{5}$
The remaining two functions can be found by using the reciprocal identities.
$\csc t=\frac{1}{\sin t}=\frac{1}{\frac{5}{8}}=\frac{8}{5}$
$\sec t=\frac{1}{\cos t}=\frac{1}{\frac{\sqrt{39}}{8}}=\frac{8}{\sqrt{39}}=\frac{8}{\sqrt{39}} \cdot \frac{\sqrt{39}}{\sqrt{39}}=\frac{8 \sqrt{39}}{39}$

## Apply the Skills

4. Given $\sin t=\frac{3}{7}$ and $\cos t=\frac{2 \sqrt{10}}{7}$ use the reciprocal and quotient identities to find the values of the other trigonometric functions of $t$.

Consider a real number $t$ corresponding to the point $P(x, y)$ on the unit circle. Since $\cos t=x$ and $\sin t=y$, point $P$ can be labeled $P(\cos t, \sin t)$. See Figure 4-25. Furthermore, since $P$ is on the unit circle, it satisfies the equation $x^{2}+y^{2}=1$. So,
$(\cos t)^{2}+(\sin t)^{2}=1$ or $(\sin t)^{2}+(\cos t)^{2}=1$.


Figure 4-25
It is universally preferred to write the exponent associated with a trigonometric function between the name of the function and its argument. For example, $(\sin t)^{2}$ is most often written as $\sin ^{2} t$. So, the relationship $(\sin t)^{2}+(\cos t)^{2}=1$ is written as $\sin ^{2} t+\cos ^{2} t=1$. This relationship is called a "Pythagorean identity" because $\sin ^{2} t+\cos ^{2} t=1$ is a statement of the Pythagorean theorem for $\triangle O A P$.

Table 4-3 summarizes three Pythagorean identities. The second and third relationships can be derived by dividing both sides of the equation $\sin ^{2} t+\cos ^{2} t=1$ by $\cos ^{2} t$ and $\sin ^{2} t$, respectively (see Exercises 47 and 48).

TABLE 4-3 Pythagorean Identities

| Trigonometric Functions | Pythagorean Identity |
| :---: | :---: |
| $\sin t$ and $\cos t$ | $\sin ^{2} t+\cos ^{2} t=1$ |
| $\tan t$ and $\sec t$ | $\tan ^{2} t+1=\sec ^{2} t$ |
| $\cot t$ and $\csc t$ | $1+\cot ^{2} t=\csc ^{2} t$ |

## Example 5: Using the Pythagorean Identities

Given that $\tan t=\frac{12}{5}$ for $\pi<t<\frac{3 \pi}{2}$, use an appropriate identity to find the value of sect.

## Solution:

$$
\tan ^{2} t+1=\sec ^{2} t
$$

The real number $t$ corresponds to a point $P(x, y)$ on the unit circle in Quadrant III. Since the value of $\tan t$ is known, we can use the relationship $\tan ^{2} t+1=\sec ^{2} t$ to find the value of sect.

$$
\sec t= \pm \sqrt{\tan ^{2} t+1}
$$

$$
\sec t=-\sqrt{\tan ^{2} t+1}
$$

Since $t$ is between $\pi$ and $\frac{3 \pi}{2}$, it corresponds to a point $P(x, y)$ on the unit circle in Quadrant III. Since $P$ is in Quadrant III, the values of both $x$ and $y$ are negative. So, $\sec t=\frac{1}{x}$ is negative.


$$
\begin{aligned}
\sec t & =-\sqrt{\left(\frac{12}{5}\right)^{2}+1} \\
& =-\sqrt{\frac{144}{25}+\frac{25}{25}}=-\sqrt{\frac{169}{25}}=-\frac{13}{5}
\end{aligned}
$$

## Apply the Skills

5. Given that $\csc t=\frac{5}{4}$ for $\frac{\pi}{2}<t<\pi$, use an appropriate identity to find the value of $\cot t$.

Sometimes it is beneficial to express one trigonometric function in terms of another. This is demonstrated in Example 6.

## Example 6: Expressing a Trigonometric Function in Terms of Another Trigonometric Function

For a given real number $t$, express $\sin t$ in terms of cost.

## Solution:

Let $P(x, y)=(\cos t, \sin t)$ be the point on the unit circle determined by $t$.


$$
x^{2}+y^{2}=1
$$

$\cos ^{2} t+\sin ^{2} t=1 \quad$ Substitute $x=\cos t$ and $y=\sin t$.
$\sin ^{2} t=1-\cos ^{2} t$
$\sin t= \pm \sqrt{1-\cos ^{2} t} \quad$ The sign is determined by the quadrant in which $P$ lies.

## Apply the Skills

6. For a given real number $t$, express $\tan t$ in terms of $\sec t$.

## Learn: Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions

Many cyclical patterns occur in nature such as the changes of the seasons, the rise and fall of the tides, and the phases of the Moon. In many cases, we can predict these behaviors by determining the period of one complete cycle. For example, an observer on the Earth would note that the Moon changes from a new moon to a full moon and back again to a new moon in approximately 29.5 days. If $m(t)$ represents the percentage of the Moon seen on day $t$, then

$$
m(t+29.5)=m(t) \text { for all } t \text { in the domain of } m \text {. }
$$

We say that function $m$ is periodic because it repeats at regular intervals. In this example, the period is 29.5 days because this is the shortest time required to complete one full cycle.

## Definition of a Periodic Function

A function $f$ is periodic if $f(t+p)=f(t)$ for some constant $p$.
The smallest positive value $p$ for which $f$ is periodic is called the period of $f$.

The values of the six trigonometric functions of $t$ are determined by the corresponding point $P(x, y)$ on the unit circle. Since the circumference of the unit circle is $2 \pi$, adding (or subtracting) $2 \pi$ to $t$ results in the same terminal point ( $x, y$ ). Consequently, the values of the trigonometric functions are the same for $t$ and $t+2 n \pi$.

In Exercises 121 and 122, we show that the sine and cosine functions are periodic with period $2 \pi$. Likewise, their reciprocal functions, cosecant and secant, are periodic with period $2 \pi$. However, the period of the tangent and cotangent functions is $\pi$, which we show in Exercises 119 and 120.

Periodic Properties of Trigonometric Functions

| Function | Period | Property |
| :--- | :--- | :--- |
| Sine | $2 \pi$ | $\sin (t+2 \pi)=\sin t$ |
| Cosine | $2 \pi$ | $\cos (t+2 \pi)=\cos t$ |
| Cosecant | $2 \pi$ | $\csc (t+2 \pi)=\csc t$ |
| Secant | $2 \pi$ | $\sec (t+2 \pi)=\sec t$ |
| Tangent | $\pi$ | $\tan (t+\pi)=\tan t$ |
| Cotangent | $\pi$ | $\cot (t+\pi)=\cot t$ |

Given a periodic function $f$, if the period is $p$, then $f(t+p)=f(t)$. It is also true that $f(t+n p)=f(t)$ for any integer $n$. That is, adding any integer multiple of the period to a domain element of a periodic function results in the same function value. This is demonstrated in Example 7.

## Example 7: Applying the Periodic Properties of the Trigonometric Functions

Given $\sin \frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$, determine the value of $\sin \frac{49 \pi}{12}$.

## Solution:

The period of the sine function is $2 \pi$ or equivalently $\frac{24 \pi}{12}$. Therefore, adding or subtracting any multiple of $\frac{24 \pi}{12}$ to the argument $\frac{\pi}{12}$ results in the same value of the sine function.

$$
\begin{aligned}
\sin \frac{49 \pi}{12} & =\sin \left(\frac{\pi}{12}+\frac{48 \pi}{12}\right)=\sin \left[\frac{\pi}{12}+2\left(\frac{24 \pi}{12}\right)\right]=\sin \left(\frac{\pi}{12}+2(2 \pi)\right) \\
& =\sin \frac{\pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}
$$

## Apply the Skills

7. Given $\sin \frac{\pi}{8}=\frac{\sqrt{2-\sqrt{2}}}{2}$, determine the value of $\sin \left(-\frac{15 \pi}{8}\right)$.

Recall that a function $f$ is even if $f(-x)=f(x)$ and odd if $f(-x)=-f(x)$. Suppose that $t$ is a real number associated with a point $P(x, y)$ on the unit circle. Then $-t$ is associated with the point $Q(x,-y)$. See Figure 4-26.


Figure 4-26
Notice that $\cos (-t)=\cos t=x$, so the cosine function is an even function. The sine function, however, is an odd function because $\sin t=y$, but $\sin (-t)=-y$. So, $\sin (-t)=-\sin t$. The tangent function is also an odd function because $\tan (-t)=\frac{-y}{x}$ and $\tan t=\frac{y}{x}$. Therefore, $\tan (-t)=-\tan t$. The functions secant, cosecant, and cotangent carry the same even and odd properties as their reciprocals: cosine, sine, and tangent, respectively.

Even and Odd Properties of Trigonometric Functions

| Function | Evaluate at $\boldsymbol{t}$ and $-\boldsymbol{t}$ | Property |
| :--- | :--- | :--- |
| Sine | $\sin t=y$ and $\sin (-t)=-y$ | $\sin (-t)=-\sin t$ (odd function) |
| Cosine | $\cos t=x$ and $\cos (-t)=x$ | $\cos (-t)=\cos t$ (even function) |
| Cosecant | $\csc t=\frac{1}{y}$ and $\csc (-t)=\frac{1}{-y}$ | $\csc (-t)=-\csc t$ (odd function) |
| Secant | $\sec t=\frac{1}{x}$ and $\sec (-t)=\frac{1}{x}$ | $\sec (-t)=\sec t$ (even function) |
| Tangent | $\tan t=\frac{y}{x}$ and $\tan (-t)=\frac{-y}{x}$ | $\tan (-t)=-\tan t$ (odd function) |
| Cotangent | $\cot t=\frac{x}{y}$ and $\cot (-t)=\frac{x}{-y}$ | $\cot (-t)=-\cot t$ (odd function) |

## Example 8: Simplifying Expressions Using the Even-Odd Properties of Trigonometric Functions

Use the properties of the trigonometric functions to simplify.
a. $4 \cos t+\cos (-t)$
b. $\tan (-3 t)-\tan (3 t+\pi)$

## Solution:

a. $4 \cos t+\cos (-t) \quad$ The cosine function is even. $\cos (-t)=\cos t$

$$
\begin{aligned}
& =4 \cos t+\cos t \\
& =5 \cos t
\end{aligned}
$$

b. $\tan (-3 t)-\tan (3 t+\pi)$ The tangent function is odd. $\tan (-3 t)=-\tan 3 t$

$$
\begin{array}{ll}
=-\tan 3 t-\tan 3 t & \begin{array}{l}
\text { The period of the tangent function is } \pi . \text { Therefore, } \\
\\
=-2 \tan 3 t
\end{array} \\
\tan (3 t+\pi)=\tan 3 t .
\end{array}
$$

## Check Your Work

A factor from the argument of a function cannot be factored out in front of the function. For example,

$$
f(2 x) \neq 2 f(x)
$$

In step 2 of Example 8(b), the value -1 that appears in front of the first term was not factored out, but rather is a result of the odd function property of the tangent function.

## Apply the Skills

8. Use the properties of the trigonometric functions to simplify.
a. $\sec t-2 \sec (-t)$
b. $\sin (2 t+2 \pi)-\sin (-2 t)$

## Learn: Approximate Trigonometric Functions on a Calculator

From Figure 4-23 (on page 34), we display the coordinates of the points on the unit circle that correspond to $t$ values in multiples of $\frac{\pi}{6}$ and $\frac{\pi}{4}$. From this information we can find the values of the six trigonometric functions for these real numbers. However, it is often the case that the exact value of a trigonometric function for a real number $t$ cannot be found analytically. In such a case, we can use a calculator to approximate the function values.

## Example 9: Approximating Trigonometric Functions Using a Calculator

Use a calculator to approximate the function values. Round to 4 decimal places.
a. $\cos \frac{2 \pi}{7}$
b. $\csc 0.92$

## Solution:

The real number $t$ corresponds to a point $P(x, y)$ on the unit circle. The value of $t$ is the same as the radian measure of the central angle $\theta$ formed by the positive $x$-axis and the ray from the origin to point $P$. Therefore, we must first be sure that the calculator is in radian mode.


## Apply the Skills

9. Use a calculator to approximate the function values. Round to 4 decimal places.
a. $\tan 1.4$
b. $\cot \frac{\pi}{8}$

## Point of Interest (Cyclical Data)

In September 1859, a gigantic coronal mass ejection erupted from the Sun, sending a huge amount of electromagnetic radiation toward the Earth. This geomagnetic storm was the largest on record to have struck the Earth, causing telegraph lines to short out and compasses to go haywire. The aurora borealis or "northern lights" following the event could be seen as far south as Cuba.

Solar storms are natural phenomena that occur as a result of the rise and fall of the Sun's magnetic activity. Although the cause of solar storms is not completely understood, they seem to be cyclical. Scientists have established a correlation between solar storms and the number of sunspots that cycle over a period of approximately 11 yr . The effects of a massive solar storm today could bring potentially devastating disruption to power grids, satellite communication, and air travel. As a result, solar activity and "space weather" are studied intensely by NASA and NOAA.

Mean Number of Sunspots by Year

(Source: SILSO data/image, Royal Observatory of Belgium, Brussels)

## Practice 4-2

## Practice Exercises

## Prerequisite Review

For Exercises R.1-R.3, determine if the function is even, odd, or neither.
R.1. $k(x)=11 x^{3}+12 x$
R.2. $r(x)=\sqrt{25-(x+2)^{2}}$
R.3. $q(x)=\sqrt{23+x^{2}}$

For Exercises R.4-R.5, write the domain of the function in set-builder notation.
R.4. $p(v)=\frac{v+3}{v+4}$
R.5. $k(q)=\frac{q^{2}}{2 q^{2}+3 q-27}$

## Concept Connections

1. The graph of $x^{2}+y^{2}=1$ is known as the $\qquad$ circle. It has a radius of length $\qquad$ and center at the $\qquad$ .
2. The circumference of the unit circle is $\qquad$ So, the value $t=\frac{\pi}{2}$ represents $\qquad$ of a revolution.
3. When $\tan t=0$, the value of $\cot t$ is $\qquad$ .
4. The value of $\sec t=\frac{1}{\cos t^{*}}$ Therefore, when $\cos t=0$, the value of $\sec t$ is $\qquad$ .
5. On the interval $[0,2 \pi)$, $\sin t=0$ for
$t=$ $\qquad$ and $t=$ $\qquad$ Since $\csc t=\frac{1}{\sin t}$, the value of $\csc t$ is $\qquad$ for these values of $t$.
6. The domain of the trigonometric functions
$\qquad$ and $\qquad$ is all real
numbers.
7. Which trigonometric functions have domain $\left\{t \left\lvert\, t \neq \frac{(2 n+1) \pi}{2}\right.\right.$ for all integers $\left.n\right\} ?$
8. A function $f$ is $\qquad$
if $f(t+p)=f(t)$ for some constant $p$.
9. The period of the tangent and cotangent functions is $\qquad$ The period of the sine, cosine, cosecant, and secant functions is $\qquad$ .
10. The cosine function is an $\qquad$
function because $\cos (-t)=\cos t$. The sine function is an $\qquad$ function because $\sin (-t)=-\sin t$.

## Learn: Evaluate Trigonometric Functions Using the Unit Circle

For Exercises 11-14, determine if the point lies on the unit circle.
11. $\left(\frac{\sqrt{10}}{5}, \frac{\sqrt{15}}{5}\right)$
12. $\left(\frac{\sqrt{61}}{8},-\frac{\sqrt{2}}{8}\right)$
13. $\left(-\frac{\sqrt{7}}{10},-\frac{2 \sqrt{23}}{10}\right)$
14. $\left(-\frac{2 \sqrt{34}}{17}, \frac{3 \sqrt{17}}{17}\right)$

For Exercises 15-18, the real number $t$ corresponds to the point $P$ on the unit circle. Evaluate the six trigonometric functions of $t$. (See Example 1, p. 472)
15.

16.

17.

18.

19. Fill in the ordered pairs on the unit circle corresponding to each real number $t$.


For Exercises 20-23, identify the coordinates of point $P$. Then evaluate the six trigonometric functions of $t$. (See Example 2, p. 477)
20.

21.

22.

23.


For Exercises 24-26, identify the ordered pairs on the unit circle corresponding to each real number $t$.
24. a. $t=\frac{2 \pi}{3}$
b. $t=-\frac{5 \pi}{4}$
c. $t=\frac{5 \pi}{6}$
25. a. $t=\frac{7 \pi}{6}$
b. $t=-\frac{3 \pi}{4}$
c. $t=\frac{4 \pi}{3}$
26. a. $t=\frac{5 \pi}{3}$
b. $t=\frac{7 \pi}{4}$
c. $t=-\frac{\pi}{6}$

For Exercises 27-32, evaluate the trigonometric function at the given real number.
27. $f(t)=\cos t ; t=\frac{2 \pi}{3}$
28. $g(t)=\sin t ; t=\frac{5 \pi}{4}$
29. $h(t)=\cot t ; t=\frac{11 \pi}{6}$
30. $s(t)=\sec t ; t=\frac{5 \pi}{3}$
31. $z(t)=\csc t ; t=\frac{7 \pi}{4}$
32. $r(t)=\tan t ; t=\frac{5 \pi}{6}$

## Learn: Identify the Domains of the Trigonometric Functions

For Exercises 33-38, select the domain of the trigonometric function.
a. All real numbers
b. $\left\{t \left\lvert\, t \neq \frac{(2 n+1) \pi}{2}\right.\right.$ for all integers $\left.n\right\}$
c. $\quad\{t \mid t \neq n \pi$ for all integers $n\}$
33. $f(t)=\sin t$
34. $f(t)=\tan t$
35. $f(t)=\cot t$
36. $f(t)=\cos t$
37. $f(t)=\sec t$
38. $f(t)=\csc t$

For Exercises 39-42, evaluate the function if possible. (See Example 3, p. 479)
39. a. $\sin 0$
b. $\cot \pi$
c. $\tan 3 \pi$
d. $\sec \pi$
e. $\csc 0$
f. $\cos \pi$
40. a. $\cos \left(-\frac{\pi}{2}\right)$
b. $\csc \frac{\pi}{2}$
c. $\cot \frac{5 \pi}{2}$
d. $\tan \frac{\pi}{2}$
e. $\sec \frac{3 \pi}{2}$
f. $\sin \frac{\pi}{2}$
41. a. $\sin \frac{3 \pi}{2}$
b. $\cos \frac{7 \pi}{2}$
c. $\tan \frac{3 \pi}{2}$
d. $\csc \left(-\frac{\pi}{2}\right)$
e. $\sec 1.5 \pi$
f. $\cot \frac{\pi}{2}$
42. a. $\cot 0$
b. $\cos 2 \pi$
c. $\csc \pi$
d. $\tan 0$
e. $\sin (-3 \pi)$
f. $\sec 0$

## Learn: Use Fundamental Trigonometric Identities

For Exercises 43-46, given the values for $\sin t$ and $\cos t$, use the reciprocal and quotient identities to find the values of the other trigonometric functions of $t$.
(See Example 4, p. 480 )
43. $\sin t=\frac{\sqrt{5}}{3}$ and $\cos t=\frac{2}{3}$
44. $\sin t=\frac{3}{4}$ and $\cos t=\frac{\sqrt{7}}{4}$
45. $\sin t=-\frac{\sqrt{39}}{8}$ and $\cos t=-\frac{5}{8}$
46. $\sin t=\frac{28}{53}$ and $\cos t=-\frac{45}{53}$

For Exercises 47-48, derive the given identity from the Pythagorean identity, $\sin ^{2} t+\cos ^{2} t=1$.
47. $\tan ^{2} t+1=\sec ^{2} t$
48. $1+\cot ^{2} t=\csc ^{2} t$

For Exercises 49-54, use an appropriate Pythagorean identity to find the indicated value. (See Example 5, p. 481)
49. Given $\cos t=-\frac{7}{25}$ for $\frac{\pi}{2}<t<\pi$, find the value of $\sin t$.
50. Given $\sin t=-\frac{8}{17}$ for $\pi<t<\frac{3 \pi}{2}$, find the value of cost.
51. Given $\cot t=\frac{45}{28}$ for $\pi<t<\frac{3 \pi}{2}$, find the value of $\csc t$.
52. Given $\csc t=-\frac{41}{40}$ for $\frac{3 \pi}{2}<t<2 \pi$, find the value of $\cot t$.
53. Given $\tan t=-\frac{11}{60}$ for $\frac{3 \pi}{2}<t<2 \pi$, find the value of $\sec t$.
54. Given $\sec t=-\frac{37}{35}$ for $\frac{\pi}{2}<t<\pi$, find the value of $\tan t$.
55. Write $\sin t$ in terms of $\cos t$ for
(See Example 6, p. 482)
a. $t$ in Quadrant $I$.
b. $t$ in Quadrant III.
56. Write $\tan t$ in terms of $\sec t$ for
a. $t$ in Quadrant II.
b. $t$ in Quadrant IV.
57. Write $\cot t$ in terms of $\csc t$ for
a. $t$ in Quadrant I.
b. $t$ in Quadrant III.
58. Write $\cos t$ in terms of $\sin t$ for
a. $t$ in Quadrant II.
b. $t$ in Quadrant IV.

## Learn: Apply the Periodic and Even and Odd Function Properties of Trigonometric Functions

59. Given that $\cos \frac{29 \pi}{12}=\frac{\sqrt{6}-\sqrt{2}}{4}$, determine the value of $\cos \frac{5 \pi}{12}$. (See Example 7, p. 484)
60. Given that $\sec \frac{11 \pi}{12}=\sqrt{2}-\sqrt{6}$, determine the value of $\sec \left(-\frac{13 \pi}{12}\right)$.
61. Given $\tan \left(-\frac{\pi}{8}\right)=1-\sqrt{2}$, determine the value of $\cot \frac{7 \pi}{8}$.
62. Given that $\sin \frac{\pi}{10}=\frac{\sqrt{5}-1}{4}$, determine the value of $\csc \frac{21 \pi}{10}$.

For exercises 63-66, use the periodic properties of the trigonometric functions to simplify each expression to a single function of $t$.
63. $\sin (t+2 \pi) \cdot \cot (t+\pi)$
64. $\sin (t+2 \pi) \cdot \sec (t+2 \pi)$
65. $\csc (t+2 \pi) \cdot \cos (t+2 \pi)$
66. $\tan (t+\pi) \cdot \csc (t+2 \pi)$

For Exercises 67-74, use the even-odd and periodic properties of the trigonometric functions to simplify. (See Example 8, p. 485)
67. $\csc t-4 \csc (-t)$
68. $\tan (-t)-3 \tan t$
69. $-\cot (-t+\pi)-\cot t$
70. $\sec (t+2 \pi)-\sec (-t)$
71. $-2 \sin (3 t+2 \pi)-3 \sin (-3 t)$
72. $\cot (-3 t)-3 \cot (3 t+\pi)$
73. $\cos (-2 t)-\cos 2 t$
74. $\sec (-2 t)+3 \sec 2 t$

## Learn: Approximate Trigonometric Functions on a Calculator

Use a calculator to approximate the function values. Round to 4 decimal places. (See Example 9, p. 486)
75. a. $\quad \sin (-0.15)$
b. $\cos \frac{2 \pi}{5}$
76. a. $\cos \left(-\frac{7 \pi}{11}\right)$
b. $\sin 0.96$
77. a. $\cot \frac{12 \pi}{7}$
b. $\sec 5.43$
78. a. $\csc 7.58$
b. $\tan \frac{3 \pi}{8}$

## Mixed Exercises

For Exercises 79-80, evaluate $\sin t, \cos t$, and $\tan t$ for the real number $t$.
79. a. $t=\frac{2 \pi}{3}$
b. $t=-\frac{4 \pi}{3}$
80.a. $t=-\frac{11 \pi}{6}$
b. $t=\frac{\pi}{6}$

For Exercises 81-86, identify the values of $t$ on the interval $[0,2 \pi]$ that make the function undefined (if any).
81. $y=\sin t$
82. $y=\cot t$
83. $y=\tan t$
84. $y=\cos t$
85. $y=\csc t$
86. $y=\sec t$

For Exercises 87-92, select all properties that apply to the trigonometric function.
a. The function is even.
b. The function is odd.
c. The period is $2 \pi$.
d. The period is $\pi$.
e. The domain is all real numbers.
f. The domain is all real numbers excluding odd multiples of $\frac{\pi}{2}$.
g. The domain is all real numbers excluding multiples of $\pi$.
87. $f(t)=\sin (t)$
88. $f(t)=\tan (t)$
89. $f(t)=\sec (t)$
90. $f(t)=\cot (t)$
91. $f(t)=\csc (t)$
92. $f(t)=\cos (t)$

For Exercises 93-98, simplify using properties of trigonometric functions.
93. $\sin ^{2}(t+2 \pi)+\cos ^{2} t+\tan ^{2}(t+\pi)$
94. $\cot (-t) \cdot \sin (t+2 \pi)+\cos ^{2} t \cdot \sec (-t+2 \pi)$
95. $\sec ^{2}\left(-\frac{\pi}{3}\right)+\tan ^{2}\left(-\frac{\pi}{3}\right)$
96. $\sin ^{2}\left(\frac{\pi}{6}\right)-\cos ^{2}\left(-\frac{\pi}{6}\right)$
97. $\sec ^{2}\left(-\frac{\pi}{4}\right)-\csc ^{2}\left(\frac{7 \pi}{6}\right)$
98. $\tan ^{2}\left(\frac{2 \pi}{3}\right)+\csc ^{2}\left(-\frac{5 \pi}{4}\right)$
99. In a small coastal town, the monthly revenue received from tourists rises and falls throughout the year. The tourist revenue peaks during the months when the town holds seafood cooking competitions. The function $f(t)=5.6 \cos \left(\frac{\pi}{3} t\right)+11.2$ represents the monthly revenue $f(t)$, in tens of thousands of dollars, for a month $t$, where $t=0$ represents April.
a. Complete the table and give the period of the function. Round to 1 decimal place.

| $t$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(t)$ |  |  |  |  |  |  |


| $t$ | 6 | 7 | 8 | 9 | 10 | 11 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(t)$ |  |  |  |  |  |  |

b. What are the months of peak revenue, and what is the revenue for those months?
100. The fluctuating brightness of a distant star is given by the function
$f(d)=3.8+0.25 \sin \left(\frac{2 \pi}{3} d\right)$
where $d$ is the number of days and $f(d)$ is the apparent brightness.
Complete the table and give the period of the function. Round to 2 decimal places.

| $d$ | 0 | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $f(d)$ |  |  |  |  |  |  |

For Exercises 101-104, identify each function as even, odd, or neither.
101. $f(t)=t^{2} \sin t$
102. $g(t)=t \cos t$
103. $z(t)=t^{3} \tan t$
104. $h(t)=t^{3}+\sec t$

## Write About It

105. Describe the changes in $\sin t$ and $\cos t$ as $t$ increases from 0 to $\frac{\pi}{2}$.
106. Describe the changes in cos $t$ and $\sec t$ as $t$ increases from 0 to $\frac{\pi}{2}$.
107. Explain why $-1 \leq \cos t \leq 1$ and $-1 \leq \sin t \leq 1$ for all real numbers $t$.
108. Do the trigonometric functions satisfy the definition of a function learned in Chapter 1? Explain your answer.

## Expanding Your Skills

For Exercises 109-114, use the figure to estimate the value of (a) $\sin t$ and (b) cos $t$ for the given value of $\boldsymbol{t}$.

109. $t=0.5$
110. $t=1.25$
111. $t=3.75$
112. $t=2.75$
113. $t=5$
114. $t=3$

For Exercises 115-120, use the figure from Exercises 109-114 to approximate the solutions to the equation over the interval [ $0,2 \pi$ ).
115. $\sin t=0.2$
116. $\sin t=-0.4$
117. $\cos t=-0.4$
118. $\cos t=0.6$
119. Show that $\tan (t+\pi)=\tan t$.
120. Show that $\cot (t+\pi)=\cot t$
121. Prove that the period of $f(t)=\sin t$ is $2 \pi$.
122. Prove that the period of $f(t)=\cos t$ is $2 \pi$.
123. Given that $\tan t=a$ for $t$ in Quadrant II, write expressions for $\sin t$ and $\cos t$.
124. Given that $\cos t=a$ and that $t$ is in Quadrant III, find the values of $\sin t$ and $\tan t$.
125. For an arbitrary value of $t$, write an expression for $\sin t$ and cost in terms of $\tan t$.
126. For an arbitrary value of $t$, write an expression for $\sin t$ and $\tan t$ in terms of $\cos t$.

## Technology Connections

127. a. Complete the table to show that $\sin t \approx t$ for values of $t$ close to zero. Round to 5 decimal places.

| $t$ | $\sin t$ | $\frac{\sin t}{t}$ |
| :---: | :---: | :---: |
| 0.2 |  |  |
| 0.1 |  |  |
| 0.01 |  |  |
| 0.001 |  |  |

b. What value does the ratio $\frac{\sin t}{t}$ seem to approach as $t \longrightarrow 0$ ?
128. Consider the expression $\frac{1-\cos t}{t}$.
a. What is the value of the numerator for $t=0$ ?
b. What is the value of the denominator for $t=0$ ?
c. The expression $\frac{1-\cos t}{t}$ is undefined at $t=0$. Complete the table to investigate the value of the expression close to $t=0$. Round to 5 decimal places.

| $t$ | $\frac{1-\cos t}{t}$ |
| :---: | :---: |
| 0.2 |  |
| 0.1 |  |
| 0.01 |  |
| 0.001 |  |

Go online for more practice problems.

## What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate Trigonometric Functions of Acute Angles
- Use Fundamental Trigonometric Identities
- Use Trigonometric Functions in Applications


## Learn: Evaluate Trigonometric Functions of Acute Angles

The science of land surveying encompasses the measurement and mapping of land using mathematics and specialized equipment and technology. Surveyors provide data relevant to the shape and contour of the Earth's surface for engineering, mapmaking, and construction projects.

One technique used by surveyors to map a landscape is called triangulation. The surveyor first measures the distance between two fixed points, $A$ and $B$. Then the surveyor measures the bearing from $A$ and the bearing from $B$ to a third point $P$. From this information, the surveyor can compute the measures of the angles of the triangle formed by the points, and can use trigonometry to find the lengths of the unknown sides.


To use trigonometry in such applications, we present alternative definitions of the trigonometric functions using right triangles. Consider a right triangle with an acute angle $\theta$. The longest side in the triangle is the hypotenuse ("hyp") and is opposite the right angle (Figure 4-27). The two legs of the triangle will be distinguished by their relative positions to $\theta$. The leg that lies on one ray of angle $\theta$ is called the adjacent leg ("adj") and the leg that lies across the triangle from $\theta$ is called the opposite leg ("opp").


Figure 4-27
We now define the six trigonometric functions as functions whose input (or argument) is an acute angle $\theta$. The output value for each function will be one of the six possible ratios of the lengths of the sides of the triangle (Table 4-4).

The definitions of the six trigonometric functions of acute angles are given in Table 4-4 along with examples based on Figure 4-28.

TABLE 4-4 Definition of Trigonometric Functions of Acute Angles

| Function Name | Definition | Example |
| :---: | :---: | :---: |
| sine | $\sin \theta=\frac{\text { opp }}{\text { hyp }}$ | $\sin \theta=\frac{3 \mathrm{ft}}{5 \mathrm{ft}}=\frac{3}{5}$ |
| cosine | $\cos \theta=\frac{\mathrm{adj}}{\text { hyp }}$ | $\cos \theta=\frac{4 \mathrm{ft}}{5 \mathrm{ft}}=\frac{4}{5}$ |
| tangent | $\tan \theta=\frac{\mathrm{opp}}{\mathrm{adj}}$ | $\tan \theta=\frac{3 \mathrm{ft}}{4 \mathrm{ft}}=\frac{3}{4}$ |
| cosecant | $\csc \theta=\frac{\text { hyp }}{\text { opp }}$ | $\csc \theta=\frac{5 \mathrm{ft}}{3 \mathrm{ft}}=\frac{5}{3}$ |
| secant | $\sec \theta=\frac{\mathrm{hyp}}{\mathrm{adj}}$ | $\sec \theta=\frac{5 \mathrm{ft}}{4 \mathrm{ft}}=\frac{5}{4}$ |
| cotangent | $\cot \theta=\frac{\mathrm{adj}}{\mathrm{opp}}$ | $\cot \theta=\frac{4 \mathrm{ft}}{3 \mathrm{ft}}=\frac{4}{3}$ |

## Good Practices

The mnemonic device "SOH-CAH-TOA" may help you remember the ratios for $\sin \theta, \cos \theta$, and $\tan \theta$, respectively.

Sine: Opp over Hyp
Cosine: Adj over Hyp
Tangent: Opp over Adj

The notation $\sin \theta$ is read as "sine of theta" (likewise, $\cos \theta$ is read as "cosine of theta" and so on). Also notice that the output value of a trigonometric function is unitless because the common units of length "cancel" within each ratio.

The definitions of the trigonometric functions given in Table 4-4 are consistent with the definitions based on the unit circle from Lesson 4.2. For example, consider a real number $0<t<\frac{\pi}{2}$ that corresponds to the point $P(x, y)$ on the unit circle (Figure 4-29). Suppose that we drop a perpendicular line segment from $P$ to the $x$-axis at point $A$. Then $\triangle O A P$ is a right triangle. The lengths of the legs are $x$ and $y$, and the hypotenuse is 1 unit. Furthermore, the radian measure of $\angle A O P$ is equal to $t$. If we let $\theta$ represent the degree measure of


Figure 4-29 $\angle A O P$, we have

## Unit circle definition

```
sin}t=
cost=x
tan}t=\frac{y}{x
```


## Right triangle definition

$$
\begin{aligned}
& \sin \theta=\frac{\text { opp }}{\text { hyp }}=\frac{y}{1}=y \\
& \cos \theta=\frac{\text { adj }}{\text { hyp }}=\frac{x}{1}=x \\
& \tan \theta=\frac{\text { opp }}{\mathrm{adj}}=\frac{y}{x}
\end{aligned}
$$

It is also very important to note that the values of the trigonometric functions depend only on the measure of the angle, not the size of the triangle. For any given acute angle $\theta$, a series of right triangles can be formed by drawing a line segment perpendicular to the initial side of $\theta$ with endpoints on the initial and terminal sides of $\theta$ (Figure 4-30).
Triangles $\triangle A B C$ and $\triangle A D E$ are similar triangles with common angle $\theta$. Therefore, the corresponding sides are proportional, and the value of each trigonometric function of $\theta$ will be the same regardless of which triangle is used. For example, $\sin \theta=\frac{x}{b}=\frac{y}{d}$.


Figure 4-30
In Example 1, we find the values of the six trigonometric functions of an acute angle within a right triangle where the lengths of two legs are given.

## Example 1: Evaluating Trigonometric Functions by First Applying the Pythagorean Theorem

Suppose that a right triangle has legs of length 4 cm and 7 cm . Evaluate the six trigonometric functions for the smaller acute angle.

## Solution:

A right triangle $\triangle A B C$ meeting the conditions of this example is drawn in Figure 4-31. Since $\angle B$ is opposite the shorter leg, $\angle B$ is the smaller acute angle.


Figure 4-31
To find the values of all six trigonometric functions, we also need to know the length of the hypotenuse. The hypotenuse is always opposite the right angle.

$$
\begin{aligned}
c^{2} & =4^{2}+7^{2} \\
c^{2} & =16+49 \\
c^{2} & =65 \\
c & =\sqrt{65}
\end{aligned}
$$

Apply the Pythagorean theorem.

$$
c^{2}=16+49 \quad \text { Simplify terms with exponents }
$$

Since $c$ represents the length of a side of a triangle, take the positive square root of 65 .

Relative to angle $B$, the $4-\mathrm{cm}$ side is the opposite side. The $7-\mathrm{cm}$ side is the adjacent leg, and the hypotenuse is $c=\sqrt{65}$.

$\sin B=\frac{\mathrm{opp}}{\text { hyp }}=\frac{4}{\sqrt{65}}=\frac{4}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}}=\frac{4 \sqrt{65}}{65}$
$\cos B=\frac{\text { adj }}{\text { hyp }}=\frac{7}{\sqrt{65}}=\frac{7}{\sqrt{65}} \cdot \frac{\sqrt{65}}{\sqrt{65}}=\frac{7 \sqrt{65}}{65}$
$\tan B=\frac{\mathrm{opp}}{\mathrm{adj}}=\frac{4}{7}$
$\csc B=\frac{\text { hyp }}{\text { opp }}=\frac{\sqrt{65}}{4}$
$\sec B=\frac{\mathrm{hyp}}{\mathrm{adj}}=\frac{\sqrt{65}}{7}$
$\cot B=\frac{\mathrm{adj}}{\mathrm{opp}}=\frac{7}{4}$

## Check Your Work

In Example 1, the opposite (opp) and adjacent (adj) legs are labeled in the triangle relative to angle $B$.

## Apply the Skills

1. Suppose that a right triangle $\triangle A B C$ has legs of length 6 cm and 3 cm .

Evaluate the six trigonometric functions for angle $B$, where angle $B$ is the larger acute angle.

In Example 1, we were given the lengths of the sides of a right triangle and asked to find the values of the six trigonometric functions. This process can be reversed. Given the value of one trigonometric function of an acute angle, we can find the lengths of the sides of a representative triangle, and therefore the values of the other trigonometric functions.

## Example 2: Using a Known Value of a Trigonometric Function to Determine Other Function Values

Suppose that $\cos \theta=\frac{\sqrt{5}}{3}$ for the acute angle $\theta$. Evaluate $\tan \theta$.

## Solution:

Since $\cos \theta$ is the ratio of the length of the leg adjacent to $\theta$ and the length of the hypotenuse, it is convenient to construct a triangle with adjacent leg $\sqrt{5}$ units and hypotenuse 3 units. Using the Pythagorean theorem, we can find the length of the opposite leg.


Simplify terms with exponents.

Take the positive square root of 4 .
Once the proper ratio is identified for $\tan \theta$, rationalize the denominator.


## Apply the Skills

2. Suppose that $\sin \theta=\frac{3}{4}$ for the acute angle $\theta$. Evaluate $\sec \theta$.

In Example 3, we find the sine, cosine, and tangent of $45^{\circ}$ by using a right triangle approach.

## Example 3: Determine the Trigonometric Function Values of a $45^{\circ}$ Angle

Evaluate $\sin 45^{\circ}, \cos 45^{\circ}$, and $\tan 45^{\circ}$.

## Solution:

A right triangle with an acute angle of $45^{\circ}$ must have a second acute angle of $45^{\circ}$ because the sum of the angles must equal $180^{\circ}$. This is called an isoceles right triangle or a $45^{\circ}-45^{\circ}-90^{\circ}$ triangle. Since all such triangles are similar, we can choose one of the sides to be of arbitrary length. We have chosen the hypotenuse to be 1 unit. The sides opposite the $45^{\circ}$ angles are equal in length and have been labeled $a$.


$$
\begin{array}{rlr}
a^{2}+a^{2} & =1^{2} & \text { Apply the Pythagorean theorem. } \\
2 a^{2} & =1 & \\
a^{2} & =\frac{1}{2} & \\
a=\sqrt{\frac{1}{2}}=\frac{1}{\sqrt{2}}=\frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{2} & \begin{array}{l}
\text { Take the positive square root of } \\
\text { Rationalize the denominator. }
\end{array} \\
\sin 45^{\circ}=\frac{\text { opp }}{\text { hyp }}=\frac{\frac{\sqrt{2}}{2}}{1}=\frac{\sqrt{2}}{2} & \text { (hyp) } \\
\cos 45^{\circ}=\frac{\operatorname{adj}}{\text { hyp }}=\frac{\frac{\sqrt{2}}{2}}{1}=\frac{\sqrt{2}}{2} &
\end{array}
$$

Take the positive square root of $\frac{1}{2}$. Rationalize the denominator.

## Apply the Skills

3. Evaluate $\csc 45^{\circ}, \sec 45^{\circ}$, and $\cot 45^{\circ}$.

## Insights

The values of $\sin 45^{\circ}, \cos 45^{\circ}$, and $\tan 45^{\circ}$ found in Example 3 are consistent with the values of $\sin \frac{\pi}{4}, \cos \frac{\pi}{4}$, and $\tan \frac{\pi}{4}$ found by using the unit circle.

A right triangle with a $30^{\circ}$ angle will also have a $60^{\circ}$ angle so that the sum of the angular measures is $180^{\circ}$. Such a triangle is called a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle. Interestingly, as we will show in Example 4, the length of the shorter leg of a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle is always one-half the length of the hypotenuse.

## Example 4: Determine the Trigonometric Function Values of a $60^{\circ}$ Angle

Evaluate $\sin 60^{\circ}, \cos 60^{\circ}$, and $\tan 60^{\circ}$.

## Solution:

Suppose that we start with an equilateral triangle with sides of length $L$. All angles in the triangle are equal to $60^{\circ}$. Any altitude $a$ of this triangle bisects a $60^{\circ}$ angle as well as the opposite side. Therefore, we can create two congruent right triangles with legs of length $a$ and $\frac{1}{2} L$ and hypotenuse of length $L$.


Use the Pythagorean theorem to find the altitude $a$ in terms of $L$.

$$
\begin{array}{rlrl}
\left(\frac{1}{2} L\right)^{2}+a^{2} & =L^{2} & & \text { Apply the Pythagorean theorem. } \\
\frac{1}{4} L^{2}+a^{2} & =L^{2} & & \text { Simplify terms with exponents. } \\
a^{2} & =\frac{3}{4} L^{2} & & \text { Combine like terms. } \\
a & =\sqrt{\frac{3}{4}} L & & \text { Take the positive square root. } \\
a & =\frac{\sqrt{3}}{2} L & & \text { Simplify the radical. } \\
\sin 60^{\circ}=\frac{\text { opp }}{\text { hyp }}=\frac{\frac{\sqrt{3}}{2} L}{L}=\frac{\sqrt{3}}{2} & & \\
\cos 60^{\circ}=\frac{\text { adj }}{\text { hyp }}=\frac{\frac{1}{2} L}{L}=\frac{1}{2} & & \frac{\frac{\sqrt{3}}{2} L}{\frac{1}{2}} L \\
\tan 60^{\circ}=\frac{\text { opp }}{\text { adj }}=\frac{\frac{\sqrt{3}}{2} L}{\frac{1}{2} L}=\left(\frac{\sqrt{3}}{2}\right) \cdot\left(\frac{2}{1}\right)=\sqrt{3}
\end{array}
$$

## Apply the Skills

4. Evaluate $\sin 30^{\circ}, \cos 30^{\circ}$, and $\tan 30^{\circ}$.

From Examples 3 and 4, we can summarize the trigonometric function values of the "special" angles $30^{\circ}, 45^{\circ}$, and $60^{\circ}$, or equivalently $\frac{\pi}{6}, \frac{\pi}{4}$, and $\frac{\pi}{3}$ radians (Table 4-5).

TABLE 4-5 Trigonometric Function Values of Special Angles

| $\theta$ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | $\csc \theta$ | $\sec \theta$ | $\cot \theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $30^{\circ}=\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | 2 | $\frac{2 \sqrt{3}}{3}$ | $\sqrt{3}$ |
| $45^{\circ}=\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\sqrt{2}$ | $\sqrt{2}$ | 1 |
| $60^{\circ}=\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{2 \sqrt{3}}{3}$ | 2 | $\frac{\sqrt{3}}{3}$ |

## Good Practices

As a memory device, note the 1-2-3 pattern in the numerator for the sine function for the special angles.

$$
\begin{aligned}
& \sin 30^{\circ}=\frac{\sqrt{1}}{2} \\
& \sin 45^{\circ}=\frac{\sqrt{2}}{2} \\
& \sin 60^{\circ}=\frac{\sqrt{3}}{2}
\end{aligned}
$$

For cosine, the order is reversed, 3-2-1.

$$
\begin{aligned}
& \cos 30^{\circ}=\frac{\sqrt{3}}{2} \\
& \cos 45^{\circ}=\frac{\sqrt{2}}{2} \\
& \cos 60^{\circ}=\frac{\sqrt{1}}{2}
\end{aligned}
$$

To help you remember the values of the trigonometric functions for $30^{\circ}, 45^{\circ}$, and $60^{\circ}$, you can refer to the first quadrant of the unit circle. The values $t=\frac{\pi}{6}, t=\frac{\pi}{4}$, and $t=\frac{\pi}{3}$ correspond to the central angles $\theta=30^{\circ}, \theta=45^{\circ}$, and $\theta=60^{\circ}$, respectively (Figure 4-32).




Figure 4-32

## Example 5: Simplifying Expressions Involving Trigonometric Functions of Special Angles

Simplify the expressions.
a. $\tan 60^{\circ}-\tan 30^{\circ}$
b. $2 \sin \frac{\pi}{6} \cos \frac{\pi}{6}$

## Solution:

a. $\tan 60^{\circ}-\tan 30^{\circ}=\sqrt{3}-\frac{\sqrt{3}}{3} \quad$ Evaluate $\tan 60^{\circ}$ and $\tan 30^{\circ}$.

$$
\begin{aligned}
& =\frac{3}{3} \cdot \frac{\sqrt{3}}{1}-\frac{\sqrt{3}}{3} \\
& =\frac{2 \sqrt{3}}{3}
\end{aligned}
$$

Write the first term with a common denominator of 3.

Combine like terms.
b. $2 \sin \frac{\pi}{6} \cos \frac{\pi}{6}=2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$$
=\frac{\sqrt{3}}{2}
$$

Evaluate $\sin \frac{\pi}{6}$ and $\cos \frac{\pi}{6}$.
Multiply.

## Apply the Skills

5. Simplify the expressions.
a. $\cot 60^{\circ}-\cot 30^{\circ}$
b. $\sin \frac{\pi}{3} \cos \frac{\pi}{6}+\cos \frac{\pi}{3} \sin \frac{\pi}{6}$

## Learn: Use Fundamental Trigonometric Identities

In Lesson 4.2, we observed several relationships among the trigonometric functions that follow directly from the definitions of the functions. The reciprocal and quotient identities also follow from the right triangle defintions of trigonometric functions of acute angles (Table 4-6). For example, since $\sin \theta=\frac{\mathrm{opp}}{\text { hyp }}$ and $\csc \theta=\frac{\text { hyp }}{\text { opp }}$ for an acute angle $\theta$, we know that $\sin \theta$ and $\csc \theta$ are reciprocals.

| Trigonometric Functions | Relationship |
| :---: | :--- |
| $\csc \theta=\frac{1}{\sin \theta}$ or $\sin \theta=\frac{1}{\csc \theta}$ | $\sin \theta$ and $\csc \theta$ are reciprocals. |
| $\sec \theta=\frac{1}{\cos \theta}$ or $\cos \theta=\frac{1}{\sec \theta}$ | $\cos \theta$ and $\sec \theta$ are reciprocals. |
| $\cot \theta=\frac{1}{\tan \theta}$ or $\tan \theta=\frac{1}{\cot \theta}$ | $\tan \theta$ and $\cot \theta$ are reciprocals. |
| $\tan \theta=\frac{\sin \theta}{\cos \theta}$ | $\tan \theta$ is the ratio of $\sin \theta$ and $\cos \theta$. |
| $\cot \theta=\frac{\cos \theta}{\sin \theta}$ | $\cot \theta$ is the ratio of $\cos \theta$ and $\sin \theta$. |

We can also show that the Pythagorean identities follow from the right triangle definitions of the trigonometric functions of an acute angle $\theta$. Consider a right triangle with legs of length $a$ and $b$ and hypotenuse $c$ (Figure 4-33).


Figure 4-33
From the Pythagorean theorem, we know that $a^{2}+b^{2}=c^{2}$. Dividing both sides by the positive real number $c^{2}$ we have

$$
\begin{aligned}
\frac{a^{2}}{c^{2}}+\frac{b^{2}}{c^{2}} & =\frac{c^{2}}{c^{2}} \\
\left(\frac{a}{c}\right)^{2}+\left(\frac{b}{c}\right)^{2} & =\left(\frac{c}{c}\right)^{2} \\
(\sin \theta)^{2}+(\cos \theta)^{2} & =1 \\
\sin ^{2} \theta+\cos ^{2} \theta & =1
\end{aligned}
$$

Table 4-7 summarizes this and two other Pythagorean identities. The second and third relationships can be derived by dividing both sides of the equation $a^{2}+b^{2}=c^{2}$ by $b^{2}$ and $a^{2}$, respectively.

TABLE 4-7 Pythagorean Identities

| Trigonometric Functions | Pythagorean Identity |
| :---: | :---: |
| $\sin t$ and $\cos t$ | $\sin ^{2} t+\cos ^{2} t=1$ |
| $\tan t$ and $\sec t$ | $\tan ^{2} t+1=\sec ^{2} t$ |
| $\cot t$ and $\csc t$ | $1+\cot ^{2} t=\csc ^{2} t$ |

Visualizing the ratios of the lengths of the sides of a right triangle can also help us understand the cofunction identities.

The two acute angles in a right triangle are complementary because the sum of their measures is $90^{\circ}$. Symbolically, the complement of angle $\theta$ is $\left(90^{\circ}-\theta\right)$. In Figure 4-34, notice that side $a$ is opposite angle $\theta$ but is the adjacent leg to angle $\left(90^{\circ}-\theta\right)$. Likewise, side $b$ is adjacent to angle $\theta$ but opposite angle $\left(90^{\circ}-\theta\right)$.

From these relationships, we have

$$
\begin{aligned}
& \sin \theta=\cos \left(90^{\circ}-\theta\right)=\frac{a}{c} \\
& \tan \theta=\cot \left(90^{\circ}-\theta\right)=\frac{a}{b} \\
& \sec \theta=\csc \left(90^{\circ}-\theta\right)=\frac{c}{b}
\end{aligned}
$$



Figure 4-34

Notice that the sine of angle $\theta$ equals the cosine of the complement $\left(90^{\circ}-\theta\right)$. For this reason, the sine and cosine functions are called cofunctions. More generally, for an acute angle $\theta$, two trigonometric functions $f$ and $g$ are cofunctions if

$$
f(\theta)=g\left(90^{\circ}-\theta\right) \text { and } g(\theta)=f\left(90^{\circ}-\theta\right)
$$

## Cofunction Identities

Cofunctions of complementary angles are equal.

$$
\begin{array}{ll}
\sin \theta=\cos \left(90^{\circ}-\theta\right) & \cos \theta=\sin \left(90^{\circ}-\theta\right) \\
\sin \theta=\cos \left(\frac{\pi}{2}-\theta\right) & \cos \theta=\sin \left(\frac{\pi}{2}-\theta\right) \\
\tan \theta=\cot \left(90^{\circ}-\theta\right) & \cot \theta=\tan \left(90^{\circ}-\theta\right) \\
\tan \theta=\cot \left(\frac{\pi}{2}-\theta\right) & \csc \theta=\tan \left(\frac{\pi}{2}-\theta\right) \\
\sec \theta=\sec \left(90^{\circ}-\theta\right) \\
\sec \theta=\csc \left(\frac{\pi}{2}-\theta\right) & \csc \theta=\sec \left(\frac{\pi}{2}-\theta\right)
\end{array}
$$

Sine and cosine are cofunctions.

Tangent and cotangent are cofunctions.

Secant and cosecant are cofunctions.

## Insights

Cosine means "complement's sine."
Cotangent means "complement's tangent."
Cosecant means "complement's secant."

## Example 6: Using the Cofunction Identities

For each function value, find a cofunction with the same value.
a. $\cot 15^{\circ}=2+\sqrt{3}$
b. $\quad \cos \frac{\pi}{6}=\frac{\sqrt{3}}{2}$

## Solution:

a. The complement of $15^{\circ}$ is $\left(90^{\circ}-15^{\circ}\right)=75^{\circ}$.

Cotangent and tangent are cofunctions.
Therefore, $\cot 15^{\circ}=\tan 75^{\circ}=2+\sqrt{3}$.
b. The complement of $\frac{\pi}{6}$ is $\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\left(\frac{3 \pi}{6}-\frac{\pi}{6}\right)=\frac{2 \pi}{6}=\frac{\pi}{3}$.

Cosine and sine are cofunctions.
Therefore, $\cos \frac{\pi}{6}=\sin \frac{\pi}{3}=\frac{\sqrt{3}}{2}$.

## Apply the Skills

6. For each function value, find a cofunction with the same value.
a. $\tan 22.5^{\circ}=\sqrt{2}-1$
b. $\sec \frac{\pi}{3}=2$

## Learn: Use Trigonometric Functions in Applications

In many applications of right triangle trigonometry, angles are measured relative to an imaginary horizontal line of reference. An angle of elevation is an angle measured upward from a horizontal line of reference. An angle of depression is an angle measured downward from a horizontal line of reference (Figure 4-35).


Figure 4-35

## Example 7: Using Trigonometry to Find the Height of a Tree

Palm trees are easy to transplant relative to similarly sized broad-leaved trees. At one tree farm, palm trees are harvested once they reach a height of 20 ft . Suppose that the distance along the ground from the base of the tree to a farm worker is 22 ft . Then, an instrument called a clinometer is held at an eye level of 6 ft to measure the angle of elevation to the top of the tree. If the angle of elevation is $30.2^{\circ}$, is the tree tall enough to harvest?


## Solution:

To find the height of the tree, we will use the right triangle to find the vertical leg $y$ and then add 6 ft .

Relative to the $30.2^{\circ}$ angle, we know the adjacent side is 22 ft and we want to know the length of the opposite side $y$.


The two trigonometric functions that are defined by the opposite and adjacent legs are tangent and cotangent. To solve for $y$, we can use the relationship $\tan 30.2^{\circ}=\frac{y}{22}$ or $\cot 30.2^{\circ}=\frac{22}{y}$. Using the tangent function we have

$$
\begin{array}{rlrl}
\tan 30.2^{\circ} & =\frac{y}{22} & & \begin{array}{l}
\text { We use the tangent function because it is easily } \\
\text { approximated on a calculator. }
\end{array} \\
y & =22 \tan 30.2^{\circ} & & \text { Multiply both sides by } 22 . \\
y \approx 12.8 \mathrm{ft} & & \text { Use a calculator to approximate } 22 \mathrm{tan} 30.2^{\circ} . \\
h & \approx 12.8 \mathrm{ft}+6 \mathrm{ft} & & \text { To find the height of the tree, add } 6 \mathrm{ft} .
\end{array}
$$

## Check Your Work

To approximate $\tan 30.2^{\circ}$ on a calculator, be sure that the calculator is in degree mode.


## Apply the Skills

7. Suppose that the distance along the ground from a farm worker to the base of a palm tree is 16 ft . If the angle of elevation from an eye level of 5 ft to the top of the tree is $45.9^{\circ}$, is the tree tall enough to harvest (at least 20 ft tall)?

## Point of Interest

A variety of instruments are used to measure angles. Historically, quadrants and sextants were used for early celestial navigation. Theodolites (or transits) are used for surveying, and clinometers are used in construction to measure slope.


## Example 8: Using Trigonometry in Aeronautical Science

A pilot flying an airplane at an altitude of 1 mi sights a point at the end of a runway. The angle of depression is $3^{\circ}$. What is the distance $d$ from the plane to the point on the runway? Round to the nearest tenth of a mile.


## Solution:

The complement of the $3^{\circ}$ angle of depression is the acute angle $87^{\circ}$ within the right triangle.


Relative to the $87^{\circ}$ angle, we know that the adjacent side is 1 mi , and we want to find the length of the hypotenuse. The two trigonometric functions that are defined by the adjacent leg and hypotenuse are the cosine and secant functions.

Using the relationship $\cos 87^{\circ}=\frac{1}{d}$, we have $d=\frac{1}{\cos 87^{\circ}} \approx 19.1$.
Therefore, the plane is approximately 19.1 mi from the end of the runway.

## Apply the Skills

8. If a 15 -ft ladder is leaning against a wall at an angle of $62^{\circ}$ with the ground, how high up the wall will the ladder reach? Round to the nearest tenth of a foot.

## Practice 4-3

## Practice Exercises

## Prerequisite Review

R.1. Simplify the radical. $\sqrt{45}$
R.2. Rationalize the denominator. $\frac{14}{\sqrt{7}}$
R.3. Rationalize the denominator. $\frac{9}{\sqrt{18}}$
R.4. Use the Pythagorean theorem to find the length of the missing side $c$.

R.5. In a right triangle, one leg measures 24 ft and the hypotenuse measures 26 ft . Find the length of the missing side.
R.6. Find the measure of angle $a$.


## Concept Connections

1. In a right triangle with an acute angle $\theta$, the longest side in the triangle is the
$\qquad$ and is opposite the angle.
2. The leg of a right triangle that lies on one ray of angle $\theta$ is called the $\qquad$ leg, and the leg that lies across the triangle from $\theta$ is called the $\qquad$ leg.
3. The mnemonic device "SOH-CAH-TOA" stands for the ratios $\sin \theta=\frac{\square}{\square} \cos \theta=\frac{\square}{\square}$, and $\tan \theta=\frac{\square}{\square}$.
4. Complete the reciprocal and quotient identities. $\csc \theta=\frac{1}{\square}, \sec \theta=\frac{1}{\square}, \cot \theta=\frac{1}{\square}$ or

5. Given the lengths of two sides of a right triangle, we can find the length of the third side by using the $\qquad$ theorem.
6. An $\qquad$ right triangle is a right triangle in which the two legs are of equal length. The two acute angles in this triangle each measure $\qquad$ ${ }^{\circ}$.
7. The length of the shorter leg of a $30^{\circ}-60^{\circ}-90^{\circ}$ triangle is always
$\qquad$ the length of the hypotenuse.
8. For the six trigonometric functions $\sin \theta$, $\cos \theta, \tan \theta, \csc \theta, \sec \theta$, and $\cot \theta$, identify the three reciprocal pairs.
9. $\tan \theta$ is the $\qquad$ of $\sin \theta$ and $\cos \theta$.
10. Complete the Pythagorean identities. $\sin ^{2} \theta+$ $\qquad$ $=1$, $\qquad$ $+1=\sec ^{2} \theta$, $1+\cot ^{2} \theta=$
11. The two acute angles in a right triangle are complementary because the sum of their measures is $\qquad$ ${ }^{\circ}$.
12. The sine of angle $\theta$ equals the cosine of
$\qquad$ For this reason, the sine and cosine functions are called
$\qquad$ .

## Learn: Evaluate Trigonometric Functions of Acute Angles

## For Exercises 13-14, find the exact values of the six trigonometric functions for angle $\theta$.

13. a.

b.

14. a.

b.


For Exercises 15-18, first use the Pythagorean theorem to find the length of the missing side. Then find the exact values of the six trigonometric functions for angle $\theta$.
(See Example 1, p. 496)
15.

16.
17.

18.


For Exercises 19-22, first use the Pythagorean theorem to find the length of the missing side of the right triangle. Then find the exact values of the six trigonometric functions for the angle $\theta$ opposite the shortest side. (See Example 1, p. 496)
19. $\mathrm{Leg}=2 \mathrm{~cm}$, leg $=6 \mathrm{~cm}$
20. Leg $=3 \mathrm{~cm}$, leg $=15 \mathrm{~cm}$
21. Leg $=2 \sqrt{7} \mathrm{~m}$, hypotenuse $=2 \sqrt{11} \mathrm{~m}$
22. Leg $=5 \sqrt{3}$ in., hypotenuse $=2 \sqrt{21} \mathrm{in}$.

In Exercises 23-24, given the value of one trigonometric function of an acute angle $\theta$, find the values of the remaining five trigonometric functions of $\theta$. (See Example 2, p. 498)
23. $\tan \theta=\frac{4}{7}$
24. $\cos \theta=\frac{\sqrt{10}}{10}$

For Exercises 25-30, assume that $\theta$ is an acute angle. (See Example 2, p. 498)
25. If $\cos \theta=\frac{\sqrt{21}}{7}$, find $\csc \theta$.
26. If $\sin \theta=\frac{\sqrt{17}}{17}$, find $\cot \theta$.
27. If $\sec \theta=\frac{3}{2}$, find $\sin \theta$.
28. If $\csc \theta=3$, find $\cos \theta$.
29. If $\tan \theta=\frac{\sqrt{15}}{9}$, find $\cos \theta$.
30. If $\cot \theta=\frac{\sqrt{3}}{2}$, find $\cos \theta$.

For Exercise 31, use the isosceles right triangle and the $30^{\circ}-60^{\circ}-90^{\circ}$ triangle to complete the table. (See Examples 3-4, pp. 498-499)
31.

| $\theta$ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | $\csc \theta$ | $\sec \theta$ | $\cot \theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $30^{\circ}=\frac{\pi}{6}$ |  |  |  |  |  |  |
| $45^{\circ}=\frac{\pi}{4}$ |  |  |  |  |  |  |
| $60^{\circ}=\frac{\pi}{3}$ |  |  |  |  |  |  |



32.a. Evaluate $\sin 60^{\circ}$.
b. Evaluate $\sin 30^{\circ}+\sin 30^{\circ}$.
c. Are the values in parts (a) and (b) the same?

For Exercises 33-38, find the exact value of each expression without the use of a calculator. (See Example 5, p. 501)
33. $\sin \frac{\pi}{4} \cdot \cot \frac{\pi}{3}$
34. $\tan \frac{\pi}{6} \cdot \csc \frac{\pi}{4}$
$35.3 \cos \frac{\pi}{3}+4 \sin \frac{\pi}{6}$
36. $2 \cos \frac{\pi}{6}-5 \sin \frac{\pi}{3}$
37. $\csc ^{2} 60^{\circ}-\sin ^{2} 45^{\circ}$
38. $\cos ^{2} 45^{\circ}+\tan ^{2} 60^{\circ}$

## Learn: Use Fundamental Trigonometric Identities

For Exercises 39-44, determine whether the statement is true or false for an acute angle $\theta$ by using the fundamental identities. If the statement is false, provide a counterexample by using a special angle: $\frac{\pi}{3}, \frac{\pi}{4}$, or $\frac{\pi}{6}$.
39. $\sin \theta \cdot \tan \theta=1$
40. $\cos ^{2} \theta \cdot \tan ^{2} \theta=\sin ^{2} \theta$
41. $\sin ^{2} \theta+\tan ^{2} \theta+\cos ^{2} \theta=\sec ^{2} \theta$
42. $\csc \theta \cdot \cot \theta=\sec \theta$
43. $\frac{1}{\tan \theta} \cdot \cot \theta+1=\csc ^{2} \theta$
44. $\sin \theta \cdot \cos \theta \cdot \tan \theta+1=\cos ^{2} \theta$

For Exercises 45-50, given the function value, find a cofunction of another angle with the same value. (See Example 6, p. 503)
45. $\tan 75^{\circ}=2+\sqrt{3}$
46. $\sec \frac{\pi}{12}=\sqrt{6}-\sqrt{2}$
47. $\csc \frac{\pi}{3}=\frac{2 \sqrt{3}}{3}$
48. $\sin 15^{\circ}=\frac{\sqrt{6}-\sqrt{2}}{4}$
49. $\cos \frac{\pi}{4}=\frac{\sqrt{2}}{2}$
50. $\cot \frac{\pi}{6}=\sqrt{3}$

For Exercises 51-54, use a calculator to approximate the function values to 4 decimal places. Be sure that your calculator is in the correct mode.
51. a. $\cos 48.2^{\circ}$
b. $\sin 2^{\circ} 55^{\prime} 42^{\prime \prime}$
c. $\tan \frac{3 \pi}{8}$
52. a. $\sin 12.6^{\circ}$
b. $\tan 19^{\circ} 36^{\prime} 18^{\prime \prime}$
c. $\cos \left(\frac{5 \pi}{22}\right)$
53. a. $\csc 39.84^{\circ}$
b. $\sec \frac{\pi}{18}$
c. $\cot 0.8$
54. a. $\cot 18.46^{\circ}$
b. $\csc \frac{2 \pi}{9}$
c. $\sec 1.25$

## Learn: Use Trigonometric Functions in Applications

55. An observer at the top of a 462 -ft cliff measures the angle of depression from the top of a cliff to a point on the ground to be $5^{\circ}$. What is the distance from the base of the cliff to the point on the ground? Round to the nearest foot. (See Example 7, p. 504)
56. A lamppost casts a shadow of 18 ft when the angle of elevation of the Sun is $33.7^{\circ}$. How high is the lamppost? Round to the nearest foot.
57. A 30 -ft boat ramp makes a $7^{\circ}$ angle with the water. What is the height of the ramp above the water at the ramp's highest point?
Round to the nearest tenth of a foot.
(See Example 8, p. 506)
58. A backyard slide is designed for a child's playground. If the top of the 10 - ft slide makes an angle of $58^{\circ}$ from the vertical, how far out from the base of the steps will the slide extend? Round to the nearest tenth of a foot.

59. The Lookout Mountain Incline Railway, located in Chattanooga, Tennessee, is 4972 ft long and runs up the side of the mountain at an average incline of $17^{\circ}$. What is the gain in altitude? Round to the nearest foot.
60. A 12-ft ladder leaning against a house makes a $64^{\circ}$ angle with the ground. Will the ladder reach a window sill that is 10.5 ft up from the base of the house?
61. According to National Football League (NFL) rules, all crossbars on goalposts must be 10 ft from the ground. However, teams are allowed some freedom on how high the vertical posts on each end may extend, as long as they measure at least 30 ft . A measurement on an NFL field taken 100 yd from the goalposts yields an angle of $7.8^{\circ}$ from the ground to the top of the posts. If the crossbar is 10 ft from the ground, do the goalposts satisfy the NFL rules?

62. A scientist standing at the top of a mountain 2 mi above sea level measures the angle of depression to the ocean horizon to be $1.82^{\circ}$. Use this information to approximate the radius of the Earth to the nearest mile. (Hint: The line of sight $\overleftrightarrow{A B}$ is tangent to the Earth and forms a right angle with the radius at the point of tangency.)

63. A scenic overlook along the Pacific Coast Highway in Big Sur, California, is 280 ft above sea level. A 6 -ft-tall hiker standing at the overlook sees a sailboat and estimates the angle of depression to be $30^{\circ}$. Approximately how far off the coast is the sailboat? Round to the nearest foot.
64. An airplane traveling 400 mph at a cruising altitude of 6.6 mi begins its descent. If the angle of descent is $2^{\circ}$ from the horizontal, determine the new altitude after 15 min . Round to the nearest tenth of a mile.

## Mixed Exercises

For Exercises 65-70, given that $\sin A=\frac{4}{5}$, $\tan B=\frac{5}{12}, \cos C=\frac{8}{17}$, and $\cot D=\frac{24}{7}$, evaluate each expression. Assume that $A, B, C$, and $D$ are all on the interval $\left(0, \frac{\pi}{2}\right)$ or $\left(0^{\circ}, 90^{\circ}\right)$.
65. $\cos A \cdot \sec D \cdot \cot C$
66. $\sin B \cdot \csc C \cdot \cos D$
67. $\sin A \cdot \csc B \cdot \sec D \cdot \cos B$
68. $\csc ^{2} A \cdot \sin D \cdot \cot B$
69. $\sin \left(90^{\circ}-B\right) \cdot \tan D \cdot \cot C \cdot \csc A$
70. $\cos \left(\frac{\pi}{2}-A\right) \cdot \tan C \cdot \cos D \cdot \sin B$

For Exercises 71-75, use the fundamental trigonometric identities as needed.
71. Given that $\sin x^{\circ} \approx 0.3746$, approximate the given function values. Round to 4 decimal places.
a. $\cos (90-x)^{\circ}$
b. $\cos x^{\circ}$
c. $\tan x^{\circ}$
d. $\sin (90-x)^{\circ}$
e. $\cot (90-x)^{\circ}$
f. $\csc x^{\circ}$
72. Given that $\cos x \approx 0.6691$, approximate the given function values. Round to 4 decimal places.
a. $\sin x$
b. $\sin \left(\frac{\pi}{2}-x\right)$
c. $\tan x$
d. $\cos \left(\frac{\pi}{2}-x\right)$
e. $\sec x$
f. $\cot \left(\frac{\pi}{2}-x\right)$
73. Given that $\cos \frac{\pi}{12}=\frac{\sqrt{2}+\sqrt{6}}{4}$, give the exact function values.
a. $\sin \frac{5 \pi}{12}$
b. $\sin \frac{\pi}{12}$
c. $\sec \frac{\pi}{12}$
74. Given that $\tan 36^{\circ}=\sqrt{5-2 \sqrt{5}}$, give the exact function values.
a. $\sec 36^{\circ}$
b. $\csc 54^{\circ}$
c. $\cot 54^{\circ}$
75. Simplify each expression to a single trigonometric function.
a. $\sec \theta \cdot \tan \left(\frac{\pi}{2}-\theta\right)$
b. $\cot ^{2} \theta \cdot \csc \left(90^{\circ}-\theta\right) \cdot \sin \theta$
76. Find the lengths $a, b, y$, and $c$. Round to the nearest tenth of a centimeter.

77. Find the exact lengths $x, y$, and $z$.


## Write About It

78. Use the figure to explain why $\tan \theta=\cot \left(90^{\circ}-\theta\right)$.

79. Explain the difference between an angle of elevation and an angle of depression.

## Expanding Your Skills

For Exercises 80-85, simplify each expression to obtain one of the following expressions.
a. 1
b. $\cos \theta$
c. $\tan \theta$
80. $\frac{\cos \theta \csc \theta}{\cot \theta}$
81. $\frac{1}{\csc \theta \tan \theta}$
82. $\frac{\sin \theta \cos \theta}{1-\sin ^{2} \theta}$
83. $\sec \theta-\tan \theta \sin \theta$
84. $\frac{\tan ^{2} \theta+1}{\tan \theta+\cot \theta}$
85. $\frac{\sin ^{2} \theta \sec \theta}{1+\sec \theta}+\cos \theta$
86. An athlete is in a boat at point $A, \frac{1}{4} \mathrm{mi}$ from the nearest point $D$ on a straight shoreline. She can row at a speed of 3 mph and run at a speed of 6 mph . Her planned workout is to row to point $D$ and then run to point $C$ farther down the shoreline. However, the current pushes her at an angle of $24^{\circ}$ from her original path so that she comes ashore at point $B, 2$ mi from her final destination at point $C$. How many minutes will her trip take? Round to the nearest minute.

87. Given that the distance from $A$ to $B$ is 60 ft , find the distance from $B$ to $C$. Round to the nearest foot.

88. In the figure, $\overline{C D}=15, \overline{D E}=8, \tan \alpha=\frac{4}{3}$, and $\sin \beta=\frac{3}{5}$. Find the lengths of
a. $\overline{A C}$
b. $\overline{A D}$
c. $\overline{D B}$
d. $\overline{B E}$
e. $\overline{A B}$

89. For the given triangle, show that $a=h \cot A-h \cot B$.

90. Show that the area $A$ of the triangle is given by $A=\frac{1}{2} b c \sin A$.

91. Use a cofunction relationship to show that the product $\left(\tan 1^{\circ}\right)\left(\tan 2^{\circ}\right)\left(\tan 3^{\circ}\right)^{-\ldots} \cdot$ $\left(\tan 87^{\circ}\right)\left(\tan 88^{\circ}\right)\left(\tan 89^{\circ}\right)$ is equal to 1 .

## Technology Connections

For Exercises 92-95, use a calculator to approximate the values of the left- and righthand sides of each statement for $A=30^{\circ}$ and $B=45^{\circ}$. Based on the approximations from your calculator, determine if the statement appears to be true or false.
92. a. $\sin (A+B)=\sin A+\sin B$
b. $\quad \sin (A+B)=\sin A \cos B+\cos A \sin B$
93. a. $\tan (A-B)=\tan A-\tan B$
b. $\tan (A-B)=\frac{\tan A-\tan B}{1+\tan A \tan B}$
94. a. $\cos \left(\frac{A}{2}\right)=\sqrt{\frac{1+\cos A}{2}}$
b. $\cos \left(\frac{A}{2}\right)=\frac{1}{2} \cos A$
95. a. $\tan \left(\frac{B}{2}\right)=\frac{1-\cos B}{\sin B}$
b. $\tan \left(\frac{B}{2}\right)=\frac{\sin B}{1+\cos B}$

## Point of Interest

Washington, D.C., is a unique American city in that it was established by the United States Constitution to serve as the nation's capital. After land was ceded by Maryland and Virginia, the seat of the national government was built along the Potomac River in what is now called the District of Columbia. Interestingly, one of the surveyors tasked with mapping the land and creating the boundaries for the city was a free African American man named Benjamin Banneker.


Benjamin Banneker was born in 1731 in the then slave state of Maryland. His mother, a free Black woman, and his father, a freed man, ran a family farm near Baltimore. Benjamin's life was somewhat of an historical anomaly. He learned to read from his maternal grandmother and for a short time attended a Quaker school, but like many people of color during that time, he was largely self-educated. He was an intellectually curious man with a knack for mathematics and invention. Among his accomplishments in early life, he is known for constructing an irrigation system for the family farm and hand-carving a wooden clock that kept near perfect time.

After meeting the prominent land surveyor George Ellicott, Banneker developed a keen interest in astronomy and surveying and quickly excelled in both fields. In 1791, Ellicott hired Banneker
 to assist in surveying the territory for the nation's capital.
Banneker also used his knowledge of mathematics and astronomy to publish his own widely distributed almanac that included astronomical information (information on the tides, phases of the Moon, eclipses, and solar table calculations), along with medical treatments of the time and his own personal essays.
96. A zip line is to be built between two towers labeled $A$ and $B$ across a wetland area. To approximate the distance of the zip line, a surveyor marks a third point $C$, a distance of 175 ft from one end of the zip line and perpendicular to the zip line. The measure of $\angle A C B$ is $74.5^{\circ}$. How long is the zip line? Round to the nearest foot.

97. To determine the width of a river from point $A$ to point $B$, a surveyor walks downriver 50 ft along a line perpendicular to $\overline{A B}$ to a new position at point $C$. The surveyor determines that the measure of $\angle A C B$ is $60^{\circ}$. Find the exact width of the river from point $A$ to point $B$.


Go online for more practice problems.

Lesson 4-4
Trigonometric Functions of Any Angle

## What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate Trigonometric Functions of Any Angle
- Determine Reference Angles
- Evaluate Trigonometric Functions Using Reference Angles


## Learn: Evaluate Trigonometric Functions of Any Angle

In many applications we encounter angles that are not acute. For example, a robotic arm may have a range of motion of $360^{\circ}$, or an object may rotate in a repeated circular pattern through all possible angles. For these situations, we need to extend the definition of trigonometric functions to any angle.

In Figure 4-36, the real number $t$ corresponds to a point $Q(a, b)$ on the unit circle. Angle $\theta$ is a second quadrant angle passing through point $P(x, y)$ with terminal side passing through $Q$. Suppose that we drop perpendicular line segments from points $P$ and $Q$ to the $x$-axis to form two similar right triangles. For $\triangle O R P$, the hypotenuse $r$ is given by $r=\sqrt{x^{2}+y^{2}}$ and the lengths of the two legs are $|x|$ and $y$. For $\triangle O S Q$, the hypotenuse is 1 unit and the lengths of the legs are $|a|$ and $b$.


Figure 4-36
Since $\triangle O R P$ and $\triangle O S Q$ are similar triangles, the ratios of their corresponding sides are proportional, which implies that $\frac{b}{1}=\frac{y}{r}$. Furthermore, since $\sin t=b$, we have $\sin t=\frac{y}{r}$. We also know that there is a one-to-one correspondence between the real number $t$ and the radian measure of the central angle $\theta$ in standard position with terminal side through $P$. From this correspondence, we define $\sin \theta=\sin t=\frac{y}{r}$. Similar logic can be used to define the other five trigonometric functions for a point $P$ in any quadrant or on the $x$ - or $y$-axes. The results are summarized in Table 4-8 on page 516.

Let $\theta$ be an angle in standard position with point $P(x, y)$ on the terminal side, and let $r=\sqrt{x^{2}+y^{2}} \neq 0$ represent the distance from $P(x, y)$ to $(0,0)$.


TABLE 4-8 Trigonometric Functions of Any Angle

| Trigonometric <br> Function | Restriction |
| :---: | :---: |
| $\sin \theta=\frac{y}{r}$ | none |
| $\cos \theta=\frac{x}{r}$ | none |
| $\tan \theta=\frac{y}{x}$ | $x \neq 0$ |
| $\csc \theta=\frac{r}{y}$ | $y \neq 0$ |
| $\sec \theta=\frac{r}{x}$ | $x \neq 0$ |
| $\cot \theta=\frac{x}{y}$ | $y \neq 0$ |

Point $P$ is on the terminal side of angle $\theta$ and not at the origin, which means that $r=\sqrt{x^{2}+y^{2}} \neq 0$. Therefore, the signs of the values of the trigonometric functions depend solely on the signs of $x$ and $y$. For an acute angle, the values of the trigonometric functions are positive because $x$ and $y$ are both positive in the first quadrant. For angles in the other quadrants, we determine the signs of the trigonometric functions by analyzing the signs of $x$ and $y$ (Figure 4-37). Note that the reciprocal functions cosecant, secant, and cotangent will have the same signs as sine, cosine, and tangent, respectively, for a given value of $\theta$.



Figure 4-37

## Example 1: Evaluating Trigonometric Functions

Let $P(-2,-5)$ be a point on the terminal side of angle $\theta$ drawn in standard position. Find the values of the six trigonometric functions of $\theta$.

## Solution:

We first draw $\theta$ in standard position and label point $P$.
The distance between $P(-2,-5)$ and $(0,0)$ is
$r=\sqrt{x^{2}+y^{2}}=\sqrt{(-2)^{2}+(-5)^{2}}=\sqrt{4+25}=\sqrt{29}$.
We have $x=-2, y=-5$, and $r=\sqrt{29}$.

$\sin \theta=\frac{y}{r}=\frac{-5}{\sqrt{29}}=\frac{-5}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}}=-\frac{5 \sqrt{29}}{29}$
$\cos \theta=\frac{x}{r}=\frac{-2}{\sqrt{29}}=\frac{-2}{\sqrt{29}} \cdot \frac{\sqrt{29}}{\sqrt{29}}=-\frac{2 \sqrt{29}}{29}$
$\tan \theta=\frac{y}{x}=\frac{-5}{-2}=\frac{5}{2}$

$$
\begin{aligned}
& \csc \theta=\frac{r}{y}=\frac{\sqrt{29}}{-5}=-\frac{\sqrt{29}}{5} \\
& \sec \theta=\frac{r}{x}=\frac{\sqrt{29}}{-2}=-\frac{\sqrt{29}}{2} \\
& \cot \theta=\frac{x}{y}=\frac{-2}{-5}=\frac{2}{5}
\end{aligned}
$$

## Apply the Skills

1. Let $P(-5,-7)$ be a point on the terminal side of angle $\theta$ drawn in standard position. Find the values of the six trigonometric functions of $\theta$.

## Learn: Determine Reference Angles

For nonacute angles $\theta$, we often find values of the six trigonometric functions by using the related reference angle.

## Definition of a Reference Angle

Let $\theta$ be an angle in standard position. The reference angle for $\theta$ is the acute angle $\theta^{\prime}$ formed by the terminal side of $\theta$ and the horizontal axis.

Figure 4-38 shows the reference angles $\theta^{\prime}$ for angles on the interval $[0,2 \pi)$ drawn in standard position for each of the four quadrants.





Figure 4-38

## Example 2: Finding Reference Angles

Find the reference angle $\theta^{\prime}$.
a. $\theta=315^{\circ}$
b. $\theta=-195^{\circ}$
c. $\theta=3.5$
d. $\theta=\frac{25 \pi}{4}$

## Solution:

a. $\theta=315^{\circ}$ is a fourth quadrant angle. The acute angle it makes with the $x$-axis is $\theta^{\prime}=360^{\circ}-315^{\circ}=45^{\circ}$.

b. $\theta=-195^{\circ}$ is a second quadrant angle coterminal to $165^{\circ}$. The acute angle it makes with the $x$-axis is $\theta^{\prime}=180^{\circ}-165^{\circ}=15^{\circ}$.

c. $\theta=3.5$ is measured in radians. Since $\pi \approx 3.14$ and $\frac{3 \pi}{2} \approx 4.71$, we know that $\pi<3.5<\frac{3 \pi}{2}$ implying that 3.5 is in the third quadrant. The reference angle is $\theta^{\prime}=3.5-\pi \approx 0.3584$.
d. $\theta=\frac{25 \pi}{4}$ is a first quadrant angle coterminal to $\frac{\pi}{4}$. The angle $\frac{\pi}{4}$ is an acute angle and is its own reference angle. Therefore, $\theta^{\prime}=\frac{\pi}{4}$.


## Apply the Skills

2. Find the reference angle $\theta^{\prime}$.
a. $\theta=150^{\circ}$
b. $\theta=-157.5^{\circ}$
c. $\theta=5$
d. $\theta=\frac{13 \pi}{3}$

## Learn: Evaluate Trigonometric Functions Using Reference Angles

In Figure 4-39, an angle $\theta$ is drawn in standard position along with its accompanying reference angle $\theta^{\prime}$. A right triangle can be formed by dropping a perpendicular line segment from a point $P(x, y)$ on the terminal side of $\theta$ to the $x$-axis. The length of the vertical leg of the triangle is $|y|$, the length of the horizontal leg is $|x|$, and the hypotenuse is $r$.


Figure 4-39

Now suppose we compare the values of the cosine function of $\theta$ and $\theta^{\prime}$.

$$
\cos \theta=\frac{x}{r} \text { and } \cos \theta^{\prime}=\frac{a d j}{\text { hyp }}=\frac{|x|}{r}
$$

Notice that the ratio for $\cos \theta$ has $x$ in the numerator and the ratio for $\cos \theta^{\prime}$ has $|x|$. It follows that $\cos \theta$ and $\cos \theta^{\prime}$ are equal except possibly for their signs. A similar argument holds for the other five trigonometric functions. This leads to the following procedure to evaluate a trigonometric function based on reference angles.

## Evaluating Trigonometric Functions Using Reference Angles

To find the value of a trigonometric function of a given angle $\theta$,

1. Determine the function value of the reference angle $\theta^{\prime}$.
2. Affix the appropriate sign based on the quadrant in which $\theta$ lies.

Before we present examples of this process, take a minute to review the values of the trigonometric functions of $30^{\circ}, 45^{\circ}$, and $60^{\circ}$ (Table 4-9).

TABLE 4-9 Trigonometric Function Values of Special Angles

| $\theta$ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | $\csc \theta$ | $\sec \theta$ | $\cot \theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $30^{\circ}=\frac{\pi}{6}$ | $\frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{3}$ | 2 | $\frac{2 \sqrt{3}}{3}$ | $\sqrt{3}$ |
| $45^{\circ}=\frac{\pi}{4}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | 1 | $\sqrt{2}$ | $\sqrt{2}$ | 1 |
| $60^{\circ}=\frac{\pi}{3}$ | $\frac{\sqrt{3}}{2}$ | $\frac{1}{2}$ | $\sqrt{3}$ | $\frac{2 \sqrt{3}}{3}$ | 2 | $\frac{\sqrt{3}}{3}$ |

## Example 3: Using Reference Angles to Evaluate Functions

Evaluate the functions.
a. $\sin 240^{\circ}$
b. $\tan \left(-225^{\circ}\right)$
c. $\sec \frac{11 \pi}{6}$

## Solution:

a. $\theta=240^{\circ}$ is in Quadrant III.

The reference angle is $240^{\circ}-180^{\circ}=60^{\circ}$
Since $\sin \theta$ is negative in Quadrant III, $\sin 240^{\circ}=-\sin 60^{\circ}=-\left(\frac{\sqrt{3}}{2}\right)=-\frac{\sqrt{3}}{2}$

b. $\theta=-225^{\circ}$ is an angle in Quadrant II, coterminal to $135^{\circ}$.

The reference angle is $180^{\circ}-135^{\circ}=45^{\circ}$.
Since $\tan \theta$ is negative in Quadrant II,
$\tan \left(-225^{\circ}\right)=-\tan 45^{\circ}=-(1)=-1$

c. $\theta=\frac{11 \pi}{6}$ is in Quadrant IV.

The reference angle is $2 \pi-\frac{11 \pi}{6}=\frac{12 \pi}{6}-\frac{11 \pi}{6}=\frac{\pi}{6}$
Since $\cos \theta$ and its reciprocal $\sec \theta$ are positive in Quadrant IV,

$$
\sec \frac{11 \pi}{6}=\sec \frac{\pi}{6}=\frac{2 \sqrt{3}}{3}
$$



## Apply the Skills

3. Evaluate the functions.
a. $\cos \frac{5 \pi}{6}$
b. $\cot \left(-120^{\circ}\right)$
c. $\csc \frac{7 \pi}{4}$

## Example 4: Evaluate Trigonometric Functions

Evaluate the functions.
a. $\sec \frac{9 \pi}{2}$
b. $\sin \left(-510^{\circ}\right)$

## Solution:

a. $\frac{9 \pi}{2}$ is coterminal to $\frac{\pi}{2}$.

The terminal side of $\frac{\pi}{2}$ is on the positive $y$-axis, where we have selected the arbitrary point $(0,1)$.
$\sec \frac{9 \pi}{2}=\sec \frac{\pi}{2}=\frac{r}{x} \rightarrow \frac{1}{0}$ is undefined.

b. $-510^{\circ}$ is coterminal to $210^{\circ}$, which is a third-quadrant angle.
The reference angle is $210^{\circ}-180^{\circ}=30^{\circ}$.
Since $\sin \theta$ is negative in Quadrant III,
$\sin \left(-510^{\circ}\right)=-\sin 30^{\circ}=-\left(\frac{1}{2}\right)=-\frac{1}{2}$


## Apply the Skills

4. Evaluate the functions.
a. $\csc (11 \pi)$
b. $\cos \left(-600^{\circ}\right)$

In Example 5, we use a reference angle to determine the values of trigonometric functions based on given information about $\theta$.

## Example 5: Evaluating Trigonometric Functions

Given $\sin \theta=-\frac{4}{7}$ and $\cos \theta>0$, find $\cos \theta$ and $\tan \theta$.

## Solution:

First note that $\sin \theta<0$ and $\cos \theta>0$ in Quadrant IV.
We can label the reference angle $\theta^{\prime}$ and then draw a representative triangle with opposite leg of length 4 units and hypotenuse of length 7 units.


Using the Pythagorean theorem, we can determine the length of the adjacent leg.

$$
\begin{aligned}
a^{2}+(4)^{2} & =(7)^{2} \\
a^{2}+16 & =49 \\
a^{2} & =33 \\
a & =\sqrt{33}
\end{aligned}
$$



Choose the positive square root for the length of a side of a triangle.
$\cos \theta=\cos \theta^{\prime}=\frac{\sqrt{33}}{7}$
$\cos \theta$ is positive in Quadrant IV.
$\tan \theta=-\tan \theta^{\prime}=-\frac{4}{\sqrt{33}}=-\frac{4}{\sqrt{33}} \cdot \frac{\sqrt{33}}{\sqrt{33}}=-\frac{4 \sqrt{33}}{33}$
$\tan \theta$ is negative in Quadrant IV.

## Insights

With the reference angle and representative triangle drawn in Quadrant IV, we can find point $P(\sqrt{33},-4)$ on the terminal side of $\theta$.

$$
\begin{aligned}
& \cos \theta=\frac{x}{r}=\frac{\sqrt{33}}{7} \\
& \tan \theta=\frac{y}{x}=\frac{-4}{\sqrt{33}}
\end{aligned}
$$

## Apply the Skills

5. Given $\cos \theta=-\frac{3}{8}$ and $\sin \theta<0$, find $\sin \theta$ and $\tan \theta$.

The identities involving trigonometric functions of acute angles presented in Lesson 4.3 are also true for trigonometric functions of non-acute angles provided that the functions are well defined at $\theta$.

Pythagorean Identities
$\sin ^{2} \theta+\cos ^{2} \theta=1$
$\tan ^{2} \theta+1=\sec ^{2} \theta$
$1+\cot ^{2} \theta=\csc ^{2} \theta$

Reciprocal Identities
$\csc \theta=\frac{1}{\sin \theta}$
$\sec \theta=\frac{1}{\cos \theta}$
$\cot \theta=\frac{1}{\tan \theta}$

Quotient Identities

$$
\begin{aligned}
& \tan \theta=\frac{\sin \theta}{\cos \theta} \\
& \cot \theta=\frac{\cos \theta}{\sin \theta}
\end{aligned}
$$

In Example 6, we are given information about an angle $\theta$ and will find the values of the trigonometric functions by applying the fundamental identities and by using reference angles.

## Example 6: Using Fundamental Identities

Given $\cos \theta=-\frac{3}{5}$ for $\theta$ in Quadrant II, find $\sin \theta$ and $\tan \theta$.

## Solution:

## Using Identities

$$
\begin{aligned}
\sin ^{2} \theta+\cos ^{2} \theta & =1 \\
\sin ^{2} \theta+\left(-\frac{3}{5}\right)^{2} & =1 \\
\sin ^{2} \theta+\frac{9}{25} & =1
\end{aligned}
$$

$$
\begin{aligned}
\sin ^{2} \theta & =1-\frac{9}{25} \\
\sin ^{2} \theta & =\frac{25}{25}-\frac{9}{25} \\
\sin ^{2} \theta & =\frac{16}{25} \\
\sin \theta & = \pm \frac{4}{5}
\end{aligned}
$$

In Quadrant II, $\sin \theta>0$. Therefore, choose $\sin \theta=\frac{4}{5}$.
$\tan \theta=\frac{\sin \theta}{\cos \theta}=\frac{\frac{4}{5}}{-\frac{3}{5}}=-\frac{4}{3}$

## Alternative Approach

Label the reference angle $\theta^{\prime}$ and then draw a representative triangle with adjacent leg of length 3 units and hypotenuse of length 5 units.

Using the Pythagorean theorem, we can determine the length of the opposite leg.

$$
\begin{aligned}
(3)^{2}+b^{2} & =(5)^{2} \\
9+b^{2} & =25 \\
b^{2} & =16 \\
b & =4
\end{aligned}
$$



In Quadrant II, $\sin \theta>0$.
Therefore, $\sin \theta=\sin \theta^{\prime}=\frac{4}{5}$.
In Quadrant II, $\tan \theta<0$.
Therefore, $\tan \theta=-\tan \theta^{\prime}=-\frac{4}{3}$.

## Insights

With the reference angle and representative triangle drawn in Quadrant II, we can find point $P(-3,4)$ on the terminal side of $\theta$. Therefore, $x=-3, y=4$, and $r=5$.

$$
\begin{aligned}
\sin \theta & =\frac{y}{r}=\frac{4}{5} \\
\tan \theta & =\frac{y}{x}=\frac{4}{-3}
\end{aligned}
$$

## Apply the Skills

6. Given $\sin \theta=-\frac{5}{13}$ for $\theta$ in Quadrant IV, find $\cos \theta$ and $\tan \theta$.

## Practice Exercises

## Prerequisite Review

R.1. Find the length of the missing side of the right triangle.

R.2. Find the length of the third side using the Pythagorean theorem. Round the answer to the nearest tenth of a foot.

R.3. Use the distance formula to find the distance from $(2,5)$ to $(-3,10)$.

## Concept Connections

1. The distance from the origin to a point $P(x, y)$ is given by $\qquad$ .
2. Angles with terminal sides on the coordinate axes are referred to as $\qquad$ angles.
3. If $\theta$ is an angle in standard position, the
$\qquad$ angle for $\theta$ is the acute angle $\theta^{\prime}$ formed by the terminal side of $\theta$ and the horizontal axis.
4. The values $\tan \theta$ and $\sec \theta$ are undefined for odd multiples of $\qquad$ .
5. The values $\cot \theta$ and $\csc \theta$ are undefined for multiples of $\qquad$ .
6. For what values of $\theta$ is $\sin \theta$ greater than 1?

## Learn: Evaluate Trigonometric Functions of Any Angle

7. Let $P(x, y)$ be a point on the terminal side of an angle $\theta$ drawn in standard position and let $r$ be the distance from $P$ to the origin. Fill in the boxes to form the ratios defining the six trigonometric functions.
a. $\sin \theta=\frac{\square}{\square}$
b. $\cos \theta=\frac{\square}{\square}$
c. $\tan \theta=\frac{\square}{\square}$
d. $\csc \theta=\frac{\square}{\square}$
e. $\sec \theta=\square$
f. $\cot \theta=\frac{\square}{\square}$
8. Fill in the cells in the table with the appropriate sign for each trigonometric function for $\theta$ in Quadrants I, II, III, and IV. The signs for the sine function are done for you.

| Quadrant |  |  |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | I | II | III | IV |
| $\sin \theta$ | + | + | - | - |
| $\cos \theta$ |  |  |  |  |
| $\tan \theta$ |  |  |  |  |
| $\csc \theta$ |  |  |  |  |
| $\sec \theta$ |  |  |  |  |
| $\cot \theta$ |  |  |  |  |

For Exercises 9-14, given the stated conditions, identify the quadrant in which $\theta$ lies.
9. $\sin \theta<0$ and $\tan \theta>0$
10. $\csc \theta>0$ and $\cot \theta<0$
11. $\sec \theta<0$ and $\tan \theta<0$
12. $\cot \theta<0$ and $\sin \theta<0$
13. $\cos \theta>0$ and $\cot \theta<0$
14. $\cot \theta<0$ and $\sec \theta<0$

For Exercises 15-20, a point is given on the terminal side of an angle $\theta$ drawn in standard position. Find the values of the six trigonometric functions of $\boldsymbol{\theta}$. (See Example 1, p. 516)
15. $(5,-12)$
16. $(-8,15)$
17. $(-3,-5)$
18. $(2,-3)$
19. $\left(-\frac{3}{2}, 2\right)$
20. $\left(5,-\frac{8}{3}\right)$
21. Complete the table for the given angles.

| $\theta$ | $\sin \theta$ | $\cos \theta$ | $\tan \theta$ | $\csc \theta$ | $\sec \theta$ | $\cot \theta$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| $0^{\circ}=0$ |  |  |  |  |  |  |
| $90^{\circ}=\frac{\pi}{2}$ |  |  |  |  |  |  |
| $180^{\circ}=\pi$ |  |  |  |  |  |  |
| $270^{\circ}=\frac{3 \pi}{2}$ |  |  |  |  |  |  |
| $360^{\circ}=2 \pi$ |  |  |  |  |  |  |

22. Evaluate the expression for $A=90^{\circ}$, $B=180^{\circ}$, and $C=270^{\circ}$.

$$
\frac{\sin A+2 \cos B}{\sec B-3 \csc C}
$$

## Learn: Determine Reference Angles

For Exercises 23-30, find the reference angle for the given angle. (See Example 2, p. 517)

## 23. a. $135^{\circ}$

b. $-330^{\circ}$
c. $660^{\circ}$
d. $-690^{\circ}$
24. a. $-120^{\circ}$
b. $225^{\circ}$
c. $-1035^{\circ}$
d. $510^{\circ}$
25. a. $\frac{2 \pi}{3}$
b. $-\frac{5 \pi}{6}$
C. $\frac{13 \pi}{4}$
d. $-\frac{10 \pi}{3}$
26. a. $-\frac{5 \pi}{4}$
b. $\frac{11 \pi}{6}$
c. $\frac{17 \pi}{3}$
d. $\frac{19 \pi}{6}$
27. a. $\frac{20 \pi}{17}$
b. $-\frac{99 \pi}{20}$
c. $110^{\circ}$
d. $-422^{\circ}$
28. a. $\frac{25 \pi}{11}$
b. $\frac{-27 \pi}{14}$
c. $-512^{\circ}$
d. $1280^{\circ}$
29. a. 1.8
b. $1.8 \pi$
d. $5.1^{\circ}$
30. a. 0.6
b. $0.6 \pi$
c. 100
d. $100^{\circ}$

## Learn: Evaluate Trigonometric Functions Using Reference Angles

For Exercises 31-54, use reference angles to find the exact value. (See Examples 3-4, pp. 519-520)
31. $\sin 120^{\circ}$
33. $\cos \frac{4 \pi}{3}$
35. $\sec \left(-330^{\circ}\right)$
37. $\sec \frac{13 \pi}{3}$
39. $\cot 240^{\circ}$
41. $\tan \frac{5 \pi}{4}$
43. $\cos \left(-630^{\circ}\right)$
45. $\sin \frac{17 \pi}{6}$
47. $\sec 1170^{\circ}$
49. $\csc (-5 \pi)$
51. $\tan \left(-2400^{\circ}\right)$
53. $\cot \frac{42 \pi}{8}$
32. $\cos 225^{\circ}$
34. $\sin \frac{5 \pi}{6}$
36. $\csc \left(-225^{\circ}\right)$
38. $\csc \frac{5 \pi}{3}$
40. $\tan \left(-150^{\circ}\right)$
42. $\cot \left(-\frac{3 \pi}{4}\right)$
44. $\sin 630^{\circ}$
46. $\cos \left(-\frac{11 \pi}{4}\right)$
48. $\csc 750^{\circ}$
50. $\sec 5 \pi$
52. $\cot 900^{\circ}$
54. $\tan \frac{18 \pi}{4}$

For Exercises 55-58, find two angles between $0^{\circ}$ and $360^{\circ}$ for the given condition.
55. $\sin \theta=\frac{1}{2}$
56. $\cos \theta=-\frac{\sqrt{2}}{2}$
57. $\cot \theta=-\sqrt{3}$
58. $\tan \theta=\sqrt{3}$

For Exercises 59-62, find two angles between 0 and $2 \pi$ for the given condition.
59. $\sec \theta=\sqrt{2}$
60. $\csc \theta=-\frac{2 \sqrt{3}}{3}$
61. $\tan \theta=-\frac{\sqrt{3}}{3}$
62. $\cot \theta=1$

In Exercises 63-68, find the values of the trigonometric functions from the given information. (See Example 5, p. 521)
63. Given $\tan \theta=-\frac{20}{21}$ and $\cos \theta<0$, find $\sin \theta$ and $\cos \theta$.
64. Given $\cot \theta=\frac{11}{60}$ and $\sin \theta<0$, find $\cos \theta$ and $\sin \theta$.
65. Given $\sin \theta=-\frac{3}{10}$ and $\tan \theta>0$, find $\cos \theta$ and $\cot \theta$.
66. Given $\cos \theta=-\frac{5}{8}$ and $\csc \theta>0$, find $\sin \theta$ and $\tan \theta$.
67. Given $\sec \theta=\frac{\sqrt{58}}{7}$ and $\cot \theta<0$, find $\csc \theta$ and $\cos \theta$.
68. Given $\csc \theta=-\frac{\sqrt{26}}{5}$ and $\cos \theta<0$, find $\sin \theta$ and $\cot \theta$.

For Exercises 69-74, use fundamental trigonometric identities to find the values of the functions. (See Example 6, p. 522)
69. Given $\cos \theta=\frac{20}{29}$ for $\theta$ in Quadrant IV, find $\sin \theta$ and $\tan \theta$.
70. Given $\sin \theta=-\frac{8}{17}$ for $\theta$ in Quadrant III, find $\cos \theta$ and $\cot \theta$.
71. Given $\tan \theta=-4$ for $\theta$ in Quadrant II, find $\sec \theta$ and $\cot \theta$.
72. Given $\sec \theta=5$ for $\theta$ in Quadrant IV, find $\csc \theta$ and $\cos \theta$.
73. Given $\cot \theta=\frac{5}{4}$ for $\theta$ in Quadrant III, find $\csc \theta$ and $\sin \theta$.
74. Given $\csc \theta=\frac{7}{3}$ for $\theta$ in Quadrant II, find $\cot \theta$ and $\cos \theta$.

## Mixed Exercises

For Exercises 75-76, find the sign of the expression for $\theta$ in each quadrant.
75.

|  |  | I | II | III | IV |
| :--- | :--- | :--- | :--- | :--- | :--- |
| a. | $\sin \theta \cos \theta$ |  |  |  |  |
| b. | $\frac{\tan \theta}{\cos \theta}$ |  |  |  |  |

76. 

|  |  | I | II | III | IV |
| :---: | :---: | :---: | :---: | :---: | :---: |
| a. | $\frac{\cot \theta}{\sin \theta}$ |  |  |  |  |
| b. | $\tan \theta \sec \theta$ |  |  |  |  |

For Exercises 77-84, suppose that $\theta$ is an acute angle. Identify each statement as true or false. If the statement is false, rewrite the statement to give the correct answer for the right side.
77. $\cos \left(180^{\circ}-\theta\right)=-\cos \theta$
78. $\tan \left(180^{\circ}-\theta\right)=\tan \theta$
79. $\tan \left(180^{\circ}+\theta\right)=\tan \theta$
80. $\sin \left(180^{\circ}+\theta\right)=-\sin \theta$
81. $\csc (\pi-\theta)=-\csc \theta$
82. $\sec (\pi+\theta)=\sec \theta$
83. $\cos (\pi+\theta)=-\cos \theta$
84. $\sin (\pi+\theta)=-\sin \theta$

For Exercises 85-90, find the value of each expression.
85. $\sin 30^{\circ} \cdot \cos 150^{\circ} \cdot \sec 60^{\circ} \cdot \csc 120^{\circ}$
86. $\cos 45^{\circ} \cdot \sin 240^{\circ} \cdot \tan 135^{\circ} \cdot \cot 60^{\circ}$
87. $\cos ^{2} \frac{5 \pi}{4}-\sin ^{2} \frac{2 \pi}{3}$
88. $\sin ^{2} \frac{11 \pi}{6}+\cos ^{2} \frac{4 \pi}{3}$
89. $\frac{2 \tan \frac{11 \pi}{6}}{1-\tan ^{2} \frac{11 \pi}{6}}$
90. $\frac{\cot ^{2} \frac{4 \pi}{3}-1}{2 \cot \frac{4 \pi}{3}}$

For Exercises 91-94, verify the statement for the given values.
91. $\sin (A-B)=\sin A \cos B-\cos A \sin B$;
$A=240^{\circ}, B=120^{\circ}$
92. $\cos (B-A)=\cos B \cos A+\sin B \sin A ;$
$A=330^{\circ}, B=120^{\circ}$
93. $\tan (A+B)=\frac{\tan A+\tan B}{1-\tan A \tan B} ; A=210^{\circ}$, $B=120^{\circ}$
94. $\cot (A+B)=\frac{\cot A \cot B-1}{\cot A+\cot B} ; A=300^{\circ}$, $B=150^{\circ}$

For Exercises 95-96, give the exact values if possible. Otherwise, use a calculator and approximate the result to 4 decimal places.
95. a. $\sin 30^{\circ}$
b. $\sin (30 \pi)$
c. $\sin 30$
96. a. $\cos (0.25 \pi)$
b. $\frac{\cos \pi}{0.25}$
c. $\cos \left(25^{\circ}\right)$

## Write About It

97. Explain why neither $\sin \theta$ nor $\cos \theta$ can be greater than 1. Refer to the figure for your explanation.

98. Explain why $\tan \theta$ is undefined at $\theta=\frac{\pi}{2}$ but $\cot \theta$ is defined at $\theta=\frac{\pi}{2}$.

## Expanding Your Skills

For Exercises 99-100, for part (a), find four angles between $0^{\circ}$ and $720^{\circ}$ that satisfy the given condition. For part (b), find four angles between $0^{\circ}$ and $360^{\circ}$ that satisfy the given condition.
99. a. $\cos \theta=-\frac{\sqrt{2}}{2}$
b. $\cos 2 \theta=-\frac{\sqrt{2}}{2}$
100. a. $\tan \theta=\sqrt{3}$
b. $\tan 2 \theta=\sqrt{3}$

For Exercises 101-102, for part (a), find four angles between 0 and $3 \pi$ that satisfy the given condition. For part (b), find four angles between 0 and $\pi$ that satisfy the given condition.
101. a. $\csc \theta=\sqrt{2}$
b. $\csc 3 \theta=\sqrt{2}$
102. a. $\sin \theta=\frac{\sqrt{3}}{2}$
b. $\sin 3 \theta=\frac{\sqrt{3}}{2}$
103. a. Graph the point $(3,4)$ on a rectangular coordinate system and draw a line segment connecting the point to the origin. Find the slope of the line segment.
b. Draw another line segment from the point $(3,4)$ to meet the $x$-axis at a right angle, forming a right triangle with the $x$-axis as one side. Find the tangent of the acute angle that has the $x$-axis as its initial side.
c. Compare the results in part (a) and part (b).
104. The circle shown is centered at the origin with a radius of 1 . The segment $\overline{B D}$ is tangent to the circle at D . Match the length of each segment with the appropriate trigonometric function.
a. $\overline{A C}$
i. $\tan \theta$
b. $\overline{B D}$
ii. $\cos \theta$
c. $\overline{O B}$
iii. $\sin \theta$
d. $\overline{O C}$
iv. $\sec \theta$

105. Circle $A$, with radius $a$, and circle $B$, with radius $b$, are tangent to each other and to $\overline{P Q}$ (see figure). $\overline{P R}$ passes through the center of each circle. Let $x$ be the distance from point $P$ to a point $S$ where $\overline{P R}$ intersects circle $A$ on the left. Let $\theta$ denote $\angle R P Q$.
a. Show that $\sin _{b} \theta=\frac{a}{x+a}$ and $\sin \theta=\frac{b}{x+2 a+b}$.
b. Use the results from part (a) to show that $\sin \theta=\frac{b-a}{b+a}$.


Go online for more practice problems.

## Graphs of Sine and Cosine Functions

## What Will You Learn?

After completing this lesson, you should be able to:

- Graph $y=\sin x$ and $y=\cos x$
- Graph $y=A \sin x$ and $y=A \cos x$
- Graph $y=A \sin B x$ and $y=A \cos B x$
- Graph $y=A \sin (B x-C)+D$ and $y=A \cos (B x-C)+D$
- Model Sinusoidal Behavior


## Learn: Graph $y=\sin x$ and $y=\cos x$

Throughout this text, we have been progressively introducing categories of functions and analyzing their properties and graphs. In each case, we begin with a fundamental "parent" function such as $y=x^{2}$ and then develop interesting variations on the graph by applying transformations. For example, $y=a(x-h)^{2}+k$ is a family of quadratic functions whose properties help us model physical phenomena such as projectile motion.

In a similar manner we will graph the basic functions $y=\sin x$ and $y=\cos x$ and then analyze their variations. To begin, note that we will use variable $x$ (rather than $\theta$ or $t$ ) to represent the independent variable.

To develop the graph of $y=\sin x$ we return to the notion of a number line wrapped onto the unit circle. The points on the number line correspond to the real numbers $x$ in the domain of the sine function. The value of $\sin x$ is the $y$-coordinate of the corresponding point on the unit circle (Figure 4-40).


Figure 4-40
Table 4-10 gives several values of $x$ and the corresponding values of $\sin x$. Plotting these as ordered pairs gives the graph of $y=\sin x$ in Figure 4-41.

| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2 \pi}{3}$ | $\frac{3 \pi}{4}$ | $\frac{5 \pi}{6}$ | $\pi$ | $\frac{7 \pi}{6}$ | $\frac{5 \pi}{4}$ | $\frac{4 \pi}{3}$ | $\frac{3 \pi}{2}$ | $\frac{5 \pi}{3}$ | $\frac{7 \pi}{4}$ | $\frac{11 \pi}{6}$ | $2 \pi$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sin x$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | -1 | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 |



Figure 4-41

## Insights

The decimal approximations of the coordinates involving irrational numbers can be used to sketch the graph. For example,

$$
\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right) \approx(0.79,0.71) .
$$

Recall that the sine function is periodic with period $2 \pi$. This means that the pattern shown between 0 and $2 \pi$ repeats infinitely far to the left and right as denoted by the dashed portion of the curve (Figure 4-41). In Figure 4-42, one complete cycle of the graph of $y=\sin x$ is shown over the interval $[0,2 \pi]$. Notice the relationship between the $y$-coordinates on the unit circle and the graph of $y=\sin x$.


Figure 4-42
In a similar fashion, we can graph $y=\cos x$. Table 4-11 (see page 530) gives several values of $x$ with the corresponding values of $\cos x$. Plotting these as ordered pairs gives the graph of $y=\cos x$ in Figure 4-43.

| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2 \pi}{3}$ | $\frac{3 \pi}{4}$ | $\frac{5 \pi}{6}$ | $\pi$ | $\frac{7 \pi}{6}$ | $\frac{5 \pi}{4}$ | $\frac{4 \pi}{3}$ | $\frac{3 \pi}{2}$ | $\frac{5 \pi}{3}$ | $\frac{7 \pi}{4}$ | $\frac{11 \pi}{6}$ | $2 \pi$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\cos x$ | 1 | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{\sqrt{3}}{2}$ | -1 | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{2}}{2}$ | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{3}}{2}$ | 1 |



Figure 4-43

## Insights

When we define $\sin t=y$ and $\cos t=x$, the values of $x$ and $y$ refer to the coordinates of a point on the unit circle $x^{2}+y^{2}=1$.

When we define $y=\sin x$ and $y=\cos x$, the variable $x$ has taken the role of $t$ (a real number on a number line wrapped onto the unit circle) and $y$ is the corresponding sine or cosine value.

In Figure 4-44, the graphs of $y=\sin x$ and $y=\cos x$ are shown on the interval $[-2 \pi, 2 \pi]$ for comparison. From the graphs, we make the following observations.


Figure 4-44

## Characteristics of the Graphs of $y=\sin x$ and $y=\cos x$

- The domain is $(-\infty, \infty)$.
- The range is $[-1,1]$.
- The period is $2 \pi$.
- The graph of $y=\sin x$ is symmetric with respect to the origin. $y=\sin x$ is an odd function.
- The graph of $y=\cos x$ is symmetric with respect to the $y$-axis. $y=\cos x$ is an even function.
- The graphs of $y=\sin x$ and $y=\cos x$ differ by a horizontal shift of $\frac{\pi}{2}$.


## Learn: Graph $y=A \sin x$ and $y=A \cos x$

Figures 4-45 and 4-46 show one complete cycle of the graphs of $y=\sin x$ and $y=\cos x$. When we graph variations of these functions we want to graph the key points. These are the relative minima, relative maxima, and $x$-intercepts. Notice that the graphs of both the sine and cosine function alternate between a relative minimum or maximum and an $x$-intercept for each quarter of a period.


Figure 4-45


Figure 4-46

We begin analyzing variations of the sine and cosine graphs by graphing functions of the form $y=A \sin x$ and $y=A \cos x$.

## Example 1: Graphing $y=A \sin x$ and $y=A \cos x$

Graph the function and identify the key points on one full period.
a. $y=3 \sin x$
b. $y=-\frac{1}{2} \cos x$

## Solution:

Recall from Lesson 1.6 that the graph of $y=A \cdot f(x)$ is the graph of $y=f(x)$ with

- $A$ vertical stretch if $|A|>1$ or
- $A$ vertical shrink if $0<|A|<1$.
- A reflection across the $x$-axis if $A<0$.
a. Given $y=3 \sin x$, the value of $A$ is 3 , so the graph of $y=3 \sin x$ is the graph of $y=\sin x$ with a vertical stretch by a factor of 3 .

The dashed curve $y=\sin x$ is shown for comparison.
b. Given $y=-\frac{1}{2} \cos x$, the value of $A$ is $-\frac{1}{2}$, so the graph of $y=-\frac{1}{2}$ $\cos x$ is the graph of $y=\cos x$ with a vertical shrink and a reflection across the $x$-axis.

The dashed curve, $y=\cos x$, is shown for comparison.



## Apply the Skills

1. Graph the function and identify the key points on one full period.
a. $y=2 \cos x$
b. $y=-\frac{1}{3} \sin x$

Notice that the graph of $y=3 \sin x$ from Example 1(a) deviates 3 units above and below the $x$-axis. For the graphs of $y=|A| \sin x$ and $y=|A| \cos x$, the value of the vertical scaling factor $|A|$ is called the amplitude of the function. The amplitude represents the amount of deviation from the central position of the sine wave. The amplitude is half the distance between the maximum value of the function and the minimum value of the function. In Example 1(a), the amplitude is $|3|=3$, and in Example 1(b), the amplitude is $\left|-\frac{1}{2}\right|=\frac{1}{2}$.

## Learn: Graph $y=A \sin B x$ and $y=A \cos B x$

Recall from Lesson 1.6 that the graph of $y=f(B x)$ is the graph of $y=f(x)$ with a horizontal shrink or stretch. This means that for $y=\sin x$ and $y=\cos x$, the factor $B$ will affect the period of the graph. For example, Figures $4-47$ and $4-48$ show the graphs of $y=\sin 2 x$ and $y=\sin \frac{1}{2} x$, respectively, as compared to the parent function $y=\sin x$.


Figure 4-47


Figure 4-48

The period of the functions $y=\sin x$ and $y=\cos x$ is $2 \pi$. That is, the graphs of $y=\sin x$ and $y=\cos x$ show one complete cycle on the interval $0 \leq x \leq 2 \pi$. To determine a comparable interval for one complete cycle of the graphs of $y=\sin B x$ and $y=\cos B x$ for $B>0$, we can solve the following inequality.


This tells us that for $B>0$, the period of $y=\sin B x$ and $y=\cos B x$ is $\frac{2 \pi}{B}$.

## Amplitude and Period of the Sine and Cosine Functions

For $y=A \sin B x$ and $y=A \cos B x$ and $B>0$, the amplitude and period are Amplitude $=|A|$ and Period $=\frac{2 \pi}{B}$

To analyze the period of a sine or cosine function with a negative coefficient on $x$, we would first rewrite $y=A \sin (-B x)$ and $y=A \cos (-B x)$ using the odd and even properties. That is, for $B>0$,

Rewrite $y=\sin (-B x)$ as $y=-\sin B x$ because $y=\sin x$ is an odd function.
Rewrite $y=\cos (-B x)$ as $y=\cos B x$ because $y=\cos x$ is an even function.

## Example 2: Graphing $y=A \sin B x$

Given $f(x)=4 \sin 3 x$,
a. Identify the amplitude and period.
b. Graph the function and identify the key points on one full period.

## Solution:

a. $f(x)=4 \sin 3 x$ is in the form $f(x)=A \sin B x$ with $A=4$ and $B=3$.

The amplitude is $|A|=|4|=4$.
The period is $\frac{2 \pi}{B}=\frac{2 \pi}{3}$.
b. The period $\frac{2 \pi}{3}$ is shorter than $2 \pi$, which tells us that the graph is compressed horizontally. For $y=\sin x$, one complete cycle can be graphed for $x$ on the interval $0 \leq x \leq 2 \pi$. For $y=4 \sin 3 x$, one complete cycle can be graphed on the interval defined by $0 \leq 3 x \leq 2 \pi$.

$$
\begin{aligned}
& 0 \leq 3 x \leq 2 \pi \\
& \frac{0}{3} \leq \frac{3 x}{3} \leq \frac{2 \pi}{3} \\
& 0 \leq x \leq \frac{2 \pi}{3}
\end{aligned}
$$



Dividing the period into fourths, we have increments of $\frac{1}{4}\left(\frac{2 \pi}{3}\right)=\frac{\pi}{6}$.
Since the amplitude is 4 , the maximum and minimum points have $y$-coordinates of 4 and -4 .


## Apply the Skills

2. Given $f(x)=2 \sin 4 x$,
a. Identify the amplitude and period.
b. Graph the function and identify the key points on one full period.

## TECHNOLOGY CONNECTIONS

## Graphing a Trigonometric Function

To graph a trigonometric function, first be sure that the calculator is in radian mode. The graph of $f(x)=4 \sin 3 x$ is shown along with the graph of $g(x)=\sin x$ for comparison. Notice that the graph of $f$ deviates more from the $x$-axis than $g$ because the amplitude is greater. The period of $f$ is smaller and as a result, shows the sine wave "compressed" horizontally


## Learn: Graph $y=A \sin (B x-C)+D$ and <br> $$
y=A \cos (B x-C)+D
$$

Recall that the graph of $y=f(x-h)$ is the graph of $y=f(x)$ with a horizontal shift of $|h|$ units. If $h>0$, the shift is to the right, and if $h<0$, the shift is to the left. The graphs of $y=A \sin (B x-C)$ and $y=A \cos (B x-C)$ may have both a change in period and a horizontal shift. In Example 3, we illustrate how to graph such a function.

## Example 3: Graphing $y=A \cos (B x-C)$

Given $y=\cos \left(2 x+\frac{\pi}{2}\right)$,
a. Identify the amplitude and period.
b. Graph the function and identify the key points on one full period.

## Solution:

The function is of the form $y=A \cos (B x-C)$, where $A=1, B=2$, and $C=-\frac{\pi}{2}$.
a. $y=\cos \left(2 x+\frac{\pi}{2}\right)$ can be written as $y=1 \cdot \cos \left(2 x+\frac{\pi}{2}\right)$.

The amplitude is $|A|=|1|=1$.
The period is $\frac{2 \pi}{B}=\frac{2 \pi}{2}=\pi$.
b. To find an interval over which this function completes one cycle, solve the inequality.

$$
0 \leq 2 x+\frac{\pi}{2} \leq 2 \pi
$$

$$
-\frac{\pi}{2} \leq 2 x \leq 2 \pi-\frac{\pi}{2} \quad \text { Subtract } \frac{\pi}{2} .
$$

$$
-\frac{\pi}{2} \leq 2 x \leq \frac{4 \pi}{2}-\frac{\pi}{2} \quad \text { Write the right side with a common }
$$

$$
\text { denominator. Then divide by } 2 \text {. }
$$

| $-\frac{\pi}{2} \leq 2 x \leq \frac{3 \pi}{2}$ |
| :--- |
| "Starting" value <br> for the cycle |
| "Ending" value <br> for the cycle |

Dividing the period into fourths, we have increments of $\frac{1}{4}(\pi)=\frac{\pi}{4}$.


Period is $\pi$.


## Insights

Note that the distance between the "ending" point and "starting" point of the cycle shown is

$$
\begin{aligned}
\frac{3 \pi}{4}-\left(-\frac{\pi}{4}\right) & =\frac{4 \pi}{4} \\
& =\pi
\end{aligned}
$$

which is the period of the function.

## Apply the Skills

3. Given $y=\cos \left(3 x-\frac{\pi}{2}\right)$,
a. Identify the amplitude and period.
b. Graph the function and identify the key points on one full period.

In Example 3, the graph of $y=\cos \left(2 x+\frac{\pi}{2}\right)$ is the graph of $y=\cos x$ compressed horizontally by a factor of 2 and shifted to the left $\frac{\pi}{4}$ units. The shift to the left is called the phase shift. In general, given $y=A \sin (B x-C)$ and $y=A \cos (B x-C)$ for $B>0$, the horizontal transformations are controlled by the variables $B$ and $C$. The value of $B$ controls the horizontal shrink or stretch, and therefore the period of the function. For $B>0$, the phase shift can be determined by solving the following inequality.


The phase shift is a horizontal shift of a trigonometric function. To find the vertical shift, recall that the graph of $y=f(x)+D$ is the graph of $y=f(x)$ shifted $|D|$ units upward for $D>0$ and $|D|$ units downward if $D<0$. We are now ready to summarize the properties of the graphs of $y=A \sin (B x-C)+D$ and $y=A \cos (B x-C)+D$.

## Properties of the General Sine and Cosine Functions

Consider the graphs of $y=A \sin (B x-C)+D$ and $y=A \cos (B x-C)+D$ with $B>0$.

1. The amplitude is $|A|$.
2. The period is $\frac{2 \pi}{B}$.
3. The phase shift is $\frac{C}{B}$.
4. The vertical shift is $D$.
5. One full cycle is given on the interval $0 \leq B x-C \leq 2 \pi$.
6. The domain is $-\infty<x<\infty$.
7. The range is $-|A|+D \leq y \leq|A|+D$.

Example 4: Graphing $y=A \cos (B x-C)+D$
Given $y=2 \cos (4 x-3 \pi)+5$,
a. Identify the amplitude, period, phase shift, and vertical shift.
b. Graph the function and identify the key points on one full period.

## Solution:

a. $y=2 \cos (4 x-3 \pi)+5$ has the form $y=A \cos (B x-C)+D$, where $A=2$,
$B=4, C=3 \pi$, and $D=5$.
The amplitude is $|A|=|2|=2$.
The period is $\frac{2 \pi}{B}=\frac{2 \pi}{4}=\frac{\pi}{2}$.
The phase shift is $\frac{C}{B}=\frac{3 \pi}{4}$.
The vertical shift is $D$. Since $D=5>0$, the shift is upward 5 units.
b. To find an interval over which this function completes one cycle, solve the inequality.

| $0 \leq 4 x-3 \pi \leq 2 \pi$ |  |  |
| :---: | :---: | :---: |
|  | $3 \pi \leq 4 x \leq 2 \pi+3 \pi$ | Add 3 $\pi$. |
| $\frac{3 \pi}{4} \leq \frac{4 x}{4} \leq \frac{5 \pi}{4} \quad$ Divide by 4. |  |  |
|  |  |  |
| $\frac{3 \pi}{4}<x<\frac{5}{4}$ |  |  |
| "Starting" value <br> for the cycle <br> "Ending" value <br> for the cycle |  |  |
|  |  |  |
|  |  |  |

Dividing the period into fourths, we have increments of $\frac{1}{4}\left(\frac{\pi}{2}\right)=\frac{\pi}{8}$.

First sketch the function on the interval $\left[\frac{3 \pi}{4}, \frac{5 \pi}{4}\right]$ without the vertical shift (solid gray curve).

The dashed curve is a continuation of this pattern.

Divide the interval $\left[\frac{3 \pi}{4}, \frac{5 \pi}{4}\right]$ into fourths.
 of


Now apply the vertical shift upwards 5 units


## Apply the Skills

4. Given $y=2 \cos (3 x-\pi)+3$,
a. Identify the amplitude, period, phase shift, and vertical shift.
b. Graph the function and identify the key points on one full period.

In Example 4, notice that the $x$-intercepts of the graph before the vertical shift become the points where the graph intersects the line $y=5$ after the vertical shift. The curve oscillates above and below the line $y=5$ rather than the $x$-axis. In a sense, $y=5$ is the "central" value or "equilibrium" value of the function. This line represents the midpoint of the range, and the amplitude tells us by how much the function deviates from this line.

Example 5: Graphing $y=A \sin (B x-C)+D$
Given $y=3 \sin \left(-\frac{\pi}{4} x-\frac{\pi}{2}\right)-4$,
a. Identify the amplitude, period, phase shift, and vertical shift.
b. Graph the function and identify the key points on one full period.

## Solution:

a. First note that the coefficient on $x$ in the argument is not positive.

We want to write the function in the form $y=A \sin (B x-C)+D$, where $B>0$.

$$
\begin{aligned}
& y=3 \sin \left[-1\left(\frac{\pi}{4} x+\frac{\pi}{2}\right)\right]-4 \\
& y=-3 \sin \left(\frac{\pi}{4} x+\frac{\pi}{2}\right)-4 \\
& y=-3 \sin \left[\frac{\pi}{4} x-\left(-\frac{\pi}{2}\right)\right]+(-4) \\
& A=-3, B=\frac{\pi}{4}, C=-\frac{\pi}{2}, \text { and } D=-4
\end{aligned}
$$

Factor out -1 from the argument.
The sine function is an odd function. $\sin (-x)=-\sin (x)$

Write the equation in the form $y=A \sin (B x-C)+D$, where $B>0$.

The amplitude is $|A|=|-3|=3$.
The period is $\frac{2 \pi}{B}=\frac{2 \pi}{\frac{\pi}{4}}=2 \pi \cdot \frac{4}{\pi}=8$.
The phase shift is $\frac{C}{B}=\frac{-\frac{\pi}{2}}{\frac{\pi}{4}}=-\frac{\pi}{2} \cdot \frac{4}{\pi}=-2$.
The vertical shift is $D$. Since $D=-4<0$, the shift is downward 4 units.
b. To find an interval over which this function completes one cycle, solve the inequality.

$$
\begin{array}{cc}
0 \leq \frac{\pi}{4} x-\left(-\frac{\pi}{2}\right) \leq 2 \pi & \text { Add }-\frac{\pi}{2} \\
-\frac{\pi}{2} \leq \frac{\pi}{4} x \leq \frac{3 \pi}{2} \\
\frac{4}{\pi} \cdot\left(-\frac{\pi}{2}\right) \leq \frac{4}{\pi} \cdot\left(\frac{\pi}{4} x\right) \leq \frac{4}{\pi} \cdot\left(\frac{3 \pi}{2}\right) & \text { Multiply by } \frac{4}{\pi} . \\
-2 \leq x \leq 6 & \begin{array}{c}
\text { "Ending" value } \\
\text { for the cycle }
\end{array}
\end{array}
$$

Dividing the period into fourths, we have increments of $\frac{1}{4}(8)=2$.


First sketch the function without the vertical shift.


## Insights

For the equation $y=-3 \sin \left(\frac{\pi}{4} x+\frac{\pi}{2}\right)$, since $A=-3$, the general shape of the curve "starting" at the phase shift is an inverted sine curve.


Now apply the vertical shift downward 4 units.


## Apply the Skills

5. Given $y=-2 \sin \left(-\frac{\pi}{6} x-\frac{\pi}{2}\right)+1$,
a. Identify the amplitude, period, phase shift, and vertical shift.
b. Graph the function and identify the key points on one full period.

## Check Your Work

A factor from the argument of a function cannot be factored out in front of the function. For example,

$$
f(2 x) \neq 2 f(x) .
$$

In step 2 of Example 5(a), the value -1 that appears in front of the function was not factored out, but rather is a result of the odd function property of the sine function.

$$
\sin (-x)=-\sin (x)
$$

## Learn: Model Sinusoidal Behavior

To this point, we have taken an equation of a sine or cosine function and sketched its graph. Now we reverse the process. In Example 6, we take observed data that follow a "wavelike" pattern similar to a sine or cosine graph and build a model. When the graph of a data set is shaped like a sine or cosine graph, we say the graph is sinusoidal.

## Example 6: Modeling the Level of the Tide

The water level relative to the top of a boat dock varies with the tides. One particular day, low tide occurs at midnight and the water level is 7 ft below the dock. The first high tide of the day occurs at approximately 6:00 A.M., and the water level is 3 ft below the dock. The next low tide occurs at noon and the water level is again 7 ft below the dock.

Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $w(t)=A \cos (B t-C)+D$. The value $w(t)$ is the water level (in ft ) relative to the top of the dock, $t$ hours after midnight.


## Solution:

Plotting the water level at midnight, 6:00 A.M., and noon helps us visualize the curve. By inspection, the curve behaves like a cosine reflected across the $t$-axis and shifted down 5 units.


The amplitude of the curve is half the distance between the highest value and lowest value.
Vertical shift: $D=\frac{-7+(-3)}{2}=-5$
The midpoint of the range gives us the vertical shift.
One complete cycle takes place between $t=0$ and $t=12$ (from low tide to low tide). Therefore, the period $P$ is $12-0=12$.

Therefore, $B=\frac{2 \pi}{12}=\frac{\pi}{6}$.
Since a minimum value of the curve occurs at $t=0$, there is no phase shift for this cosine function, implying that $C=0$.

Finally, the amplitude is 2 , however, we take $A$ to be negative because the graph was reflected across the t -axis before being shifted downward.

$$
w(t)=-2 \cos \left(\frac{\pi}{6} t\right)-5
$$

Substitute $A=-2, B=\frac{\pi}{6}, C=0$, and

$$
D=-5, \text { into } w(t)=A \cos (B t-C)+D .
$$

## Apply the Skills

6. A mechanical metronome uses an inverted pendulum that makes a regular, rhythmic click as it swings to the left and right. With each swing, the pendulum moves 3 in . to the left and right of the center position. The pendulum is initially pulled to the right 3 in . and then released. It returns to its starting position in 0.8 sec . Assuming that this pattern continues indefinitely and behaves like a cosine wave, write a function of the form $x(t)=A \cos (B t-C)+D$. The value $x(t)$ is the horizontal position (in inches) relative to the center line of the pendulum.


## Point of Interest

AM (amplitude modulation) and FM (frequency modulation) radio are ways of broadcasting radio signals by sending electromagnetic waves through space from a transmitter to a receiver. Each method transmits information in the form of electromagnetic waves. AM works by modulating (or varying) the amplitude of the signal while the frequency remains constant. FM works by varying the frequency and keeping the amplitude constant. In 1873, James Maxwell showed mathematically that electromagnetic waves could propagate through free space. Today the far-reaching applications of wireless radio technology include use in televisions, computers, cell phones, and even deep-space radio communications.


## Practice Exercises <br> Prerequisite Review

For Exercises R.1-R.5, solve the equation.
R.1. $h(x)=|x|-1$
R.2. $n(x)=|x+1|$
R.3. $b(x)=2|x|$
R.4. $h(x)=\frac{1}{3}|x|$
R.5. $p(x)=-|x|$

## Concept Connections

1. The value of $\sin x$ (increases/decreases)
$\qquad$ on ( $0, \frac{\pi}{2}$ ) and (increases/ decreases) $\qquad$ on $\left(\frac{\pi}{2}, \pi\right)$.
2. The value of $\cos x$ (increases/decreases)
$\qquad$ on ( $0, \frac{\pi}{2}$ ) and (increases/ decreases) $\qquad$ on $\left(\frac{\pi}{2}, \pi\right)$.
3. The graph of $y=\sin x$ and $y=\cos x$ differ by a horizontal shift of $\qquad$ units.
4. Given $y=A \sin (B x-C)+D$ or $y=A \cos$ $(B x-C)+D$, for $B>0$ the amplitude is
$\qquad$ , the period is $\qquad$ , the phase shift is $\qquad$ , and the vertical shift is $\qquad$ .
5. The sine function is an (even/odd) $\qquad$ function because $\sin (-x)=$ $\qquad$ The cosine function is an (even/odd)
$\qquad$ function because
$\cos (-x)=$ $\qquad$
6. Given $B>0$, how would the equation $y=A$ $\sin (-B x-C)+D$ be rewritten to obtain a positive coefficient on $x$ ?
7. Given $B>0$, how would the equation $y=A \cos (-B x-C)+D$ be rewritten to obtain a positive coefficient on $x$ ?
8. Given $y=\sin (B x)$ and $y=\cos (B x)$, for $B>1$, is the period less than or greater than $2 \pi$ ? If $0<B<1$, is the period less than or greater than $2 \pi$ ?

## Learn: Graph $y=\sin x$ and $y=\cos x$

9. From memory, sketch $y=\sin x$ on the interval $[0,2 \pi]$.
10. From memory, sketch $y=\cos x$ on the interval $[0,2 \pi]$.
11. For $y=\cos x$,
a. The domain is $\qquad$ .
b. The range is $\qquad$ .
c. The amplitude is $\qquad$ .
d. The period is $\qquad$ .
e. The cosine function is symmetric to the $\qquad$ -axis.
f. On the interval $[0,2 \pi]$, the $x$-intercepts are $\qquad$ .
g. On the interval $[0,2 \pi]$, the maximum points are $\qquad$ and $\qquad$ , and the minimum point is $\qquad$ .
12. For $y=\sin x$,
a. The domain is $\qquad$ .
b. The range is $\qquad$ .
c. The amplitude is $\qquad$ .
d. The period is $\qquad$ .
e. The sine function is symmetric to the $\qquad$ .
f. On the interval [ $0,2 \pi$ ], the $x$-intercepts are $\qquad$ .
g. On the interval $[0,2 \pi]$, the maximum point is $\qquad$ and the minimum point is $\qquad$ .
13. a. Over what interval(s) taken between 0 and $2 \pi$ is the graph of $y=\sin x$ increasing?
b. Over what interval(s) taken between 0 and $2 \pi$ is the graph of $y=\sin x$ decreasing?
14. a. Over what interval(s) taken between 0 and $2 \pi$ is the graph of $y=\cos x$ increasing?
b. Over what interval(s) taken between 0 and $2 \pi$ is the graph of $y=\cos x$ decreasing?

## Learn: Graph $y=A \sin x$ and $y=A \cos x$

For Exercises 15-16, identify the amplitude of the function.
15. a. $y=7 \sin x$
b. $y=\frac{1}{7} \sin x$
c. $y=-7 \sin x$
16. a. $y=2 \cos x$
b. $y=\frac{1}{2} \cos x$
c. $y=-2 \cos x$
17. By how many units does the graph of $y=\frac{1}{4} \cos x$ deviate from the $x$-axis?
18. By how many units does the graph of $y=-5 \sin x$ deviate from the $x$-axis?

For Exercises 19-24, graph the function and identify the key points on one full period. (See Example 1, p, 531)
19. $y=5 \cos x$
20. $y=4 \sin x$
21. $y=\frac{1}{2} \sin x$
22. $y=\frac{1}{4} \cos x$
23. $y=-2 \cos x$
24. $y=-3 \sin x$

## Learn: Graph $y=A \sin B x$ and

 $y=A \cos B x$For Exercises 25-26, identify the period.
25.a. $\sin 2 x$
b. $\sin 2 \pi x$
c. $\sin \left(-\frac{2}{3} x\right)$
26.a. $\cos \frac{1}{3} x$
b. $\cos (-3 \pi x)$
c. $\cos \frac{1}{3} \pi x$

For Exercises 27-32
a. Identify the amplitude and period.
b. Graph the function and identify the key points on one full period.
(See Example 2, p. 533)
27. $y=2 \cos 3 x$
28. $y=6 \sin 4 x$
29. $y=4 \sin \frac{\pi}{3} x$
30. $y=5 \cos \frac{\pi}{6} x$
31. $y=\sin \left(-\frac{1}{3} x\right)$
32. $y=\cos \left(-\frac{1}{2} x\right)$
33. Write a function of the form $f(x)=A \cos B x$ for the given graph.

34. Write a function of the form $g(x)=A \sin B x$ for the given graph.

35. The graph shows the percentage in decimal form of the Moon illuminated for the first 40 days of a recent year. (Source: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)

a. On approximately which days during this time period did a full moon occur? (A full moon corresponds to $100 \%$ or 1.0.)
b. On which day was there a new moon (no illumination)?
c. From the graph, approximate the period of a synodic month. $A$ synodic month is the period of one lunar cycle (full moon to full moon).
36. A respiratory cycle is defined as the beginning of one breath to the beginning of the next breath. The rate of air intake $r$ (in $\mathrm{L} / \mathrm{sec}$ ) during a respiratory cycle for a physically fit male can be approximated by $r(t)=0.9 \sin \frac{\pi}{3.5} t$, where $t$ is the number of seconds into the cycle. A positive value for $r$ represents inhalation and a negative value represents exhalation.
a. How long is the respiratory cycle?
b. What is the maximum rate of air intake?
c. Graph one cycle of the function. On what interval does inhalation occur? On what interval does exhalation occur?

## Learn: Graph $y=A \sin (B x-C)+D$ and $y=A \cos (B x-C)+D$

For Exercises 37-38, identify the phase shift and indicate whether the shift is to the left or to the right.
37. a. $\cos \left(x-\frac{\pi}{3}\right)$
b. $\cos \left(2 x-\frac{\pi}{3}\right)$
c. $\cos \left(3 \pi x+\frac{\pi}{3}\right)$
38. a. $\sin \left(x+\frac{\pi}{8}\right)$
b. $\sin \left(2 \pi x-\frac{\pi}{8}\right)$
c. $\sin \left(4 x-\frac{\pi}{8}\right)$

For Exercises 39-44,
a. Identify the amplitude, period, and phase shift.
b. Graph the function and identify the key points on one full period.
(See Example 3, p. 534)
39. $y=2 \cos (x+\pi)$
40. $y=4 \sin \left(x+\frac{\pi}{2}\right)$
41. $y=\sin \left(2 x-\frac{\pi}{3}\right)$
42. $y=\cos \left(3 x-\frac{\pi}{4}\right)$
43. $y=-6 \cos \left(\frac{1}{2} x+\frac{\pi}{4}\right)$
44. $y=-5 \sin \left(\frac{1}{3} x+\frac{\pi}{6}\right)$
45. Write a function of the form $f(x)=A \cos (B x-C)$ for the given graph.

46. Write a function of the form
$f(x)=A \sin (B x-C)$ for the given graph.

47. Given $y=-2 \sin \left(2 x-\frac{\pi}{6}\right)-7$,
a. Is the period less than or greater than $2 \pi$ ?
b. Is the phase shift to the left or right?
c. Is the vertical shift upward or downward?
48. Given $y=\cos \left(\frac{1}{2} x+\pi\right)+4$,
a. Is the period less than or greater than $2 \pi$ ?
b. Is the phase shift to the left or right?
c. Is the vertical shift upward or downward?

For Exercises 49-52, rewrite the equation so that the coefficient on $x$ is positive.
49. $y=\cos \left(-2 x+\frac{\pi}{6}\right)-4$
50. $y=4 \cos (-3 x-\pi)+5$
51. $y=\sin \left(-2 x+\frac{\pi}{6}\right)-4$
52. $y=4 \sin (-3 x-\pi)+5$
53. Given $y=2 \sin \left(-\frac{\pi}{6} x+\frac{\pi}{2}\right)-3$, is the phase shift to the right or left?
54. Given $y=-4 \cos \left(-\frac{\pi}{6} x-\frac{\pi}{2}\right)+1$, is the phase shift to the right or left?

## For Exercises 55-68,

a. Identify the amplitude, period, phase shift, and vertical shift.
b. Graph the function and identify the key points on one full period.
(See Examples 4-5, pp. 536-538)
55. $h(x)=3 \sin (4 x-\pi)+5$
56. $g(x)=2 \sin (3 x-\pi)-4$
57. $f(x)=4 \cos \left(3 x-\frac{\pi}{2}\right)-1$
58. $k(x)=5 \cos \left(2 x-\frac{\pi}{2}\right)+1$
59. $y=\frac{1}{2} \sin \left(-\frac{1}{3} x\right)$
60. $y=\frac{2}{3} \sin \left(-\frac{1}{2} x\right)$
61. $v(x)=1.6 \cos (-\pi x)$
62. $m(x)=2.4 \cos (-4 \pi x)$
63. $y=2 \sin (-2 x-\pi)+5$
64. $y=3 \sin (-4 x-\pi)-7$
65. $p(x)=-\cos \left(-\frac{\pi}{2} x-\pi\right)+2$
66. $q(x)=-\cos \left(-\frac{\pi}{3} x-\pi\right)-2$
67. $y=\sin \left(-\frac{\pi}{4} x-\frac{\pi}{2}\right)-3$
68. $y=\sin \left(-\frac{\pi}{3} x-\frac{\pi}{2}\right)+4$
69. The temperature $T$ (in ${ }^{\circ} \mathrm{F}$ ) for a midwestern city over a several day period in April can be approximated by
$T(t)=-5.9 \cos (0.262 t-1.245)+48.2$, where $t$ is the number of hours since midnight on day 1 .
a. What is the period of the function? Round to the nearest hour.
b. What is the significance of the term 48.2 in this model?
c. What is the significance of the factor 5.9 in this model?
d. What was the minimum temperature for the day? When did it occur?
e. What was the maximum temperature for the day? When did it occur?
70. The duration of daylight and darkness varies during the year due to the angle of the Sun in the sky. The model $d(t)=2.65 \sin (0.51 t-1.32)+12$ approximates the amount of daylight $d(t)$ (in hours) for a northern city as a function of the time $t$ (in months) after January 1 for a recent year; that is, $t=0$ is January $1, t=1$ is February 1, and so on. The model $y=n(t)$ represents the amount of darkness as a function of $t$.

a. Describe the relationship between the graphs of the functions and the line $y=12$.
b. Use the result of part (a) and a transformation of $y=d(t)$ to write an equation representing $n$ as a function of $t$.
c. What do the points of intersection of the two graphs represent?
d. What do the relative minima and relative maxima of the graphs represent?
e. What does $T(t)=d(t)+n(t)$ represent?

## Learn: Model Sinusoidal Behavior

71. The probability of precipitation in Modesto, California, varies from a peak of 0.34 (34\%) in January to a low of 0.04 (4\%) in July. Assume that the percentage of precipitation varies monthly and behaves like a cosine curve.
a. Write a function of the form
$P(t)=A \cos (B t-C)+D$ to model the precipitation probability. The value $P(t)$ is the probability of precipitation (as a decimal), for month $t$, with January as $t=1$.
b. Graph the function from part (a) on the interval $[0,13]$ and plot the points $(1,0.34),(7,0.04)$, and $(13,0.34)$ to check the accuracy of your model.
72. The monthly high temperature for Atlantic City, New Jersey, peaks at an average high of $86^{\circ}$ in July and goes down to an average high of $64^{\circ}$ in January. Assume that this pattern for monthly high temperatures continues indefinitely and behaves like a cosine wave.
a. Write a function of the form $H(t)=A \cos (B t-C)+D$ to model the average high temperature. The value $H(t)$ is the average high temperature for month $t$, with January as $t=0$.
b. Graph the function from part (a) on the interval $[0,13]$ and plot the points $(0,64)$, $(6,86)$, and $(12,64)$ to check the accuracy of your model.
73. An adult human at rest inhales and exhales approximately 500 mL of air (called the tidal volume) in approximately 5 sec . However, at the end of each exhalation, the lungs still contain a volume of air, called the functional residual capacity (FRC), which is approximately 2000 mL .
(See Example 6, p. 540)
a. What volume of air is in the lungs after inhalation?
b. What volume of air is in the lungs after exhalation?
c. What is the period of a complete respiratory cycle?
d. Write a function $V(t)=A \cos B t+D$ to represent the volume of air in the lungs $t$ seconds after the end of an inhalation.
e. What is the average amount of air in the lungs during one breathing cycle?
f. During hyperventilation, breathing is more rapid with deep inhalations and exhalations. What parts of the equation from part (d) change?
74. The times for high and low tides are given in the table for a recent day in Jacksonville Beach, Florida. The times are rounded to the nearest hour, and the tide levels are measured relative to mean sea level (MSL).
a. Write a model $h(t)=A \cos (B t-C)$ to represent the tide level $h(t)$ (in feet) in terms of the amount of time $t$ elapsed since midnight.
b. Use the model from part (a) to estimate the tide level at 3:00 P.M.

|  | Time (hr <br> after <br> midnight) | Height <br> Relative to <br> MSL (ft) |
| :--- | :---: | :---: |
| High tide | 0 | 3.4 |
| Low tide | 6 | -3.4 |
| High tide | 12 | 3.4 |
| Low tide | 18 | -3.4 |
| High tide | 24 | 3.4 |

75. The data in the table represent the monthly power bills (in dollars) for a homeowner.
a. Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $A(t)=a \sin (b t+c)+d$, where $A(t)$ represents the amount of the bill for month $t(t=1$ represents January, $t=2$ represents February, and so on).
b. Graph the data and the resulting function.

| Month, $\boldsymbol{t}$ | 1 | 2 | 3 | 4 | 5 | 6 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Amount (\$) | 104.73 | 66.13 | 48.99 | 56.04 | 85.51 | 98.57 |


| Month, $\boldsymbol{t}$ | 7 | 8 | 9 | 10 | 11 | 12 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Amount (\$) | 125.08 | 124.48 | 113.93 | 81.06 | 63.30 | 71.85 |

76. The data in the table represent the duration of daylight $d(t)$ (in hours) for Houston, Texas, for the first day of the month, $t$ months after January 1 for a recent year. (Source: Astronomical Applications Department, U.S. Naval Observatory: http://aa.usno.navy.mil)
a. Enter the data in a graphing utility and use the sinusoidal regression tool (SinReg) to find a model of the form $d(t)=a \sin (b t+c)+d$.
b. Graph the data and the resulting function.

| Month, $\boldsymbol{t}$ | 1 | 2 | 3 | 4 | 5 |
| :--- | :---: | :---: | :---: | :---: | :---: |
| Amount (hr) | 10.28 | 10.80 | 11.57 | 12.48 | 13.65 |


| Month, $\boldsymbol{t}$ | 6 | 7 | 8 | 9 | 10 | 11 |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| Amount (hr) | 14.02 | 13.55 | 12.73 | 11.87 | 11.00 | 10.38 |

## Mixed Exercises

For Exercises 77-78, write the range of the function in interval notation.
77. a. $y=8 \cos (2 x-\pi)+4$
b. $y=-3 \cos \left(x+\frac{\pi}{3}\right)-5$
78. a. $y=-6 \sin \left(3 x-\frac{\pi}{2}\right)-2$
b. $y=2 \sin (3 x+2 \pi)+12$
79. Given $f(x)=\cos x$ and $h(x)=3 x+2$, find $(h \circ f)(x)$ and for $(h \circ f)(x)$,
a. Find the amplitude.
b. Find the period.
c. Write the domain in interval notation.
d. Write the range in interval notation.
80. Given $g(x)=\sin x$ and $k(x)=6 x$, find $(g \circ k)(x)$ and for $(g \circ k)(x)$,
a. Find the amplitude.
b. Find the period.
c. Write the domain in interval notation.
d. Write the range in interval notation.
81. Write a function of the form $y=A$ sin $(B x-C)+D$ that has period $\frac{\pi}{3}$, amplitude 4, phase shift $\frac{\pi}{2}$, and vertical shift 5 .
82. Write a function of the form $y=A \cos$ $(B x-C)+D$ that has period $\frac{\pi}{4}$, amplitude 2 , phase shift $-\frac{\pi}{3}$, and vertical shift 7.
83. Write a function of the form $y=A \cos$ $(B x-C)+D$ that has period 16 , phase shift -4 , and range $3 \leq y \leq 7$.
84. Write a function of the form $y=A \sin (B x-C)+$ $D$ that has period 8 , phase shift -2 , and range $-14 \leq y \leq-6$.
85. Describe the transformation applied to the dashed curve to make the solid curve. Choose from: change in amplitude, change in period, phase shift, or vertical shift. Then write an equation for the solid curve in terms of the cosine function.
a.

b.

c.

d.

86. Describe the transformations (there may be more than one) applied to the dashed curve to make the solid curve. Choose from: change in amplitude, change in period, phase shift, or vertical shift. Then write an equation for the solid curve in terms of the sine function.
a.

b.

c.

d.

87. Write an equation of the form $y=A \sin B x$ to model the graph.

88. Write an equation of the form $y=A \cos B x$ to model the graph.


For Exercises 89-90,
a. Write an equation of the form $y=A \cos (B x-C)+D$ with $A>0$ to model the graph.
b. Write an equation of the form
$y=A \sin (B x-C)+D$ with $A>0$ to model the graph.
89.

90.


For Exercises 91-94, explain how to graph the given function by performing transformations on the "parent" graphs $y=\sin x$ and $y=\cos x$.
91. a. $y=\sin 2 x$
b. $y=2 \sin x$
92. a. $y=\frac{1}{3} \cos x$
b. $y=\cos \frac{1}{3} x$
93. a. $y=\sin (x+2)$
b. $y=\sin x+2$
94.a. $y=\cos x-4$
b. $y=\cos (x-4)$

## Write About It

95. Is $f(x)=\sin x$ one-to-one? Explain why or why not.
96. Is $f(x)=\cos x$ one-to-one? Explain why or why not.
97. If $f$ and $g$ are both periodic functions with period $P$, is $(f+g)(x)$ also periodic? Explain why or why not.

## Expanding Your Skills

98. Explain why a function that is increasing on its entire domain cannot be periodic.

For Exercises 99-100, find the average rate of change on the given interval. Give the exact value and an approximation to 4 decimal places. Verify that your results are reasonable by comparing the results to the slopes of the lines given in the graph.
99. $f(x)=\sin x$

a. $\left[0, \frac{\pi}{6}\right]$
b. $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$
c. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$
100. $f(x)=\cos x$

a. $\left[0, \frac{\pi}{6}\right]$
b. $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$
C. $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$

For Exercises 101-104, use your knowledge of the graphs of the sine function and linear functions to determine the number of solutions to the equation.
101. $\sin x=x-2$
102. $\cos x=-x$
103. $\sin 2 x=-2$
104. $2 \sin 2 x=-2$

For Exercises 105-106, graph the piecewisedefined function.
105. $g(x)=\left\{\begin{array}{c}\sin x \text { for } 0 \leq x \leq \pi \\ -\sin x \text { for } \pi<x \leq 2 \pi\end{array}\right.$
106. $f(x)=\left\{\begin{array}{l}\cos x \text { for } 0 \leq x \leq \frac{\pi}{4} \\ \sin x \text { for } \frac{\pi}{4}<x \leq \frac{\pi}{2}\end{array}\right.$

## Technology Connections

107. Functions $a$ and $m$ approximate the duration of daylight, respectively, for Albany, New York, and Miami, Florida, for a given year for day $t$. The value $t=1$ represents January $1, t=2$ represents February 1 , and so on.

$$
\begin{aligned}
& a(t)=12+3.1 \sin \left[\frac{2 \pi}{365}(t-80)\right] \\
& m(t)=12+1.6 \sin \left[\frac{2 \pi}{365}(t-80)\right]
\end{aligned}
$$

a. Graph the two functions with a graphing utility and comment on the difference between the two graphs.
b. Both functions have a constant term of 12. What does this represent graphically and in the context of this problem?
c. What do the factors 3.1 and 1.6 represent in the two functions?
d. What is the period of each function?
e. What does the horizontal shift of 80 units represent in the context of this problem?
f. Use the Intersect feature to approximate the points of intersection.
g. Interpret the meaning of the points of intersection.

For Exercises 108-109, we demonstrate that trigonometric functions can be approximated by polynomial functions over a given interval in the domain.

Graph functions $f, g, h$, and $k$ on the viewing window $-4 \leq x \leq 4,-4 \leq y \leq 4$. Then use a Table feature on a graphing utility to evaluate each function for the given values of $x$. How do functions $g$, $h$, and $k$ compare to function $f$ for $x$ values farther from 0 ? [Hint: For a given natural number $n$, the value $n!$, read as " $n$ factorial," is defined as $n!=n(n-1)(n-2) \cdots 1$. For example, $3!=3 \cdot 2 \cdot 1=6$.]
108.

| Function | $x=$ | $x=$ | $x=$ | $x=$ |
| :--- | :--- | :--- | :--- | :--- |
| $f(x)=\cos x$ |  |  |  |  |
| $g(x)=1-\frac{x^{2}}{2!}$ |  |  |  |  |
| $h(x)=1-\frac{x^{2}}{2!}+\frac{x^{4}}{4!}$ |  |  |  |  |
| $k(x)=\overline{x^{2}}$ <br> $1-\frac{x^{4}}{2!}+\frac{x^{6}}{4!}-\frac{x^{6}}{6!}$ |  |  |  |  |

109. 

| Function | $x=$ <br> 0.1 | $x=$ | $x=$ | $x=$ |
| :--- | :--- | :--- | :--- | :--- |
| $f(x)=\sin x$ |  |  |  |  |
| $g(x)=x-\frac{x^{3}}{3!}$ |  |  |  |  |
| $h(x)=x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}$ |  |  |  |  |
| $k(x)=$ <br> $x-\frac{x^{3}}{3!}+\frac{x^{5}}{5!}-\frac{x^{7}}{7!}$ |  |  |  |  |
| $x$ |  |  |  |  |

For Exercises 110-111, use a graph to solve the equation on the given interval.
110. $\cos \left(2 x-\frac{\pi}{3}\right)=0.5$ on $[0, \pi]$

Viewing window: $\left[0, \pi, \frac{\pi}{3}\right]$ by $\left[-1,1, \frac{1}{2}\right]$
111. $\sin \left(2 x+\frac{\pi}{4}\right)=1$ on $[0,2 \pi]$

Viewing window: $\left[0,2 \pi, \frac{\pi}{8}\right]$ by $[-2,2,1]$
For Exercises 112-113, use a graph to solve the equation on the given interval. Round the answer to $\mathbf{2}$ decimal places.
112. $\sin \left(x-\frac{\pi}{4}\right)=-\mathrm{e}^{x}$ on $[-\pi, \pi]$

Viewing window: $\left[-\pi, \pi, \frac{\pi}{2}\right]$ by $[-2,2,1]$
113. $6 \cos \left(x+\frac{\pi}{6}\right)=\ln x$ on $[0,2 \pi]$

Viewing window: $\left[0,2 \pi, \frac{\pi}{2}\right]$ by $[-7,7,1]$

Variable Amplitude: The graphs of functions such as $f(x)=a \cos x$ and $g(x)=a \sin x$ are curves with a constant amplitude, $a$. If we have a nonconstant coefficient $a(x)$, we will have a variable amplitude with the graph of sine or cosine bounded by the graphs of $y=a(x)$ and $y=-a(x)$. Use this information for Exercises 114-115.
114. Graph $f(x)=0.5 x \sin x$ on the viewing window $[-5 \pi, 5 \pi, \pi]$ by $[-8,8,1]$ and give the equations of the bounding functions.
115. Graph $g(x)=\cos x \sin 12 x$ on the viewing window $[0,2.5 \pi, 0.25 \pi]$ by $[-1,1,0.5]$ and give the equations of the bounding functions.

Go online for more practice problems.

## What Will You Learn?

After completing this lesson, you should be able to:

- Graph the Secant and Cosecant Functions
- Graph the Tangent and Cotangent Functions


## Learn: Graph the Secant and Cosecant Functions

In this lesson, we turn our attention to the graphs of the secant, cosecant, tangent, and cotangent functions. Each of these functions has vertical asymptotes and, for this reason, we often use them to model patterns involving unbounded behavior.

For example, the distance $d(x)$ between an observer 30 ft from a straight highway and a police car traveling down the highway is given by $d(x)=30 \mathrm{sec} x$. The independent variable $x$ is the angle formed from the perpendicular line from the road to the observer and from the observer to the police car (Figure 4-49). Notice that for $x$ values close to $90^{\circ}\left(\frac{\pi}{2}\right.$ radians $)$, the distance $d(x)$ approaches infinity. This means that the graph of $d(x)=30 \sec x$ will have a vertical asymptote at $x=\frac{\pi}{2}$.


Figure 4-49
To graph the cosecant and secant functions, we first review their reciprocal relationships.

$$
y=\csc x=\frac{1}{\sin x} \quad \text { and } \quad y=\sec x=\frac{1}{\cos x}
$$

- If $\sin x=0$, then $\frac{1}{\sin x}$ is undefined, and the graph of $y=\csc x$ has a vertical asymptote (the numerator is nonzero and the denominator is zero). This occurs when $x$ is a multiple of $\pi: x=n \pi$ for all integers $n$.
- If $\cos x=0$, then $\frac{1}{\cos x}$ is undefined, and the graph of $y=\sec x$ has a vertical asymptote (the numerator is nonzero and the denominator is zero). This occurs when $x$ is an odd multiple of $\frac{\pi}{2}: x=\frac{(2 n+1) \pi}{2}$ for all integers $n$.
To sketch the graph of $y=\csc x$, we construct a table with several values of $x$ and the corresponding values of $\sin x$ and $\csc x$ (Table 4-12). The cosecant function is an odd function, so the points plotted in Quadrants I and II each have a mirror image in Quadrants III and IV. Furthermore, both the sine and cosecant functions are periodic, and their graphs show a repeated pattern in intervals of $2 \pi$ (Figure 4-50).

TABLE 4-12 Values of $\operatorname{Sin} x$ and $\operatorname{Csc} x$ for Common Values of $x$

| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2 \pi}{3}$ | $\frac{3 \pi}{4}$ | $\frac{5 \pi}{6}$ | $\pi$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\sin x$ | 0 | $\frac{1}{2}$ | $\frac{\sqrt{2}}{2} \approx 0.71$ | $\frac{\sqrt{3}}{2} \approx 0.87$ | 1 | $\frac{\sqrt{3}}{2} \approx 0.87$ | $\frac{\sqrt{2}}{2} \approx 0.71$ | $\frac{1}{2}$ | 0 |
| $\csc x$ | Undefined | 2 | $\sqrt{2} \approx 1.4$ | $\frac{2 \sqrt{3}}{3} \approx 1.2$ | 1 | $\frac{2 \sqrt{3}}{3} \approx 1.2$ | $\sqrt{2} \approx 1.4$ | 2 | Undefined |



Figure 4-50
Notice that the relative maxima and minima from the sine function correspond to the relative minima and maxima for the cosecant function. The $x$-intercepts of the sine graph give the location of the vertical asymptotes for the graph of $y=\csc x$.

The graphs of the secant function and cosine function are similarly related. The secant function is an even function, and therefore its graph is symmetric with respect to the $y$-axis. The asymptotes of the graph of $y=\sec x$ occur at odd multiples of $\frac{\pi}{2}$ where $\cos x=0$ (Figure 4-51).


Figure 4-51

Table 4-13 summarizes key properties of the graphs of $y=\csc x$ and $y=\sec x$.
TABLE 4-13 Graphs of the Cosecant and Secant Functions

| Function | $y=\csc x$ | $y=\sec x$ |
| :---: | :---: | :---: |
| Domain | $\{x \mid x \neq n \pi$ for all integers $n\}$ | $\left\{x \left\lvert\, x \neq \frac{(2 n+1) \pi}{2}\right. \text { for all integers } n\right\}$ |
| Range | $\{y \mid y \leq-1$ or $y \geq 1\}$ | $\{y \mid y \leq-1$ or $y \geq 1\}$ |
| Amplitude | None $(y=\csc x$ increases and decreases without bound) | None ( $y=\sec x$ increases and decreases without bound) |
| Period | $2 \pi$ | $2 \pi$ |
| Vertical Asymptotes | $x=n \pi$ (multiples of $\pi$ ) | $x=\frac{(2 n+1) \pi}{2}\left(\text { odd multiples of } \frac{\pi}{2}\right)$ |
| Symmetry | Origin (The cosecant function is an odd function.) | $y$-axis (The secant function is an even function.) |

To graph variations of $y=\csc x$ or $y=\sec x$, we use the graph of the related reciprocal function for reference.

## Example 1: Graphing a Variation of $y=\sec x$

Graph $y=3 \sec 2 x$.

## Solution:

We first graph the reciprocal function $y=3 \cos 2 x$ (shown in gray) using the techniques from Lesson 4.5.
The amplitude is $|3|=3$.
The period is $\frac{2 \pi}{2}=\pi$.
There is no phase shift or vertical shift.


The graph of $y=3 \sec 2 x$ (shown in blue) has vertical asymptotes at the odd multiples of $\frac{\pi}{4}$ where the graph of $y=3 \cos 2 x$ has $x$-intercepts.

## Apply the Skills

1. Graph $y=2 \sec 4 x$.

In Example 1 we built the graph of $y=3 \sec 2 x$ by using reference points ( $x$-intercepts and relative extrema) from the graph of $y=3 \cos 2 x$. To graph a cosecant or secant function involving a vertical shift, we recommend first graphing the function without the vertical shift. This is to make use of the $x$-intercepts of the reciprocal function to find the vertical asymptotes of the cosecant or secant function. This is demonstrated in Example 2.

Example 2: Graphing a Cosecant Function with a Vertical Shift
Graph $y=\csc \left(x-\frac{\pi}{4}\right)+3$.

## Solution:

First graph the function without the vertical shift, $y=\csc \left(x-\frac{\pi}{4}\right)$.
For the related reciprocal function $y=\sin \left(x-\frac{\pi}{4}\right)$, we have:

The amplitude is 1.
The period is $2 \pi$.


The phase shift is $\frac{\pi}{4}$.
The $x$-intercepts occur at
$x=\ldots,-\frac{3 \pi}{4}, \frac{\pi}{4}, \frac{5 \pi}{4}, \frac{9 \pi}{4}, \ldots$.
The vertical asymptotes of $y=\csc \left(x-\frac{\pi}{4}\right)$ pass through the $x$-intercepts of the related sine function.


Next, apply the vertical shift 3 units upward to graph
$y=\csc \left(x-\frac{\pi}{4}\right)+3$
Notice that the range of $y=\csc \left(x-\frac{\pi}{4}\right)+3$ is $(-\infty, 2] \cup[4, \infty)$.


## Apply the Skills

2. Graph $y=\csc (x+\pi)-2$.

## Learn: Graph the Tangent and Cotangent Functions

We now investigate the graph $y=\tan x$ and begin with several observations. The graph of $y=\tan x=\frac{\sin x}{\cos x}$ will have

- An $x$-intercept where the numerator equals zero. The value $\sin x=0$ when $x$ is a multiple of $\pi$; that is, $x=n \pi$ for all integers $n$ (Figure 4-52).
- A vertical asymptote where the denominator equals zero. The value $\cos x=0$ when $x$ is an odd multiple of $\frac{\pi}{2}$; that is, at $x=\frac{(2 n+1) \pi}{2}$ for all integers $n$ (Figure 4-52).


Figure 4-52
The tangent function is an odd function and is symmetric with respect to the origin. Therefore, each Quadrant I point given in Table 4-14 has a mirror image in Quadrant III (Figure 4-53). The period of the tangent function is $\pi$, so one complete period occurs between two consecutive vertical asymptotes. We can then sketch more cycles to the left and right as desired.

TABLE 4-14 Values of Tan $x$ for Common Values of $x$

| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\tan x$ | 0 | $\frac{\sqrt{3}}{3} \approx 0.58$ | 1 | $\sqrt{3} \approx 1.73$ | Undefined |



Figure 4-53
We need to use particular care graphing the function near the asymptotes; for example, we can find a value of $\tan x$ arbitrarily large if $x$ is taken sufficiently close to $\frac{\pi}{2}$ from the left. Therefore, $\tan x$ approaches $\infty$ as $x$ approaches $\frac{\pi}{2}$ from the left (Table 4-15). Likewise, $\tan x$ approaches $-\infty$ as $x$ approaches $\frac{\pi}{2}$ from the right.


To graph the cotangent function, we make the following observations. The graph of $y=\cot x=\frac{\cos x}{\sin x}$ will have

- An $x$-intercept where the numerator equals zero. The value $\cos x=0$ when $x$ is an odd multiple of $\frac{\pi}{2}$; that is, at $x=\frac{(2 n+1) \pi}{2}$ or all integers $n$.
- A vertical asymptote where the denominator equals zero. The value $\sin x=0$ when $x$ is a multiple of $\pi$; that is, at $x=n \pi$ for all integers $n$.

The cotangent function is an odd function and is symmetric with respect to the origin. Therefore, each Quadrant I point given in Table 4-16 has a mirror image in Quadrant III (Figure 4-54). The period of the cotangent function is $\pi$, so one complete period occurs between two consecutive vertical asymptotes. We can then sketch more cycles to the left and right as desired.

TABLE 4-16 Values of $\operatorname{Cot} x$ for Common Values of $x$

| $x$ | 0 | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\cot x$ | Undefined | $\sqrt{3} \approx 1.73$ | 1 | $\frac{\sqrt{3}}{3} \approx 0.58$ | 0 |

Take a few minutes to review the key characteristics of the tangent and cotangent functions and their graphs.


Figure 4-54

## Graphs of the Tangent and Cotangent Functions

| Function | $y=\tan x$ | $y=\cot x$ |
| :---: | :---: | :---: |
| Domain | $\left\{x \left\lvert\, x \neq \frac{(2 n+1) \pi}{2}\right.\right.$ for all integers $\left.n\right\}$ | $\{x \mid x \neq n \pi$ for all integers $n\}$ |
| Range | All real numbers | All real numbers |
| Amplitude | None ( $y=\tan x$ is unbounded.) | None ( $y=\cot x$ is unbounded.) |
| Period | $\pi$ | $\pi$ |
| Vertical Asymptotes | $x=\frac{(2 n+1) \pi}{2}\left(\text { odd multiples of } \frac{\pi}{2}\right)$ | $x=n \pi$ (multiples of $\pi$ ) |
| Symmetry | Origin (The tangent function is an odd function.) | Origin (The cotangent function is an odd function.) |

Now consider variations on the tangent and cotangent functions $y=A \tan (B x-C)+D$ and $y=A \cot (B x-C)+D$ with $B>0$. (If $B<0$, we can rewrite the equations using the odd function property of the tangent and cotangent functions.)

- $|A|$ is the vertical scaling factor.
- If $A<0$, the graph is reflected across the $x$-axis.
- The period is $\frac{\pi}{B}$.
- The phase shift is $\frac{C}{B}$.
- The vertical shift is $D$.

To graph variations of the tangent or cotangent functions, we graph the functions first without the vertical shift using the guidelines in the box below. The vertical shift is applied once the other transformations are accounted for.

## Guidelines to $G r a p h ~ y=A \tan (B x-C)$ and $y=A \cot (B x-C)$

1. Find an interval between two consecutive vertical asymptotes and graph the asymptotes as dashed lines.

- For $y=A \tan (B x-C)$ and $B>0$, solve the inequality $-\frac{\pi}{2}<B x-C<\frac{\pi}{2}$.
- For $y=A \cot (B x-C)$ and $B>0$, solve the inequality $0<B x-C<\pi$.

2. Plot an $x$-intercept halfway between the asymptotes from step 1 .
3. Sketch the general shape of the "parent" function between the asymptotes (if $A<0$, the parent function is reflected across the $x$-axis). If more accuracy is desired, consider evaluating the function for $x$ values midway between the $x$-intercept and the asymptotes found in step 1 .

- For $y=A \tan x$, the $y$-coordinates for $x$ values midway between the $x$-intercept and the asymptotes will be $-A$ and $A$.
- For $y=A \cot x$, the $y$-coordinates for $x$ values midway between the $x$-intercept and the asymptotes will be $A$ and $-A$.

4. Sketch additional cycles to the right or left as desired.

Example 3: Graphing a Variation of $y=\tan x$
Graph $f(x)=3 \tan \frac{\pi}{4} x$.

## Solution:

One complete period of the "parent" function $y=\tan x$ occurs between consecutive asymptotes on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. To find the related interval for $f(x)=3 \tan \frac{\pi}{4} x$, solve the following inequality.

$$
-\frac{\pi}{2}<\frac{\pi}{4} x<\frac{\pi}{2} \quad \text { Multiply by } \frac{4}{\pi} .
$$

$$
\frac{4}{\pi} \cdot\left(-\frac{\pi}{2}\right)<\frac{4}{\pi} \cdot\left(\frac{\pi}{4} x\right)<\frac{4}{\pi} \cdot\left(\frac{\pi}{2}\right)
$$



One complete cycle of $f(x)=$ $3 \tan \frac{\pi}{4} x$ occurs between the asymptotes $x=-2$ and $x=2$.
The midpoint of the interval $(-2,2)$ is $\frac{-2+2}{2}=0$.
The function has an $x$-intercept at ( 0,0 ).


$$
\begin{aligned}
f(-1) & =3 \tan \left[\frac{\pi}{4}(-1)\right]=3 \tan \left(-\frac{\pi}{4}\right)=3(-1) \\
& =-3
\end{aligned}
$$

$$
f(1)=3 \tan \left[\frac{\pi}{4}(1)\right]=3 \tan \left(\frac{\pi}{4}\right)=3(1)=3
$$



## Apply the Skills

3. Graph $y=4 \tan \frac{\pi}{2} x$.

If the graph of a tangent or cotangent function involves a vertical shift, we recommend graphing the function without the vertical shift first.

Example 4: Graphing a Variation of $y=\cot x$
Graph $y=\cot \left(x+\frac{\pi}{4}\right)+2$.
Solution:
We first graph $g(x)=\cot \left(x+\frac{\pi}{4}\right)$.
One complete period of the "parent" function $y=\cot x$ occurs between consecutive asymptotes on the interval $(0, \pi)$. To find the related interval for $g(x)=\cot \left(x+\frac{\pi}{4}\right)$, solve the following inequality.


Subtract $\frac{\pi}{4}$.
One complete cycle of $g(x)=$ $\cot \left(x+\frac{\pi}{4}\right)$ occurs between the asymptotes $x=-\frac{\pi}{4}$ and $x=\frac{3 \pi}{4}$.

The midpoint of the interval $\left(-\frac{\pi}{4}, \frac{3 \pi}{4}\right)$
is $\frac{1}{2}\left(-\frac{\pi}{4}+\frac{3 \pi}{4}\right)=\frac{1}{2}\left(\frac{2 \pi}{4}\right)=\frac{\pi}{4}$.
$g(x)=\cot \left(x+\frac{\pi}{4}\right)$ has an $x$-intercept at $\left(\frac{\pi}{4}, 0\right)$.

$g(0)=\cot \left[0+\frac{\pi}{4}\right]=\cot \left(\frac{\pi}{4}\right)=1 \quad$ If more accuracy in the graph is desired, plot $g\left(\frac{\pi}{2}\right)=\cot \left[\frac{\pi}{2}+\frac{\pi}{4}\right]=\cot \left(\frac{3 \pi}{4}\right)=-1$ a few more points. The values $x=0$ and $x=\frac{\pi}{2}$ are midway between the $x$-intercept and the vertical asymptotes.


## Apply the Skills

4. Graph $y=\cot \left(x+\frac{\pi}{2}\right)-3$.

## Practice Exercises

## Prerequisite Review

R.1. Write the domain of the function in interval notation.

$$
f(x)=\frac{x^{2}-16}{x-4}
$$

R.2. Refer to the graph of the function to complete the statement.

a. As $x \longrightarrow-2^{-}, f(x) \longrightarrow$.
b. As $x \longrightarrow-2^{+}, f(x) \longrightarrow$
R.3. Determine the vertical asymptotes of the graph of the function.

$$
f(x)=\frac{6}{x-5}
$$

## Concept Connections

1. At values of $x$ for which $\sin x=0$, the graph of $y=\csc x$ will have a
$\qquad$ This occurs for $x=$ ___ for all integers $n$.
2. At values of $x$ for which $\cos x=0$, the graph of $y=\sec x$ will have a $\qquad$
$\qquad$ This occurs for $x$ at odd multiples of $\qquad$
3. The relative maxima on the graph of $y=\sin x$ correspond to the $\qquad$
$\qquad$ on the graph of $y=\csc x$.
4. The graph of $y=\csc x$ is symmetric with respect to the $\qquad$ The graph of $y=\sec x$ is symmetric with respect to the
$\qquad$ -axis.
5. If a function is an odd function, then each point $(x, y)$ in Quadrant I will have a corresponding point ( $\qquad$ ) in Quadrant
6. The range of $y=\tan x$ and $y=\cot x$ is
$\qquad$
7. The graphs of both $y=\tan x$ and $y=\cot x$ are symmetric with respect to the
$\qquad$ _.
8. For the functions $y=A \tan (B x-C)$ and $y=A \cot (B x-C)$ with $B>0$, the vertical scaling factor is $\qquad$ the period is $\qquad$ and the phase shift is

## Learn: Graph the Secant and Cosecant Functions

9. Sketch the graph of $y=\csc x$ from memory. Use the graph of $y=\sin x$ for reference.
10. Sketch the graph of $y=\sec x$ from memory. Use the graph of $y=\cos x$ for reference.

For Exercises 11-16, identify the statements among a-h that follow directly from the given condition about $\boldsymbol{x}$.
a. $\csc x$ is undefined.
b. $\sec x$ is undefined.
c. The graph of $y=\sec x$ has a relative maximum at $x$.
d. The graph of $y=\csc x$ has a relative minimum at $x$.
e. The graph of $y=\sec x$ has a vertical asymptote.
f. The graph of $y=\csc x$ has a vertical asymptote.
g. The graph of $y=\csc x$ has a relative maximum at $x$.
h. The graph of $y=\sec x$ has a relative minimum at $x$.
11. $\sin x=0$
12. $\cos x=0$
13. The graph of $y=\cos x$ has a relative maximum at $x$.
14. The graph of $y=\sin x$ has a relative minimum at $x$.
15. The graph of $y=\cos x$ has a relative minimum at $x$.
16. The graph of $y=\sin x$ has a relative maximum at $x$.

For Exercises 17-32, graph one period of the function. (See Examples 1-2, pp. 554-555)
17. $y=2 \csc x$
18. $y=\frac{1}{4} \sec x$
19. $y=-5 \sec x$
20. $y=-\frac{1}{3} \csc x$
21. $y=3 \csc \frac{x}{3}$
22. $y=-4 \sec \frac{x}{2}$
23. $y=\sec 2 \pi x$
24. $y=\csc 3 \pi x$
25. $y=\csc \left(x-\frac{\pi}{4}\right)$
26. $y=\sec \left(x+\frac{\pi}{3}\right)$
27. $y=-2 \sec (2 \pi x+\pi)$
28. $y=-\csc \left(\frac{\pi}{3} x+\frac{\pi}{2}\right)$
29. $y=2 \csc \left(2 x+\frac{\pi}{4}\right)+1$
30. $y=3 \sec \left(x-\frac{\pi}{3}\right)-3$
31. $y=\sec \left(-x-\frac{\pi}{4}\right)-2$
32. $y=-\csc (-x+\pi)+4$

For Exercises 33-34, write the range of the function in interval notation.
33. a. $y=-4 \csc (2 x-\pi)+7$
b. $y=2 \csc \left(\frac{\pi}{3} x\right)-10$
34. a. $y=-3 \sec (5 \pi x)-1$
b. $y=5 \sec \left(3 x-\frac{\pi}{2}\right)-2$
35. Write a function of the form $y=A \sec B x$ for the given graph.

36. Write a function of the form $y=\csc (B x-C)$ for the given graph.

37. A plane flying at an altitude of 5 mi travels on a path directly over a radar tower.

a. Express the distance $d(\theta)$ (in miles) between the plane and the tower as a function of the angle $\theta$ in standard position from the tower to the plane.
b. If $d(\theta)=5$, what is the measure of the angle and where is the plane located relative to the tower?
c. Can the value of $\theta$ be $\pi$ ? Explain your answer in terms of the function $d$.
38. The distance $d(x)$ (in feet) between an observer 30 ft from a straight highway and a police car traveling down the highway is given by $d(x)=30 \sec x$, where $x$ is the angle (in degrees) between the observer and the police car.

a. Use a calculator to evaluate $d(x)$ for the given values of $x$. Round to the nearest foot.

| $x$ | 45 | 60 | 70 | 80 | 89 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $d(x)$ |  |  |  |  |  |

b. Try experimenting with values of $x$ closer to $90^{\circ}$. What happens as $x \longrightarrow 90^{\circ}$ ?

## Learn: Graph the Tangent and Cotangent Functions

39. a. Graph $y=\tan x$ on the interval $[-\pi, \pi]$.
b. How many periods of the tangent function are shown on the interval $[-\pi, \pi]$ ?
40. a. Graph $y=\cot x$ on the interval $[-\pi, \pi]$.
b. How many periods of the cotangent function are shown on the interval $[-\pi, \pi]$ ?

## For Exercises 41-42, graph one complete

 period of the function. Identify the $x$-intercept and evaluate the function for $x$ values midway between the $x$-intercept and the asymptotes. (See Example 3, p. 559)41. a. $y=\frac{1}{2} \tan \pi x$
42. a. $y=2 \cot \pi x$
b. $y=-3 \tan \pi x$
b. $y=-\frac{1}{4} \cot \pi x$

For Exercises 43-58, graph the function. (See Examples 3-4, pp. 559-560)
43. $y=\tan 2 x$
44. $y=\cot 3 x$
45. $y=\cot (-2 x)$
46. $y=\tan \left({ }^{-} x\right)$
47. $y=\tan \left(\frac{1}{2} x\right)$
48. $y=\cot \left(\frac{1}{3} x\right)$
49. $y=4 \cot 2 x$
50. $y=-3 \tan 4 x$
51. $y=-2 \tan \frac{\pi}{3} x$
52. $y=5 \cot \frac{\pi}{4} x$
53. $y=\cot \left(2 x+\frac{\pi}{3}\right)$
54. $y=\tan \left(3 x-\frac{\pi}{4}\right)$
55. $y=-\tan 3 x+4$
56. $y=-\cot 2 x-3$
57. $y=3 \tan (\pi x+\pi)-2$
58. $y=2 \cot \left(2 \pi x-\frac{\pi}{2}\right)+1$
59. Write a function of the form $y=\tan (B x-C)$ for the given graph.

60. Write a function of the form $y=\cot (B x)$ for the given graph.


## Mixed Exercises

For Exercises 61-64, given $y=f(x)$ and $y=g(x)$,
a. Find $(f \circ g)(x)$ and graph the resulting function.
b. Find $(g \circ f)(x)$ and graph the resulting function.
61. $f(x)=\tan x$ and $g(x)=\frac{x}{4}$
62. $f(x)=-2 x$ and $g(x)=\cot x$
63. $f(x)=-3 x$ and $g(x)=\csc x$
64. $f(x)=\frac{x}{3}$ and $g(x)=\sec x$

For Exercises 65-68, complete the statements for the function provided.
65. $f(x)=\tan x$
a. As $x \rightarrow-\frac{\pi^{-}}{2}, f(x) \longrightarrow$
b. As $x \rightarrow-\frac{\pi^{+}}{}{ }^{+}, f(x) \longrightarrow$
$\qquad$
$\qquad$
66. $f(x)=\cot x$
a. As $x \longrightarrow 0^{-}, f(x) \longrightarrow$ $\qquad$
b. As $x \longrightarrow 0^{+}, f(x) \longrightarrow$ $\qquad$
67. $f(x)=\csc x$
a. As $x \longrightarrow 0^{-}, f(x) \longrightarrow$ $\qquad$
b. As $x \longrightarrow 0^{+}, f(x) \longrightarrow$ $\qquad$
68. $f(x)=\sec x$
a. As $x \rightarrow \frac{\pi^{-}}{2}, f(x) \longrightarrow$
b. As $x \rightarrow \frac{\pi^{+}}{}{ }^{+}, f(x) \longrightarrow$

## Write About It

69. Explain how to find two consecutive vertical asymptotes of $y=A \tan (B x-C)$ for $B>0$.
70. Explain how to find two consecutive vertical asymptotes of $y=A \cot (B x-C)$ for $B>0$.
71. Explain how to graph $y=A \sec (B x-C)+D$.
72. Explain how to graph $y=A \csc (B x-C)+D$.

## Expanding Your Skills

For Exercises 73-76, solve each equation for $\boldsymbol{x}$ on the interval $[0,2 \pi)$.
73. $\tan x=1$
74. $\sec x=-1$
75. $\csc x=1$
76. $\cot x=-1$

For Exercises 77-78, graph the piecewisedefined function.
77. $f(x)=\left\{\begin{array}{ccc}-\frac{3 \sqrt{3}}{2 \pi} x-\frac{3 \sqrt{3}}{2} & \text { for } & x \leq-\frac{\pi}{3} \\ \tan x & \text { for } & -\frac{\pi}{3}<x<\frac{\pi}{3} \\ \sqrt{3} & \text { for } & x \geq \frac{\pi}{3}\end{array}\right.$

Hint: Graph the function on the interval $[-\pi, \pi]$. Use increments of $\frac{\pi}{3}$ on the $x$-axis and increments of $\sqrt{3}$ on the $y$-axis.
78. $g(x)=\left\{\begin{array}{ccc}|x|+1 & \text { for } & x \leq-\frac{\pi}{3} \\ \sec x & \text { for } & -\frac{\pi}{3}<x<\frac{\pi}{3} \\ |x|+1 & \text { for } & x \geq \frac{\pi}{3}\end{array}\right.$

Hint: Graph the function on the interval $[-\pi, \pi]$.
79. Show that the maximum length $L$ (in feet) of a beam that can fit around the corner shown in the figure is $L=5 \sec \theta+4 \csc \theta$.


## Technology Connections

80. Graph the functions $y=\tan x$ and $y=x+\frac{x^{3}}{3}+\frac{2 x^{5}}{15}$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. How do the functions compare for values of $x$ taken close to 0 ?
81. Graph the functions $y=\sec x$ and $y=1+\frac{x^{2}}{2}+\frac{5 x^{4}}{24}$ on the interval $[-\pi, \pi]$. How do the functions compare for values of $x$ taken close to 0 ?
82. Given $f(x)=x^{2}, g(x)=\tan x$, and $h(x)=\sec x$,
a. Find $(f \circ h)(x)$.
b. Graph $g(x)$ and $(f \circ h)(x)$ together using the ZTRIG window. The relationship between the two graphs will be studied in calculus. For a given value of $x$ in the domain of $g(x)=\tan x, y=\sec ^{2} x$ gives the slope of a line tangent to $g$ at $x$.
83. Given $r(x)=-x^{2}, s(x)=\cot x$, and $t(x)=\csc x$,
a. Find $(r \circ t)(x)$
b. Graph $s(x)$ and $(r \circ t)(x)$ together on the ZTRIG window. The relationship between the two graphs will be studied in calculus. For a given value of $x$ in the domain of $s(x)=\cot x, y=-\csc ^{2} x$ gives the slope of a line tangent to $s$ at $x$.

## Problem Recognition Exercises

## Comparing Graphical Characteristics of Trigonometric Functions

For Exercises 1-16, identify which functions shown here ( $f, g, h$, and so on) have the given characteristics.
$f(x)=\sin \left(\frac{\pi}{2} x\right)+3$
$g(x)=-3 \cos \left(\frac{1}{2} x-\frac{\pi}{4}\right)$
$h(x)=3 \sin \left(-\frac{1}{2} x-\frac{\pi}{4}\right)$
$k(x)=-3 \sec (2 x+\pi)$
$m(x)=2 \csc \left(2 x-\frac{\pi}{2}\right)-3 \quad n(x)=3 \tan \left(x-\frac{\pi}{2}\right)$
$p(x)=-2 \cot \left(\frac{1}{2} x+\pi\right)$
$t(x)=-3+2 \cos x$

1. Has an amplitude of 3
2. Has no amplitude
3. Has a period of $4 \pi$
4. Has a vertical shift upward from the parent graph
5. Has no asymptotes
6. Has no $y$-intercept
7. Has domain of all real numbers
8. Has a phase shift of $\frac{\pi}{2}$
9. Has an amplitude of 2
10. Has a period of $2 \pi$
11. Has a period of $\pi$
12. Has a vertical shift downward from the parent graph
13. Has no $x$-intercepts
14. Has a range of all real numbers
15. Has a phase shift of $-\frac{\pi}{2}$
16. Has no phase shift

GN Go online for more practice problems.

## Inverse Trigonometric Functions

## What Will You Learn?

After completing this lesson, you should be able to:

- Evaluate the Inverse Sine Function
- Evaluate the Inverse Cosine and Tangent Functions
- Approximate Inverse Trigonometric Functions on a Calculator
- Compose Trigonometric Functions and Inverse Trigonometric Functions
- Apply Inverse Trigonometric Functions
- Evaluate the Inverse Secant, Cosecant, and Cotangent Functions


## Learn: Evaluate the Inverse Sine Function

Suppose that a yardstick casts a 4 -ft shadow when the Sun is at an angle of elevation $\theta$ (Figure 4-55). It seems reasonable that we should be able to determine the angle of elevation from the relationship $\tan \theta=\frac{3}{4}$.


Figure 4-55
Until now, we have always been given an angle and then asked to find the sine, cosine, or tangent of the angle. However, finding the angle of elevation of the Sun from $\tan \theta=\frac{3}{4}$ requires that we reverse this process. Given the value of the tangent, we must find an angle that produced it. Therefore, we need to use the inverse of the tangent function.

We begin our study of the inverse trigonometric functions with the inverse of the sine function. First recall that a function must be one-to-one to have an inverse function. From the graph of $y=\sin x$ (Figure 4-56), we see that any horizontal line taken between $-1 \leq y \leq 1$ intersects the graph infinitely many times. Therefore, $y=\sin x$ is not a one-to-one function.


Figure 4-56
However, suppose we restrict the domain of $y=\sin x$ to the interval $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ (shown in blue in Figure 4-57).


Figure 4-57
The graph of the restricted sine function is one-to-one and contains the entire range of $y$ values $-1 \leq y \leq 1$.

## Insights

The horizontal line test indicates that if no horizontal line intersects the graph of a function more than once, the function is one-to-one and has an inverse function. The restricted sine function is one-to-one (shown in blue in Figure 4-57).

The inverse of this restricted sine function is called the inverse sine function and is denoted by $\mathrm{sin}^{-1}$ or arcsin (shown in red in Figure 4-58).


Figure 4-58
Recall that a point $(a, b)$ on the graph of a function $f$ corresponds to the point $(b, a)$ on its inverse. So, points on the graph of $y=\sin x \operatorname{such}$ as $\left(-\frac{\pi}{2},-1\right)$ and $\left(\frac{\pi}{2}, 1\right)$ have their coordinates reversed on the graph of $y=\sin ^{-1} x$. Also notice that the graphs of $y=\sin x$ and $y=\sin ^{-1} x$ are symmetric with respect to the line $y=x$ as expected.

## The Inverse Sine Function

The inverse sine function (or arcsine) denoted by $\sin ^{-1}$ or arcsin is the inverse of the restricted sine function $y=\sin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. Therefore,

$$
\begin{aligned}
& y=\sin ^{-1} x \Leftrightarrow \sin y=x \\
& y=\arcsin x
\end{aligned} \Leftrightarrow \sin y=x
$$

where $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

## Insights

The double arrow symbol $\Leftrightarrow$ means that the statements to the left and right of the arrow are logically equivalent. That is, one statement follows from the other and vice versa.

- $y=\sin ^{-1} x$ is read as " $y$ equals the inverse sine of $x$ " and $y=\arcsin x$ is read as " $y$ equals the arcsine of $x$."
- To evaluate $y=\sin ^{-1} x$ or $y=\arcsin x$ means to find an angle $y$ between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, inclusive, whose sine value is $x$.


## Check Your Work

The notation $\sin ^{-1} x$ represents the inverse of the sine function, not the reciprocal. That is, $\sin ^{-1} x \neq \frac{1}{\sin x}$.

## Example 1: Evaluating the Inverse Sine Function

Find the exact values or state that the expression is undefined.
a. $\sin ^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
b. $\arcsin \frac{1}{2}$
c. $\sin ^{-1} 2$

## Solution:

a. Let $y=\sin ^{-1}\left(-\frac{\sqrt{3}}{2}\right)$.

Then $\sin y=-\frac{\sqrt{3}}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
Find an angle $y$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y=-\frac{\sqrt{3}}{2}$.

$y=-\frac{\pi}{3}$. Therefore $\sin ^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$.
b. Let $y=\arcsin \frac{1}{2}$.

Then $\sin y=\frac{1}{2}$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
$y=\frac{\pi}{6}$. Therefore $\arcsin \frac{1}{2}=\frac{\pi}{6}$.

Find an angle $y$ on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ such that $\sin y=\frac{1}{2}$.

c. Let $y=\sin ^{-1} 2$.

To evaluate $y=\sin ^{-1} 2$ would mean that we find an angle $y$ such that $\sin y=2$ for $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. However, recall that $-1 \leq \sin y \leq 1$ for any angle $y$. Therefore $\sin ^{-1} 2$ is undefined.

## Apply the Skills

1. Find the exact values or state that the expression is undefined.
a. $\sin ^{-1} \frac{\sqrt{2}}{2}$
b. $\arcsin (-1)$
c. $\sin ^{-1}(-3)$

## Learn: Evaluate the Inverse Cosine and Tangent Functions

The inverse cosine function and the inverse tangent function are defined in a similar way. First, the domain of $y=\cos x$ and $y=\tan x$ must each be restricted to create a one-to-one function on an interval containing all values in the range.

The restricted cosine function is defined on $0 \leq x \leq \pi$ (Figure 4-59). The graph of the inverse cosine or "arc cosine" (denoted by $\cos ^{-1}$ or arccos) is shown in red in Figure 4-60.


Figure 4-59


Figure 4-60

The restricted tangent function is defined on $-\frac{\pi}{2}<x<\frac{\pi}{2}$ (Figure 4-61). The graph of the inverse tangent or "arctangent" (denoted by $\tan ^{-1}$ or arctan) is shown in red in Figure 4-62.


Figure 4-61


Figure 4-62

The restricted tangent function has vertical asymptotes at $x=-\frac{\pi}{2}$ and $x=\frac{\pi}{2}$. The inverse tangent function has horizontal asymptotes at $y=-\frac{\pi}{2}$ and $y=\frac{\pi}{2}$. We are now ready to summarize the definitions of the inverse functions for sine, cosine, and tangent.

## Inverse Sine Function

We first consider the restricted function $y=\sin x$, where $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ and $-1 \leq y \leq 1$.
The inverse sine function (or arcsine), denoted by $\sin ^{-1}$ or arcsin, is defined by

$$
\begin{array}{rll}
y=\sin ^{-1} x & \Leftrightarrow & \sin y=x \\
y=\arcsin x & \Leftrightarrow & \sin y=x \\
-1 \leq x \leq 1 & \text { and } & -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \\
\text { Graph } &
\end{array}
$$



## Inverse Cosine Function

We first consider the restricted function $y=\cos x$, where $0 \leq x \leq \pi$ and $-1 \leq y \leq 1$.
The inverse function (or arccosine), denoted by $\cos ^{-1}$ or arccos, is defined by

$$
\begin{aligned}
& y=\cos ^{-1} x \quad \Leftrightarrow \cos y=x \\
& y=\arccos x \Leftrightarrow \cos y=x \\
& -1 \leq x \leq 1 \quad \text { and } \quad 0 \leq y \leq \pi
\end{aligned}
$$

Graph


## Inverse Tangent Function

We first consider the restricted function $y=\tan x$ where $-\frac{\pi}{2}<x<\frac{\pi}{2}$ and $y \in \mathbb{R}$. This restricted function has vertical asymptotes: $x=-\frac{\pi}{2}, x=\frac{\pi}{2}$.

The inverse tangent function (or arc tangent), denoted by $\tan ^{-1}$ or arctan, is defined by

$$
\begin{gathered}
y=\tan ^{-1} x \Leftrightarrow \tan y=x \\
y=\arctan x \Leftrightarrow \tan y=x \\
x \in \mathbb{R} \quad \text { and } \quad-\frac{\pi}{2}<y<\frac{\pi}{2}
\end{gathered}
$$

Horizontal asymptotes: $y=-\frac{\pi}{2}, y=\frac{\pi}{2}$

## Graph



## Example 2: Evaluating Inverse Trigonometric Functions

Find the exact values.
a. $\cos ^{-1}\left(-\frac{1}{2}\right)$
b. $\tan ^{-1} \sqrt{3}$
c. $\arctan (-1)$

## Solution:

a. Let $y=\cos ^{-1}\left(-\frac{1}{2}\right)$.

Then $\cos y=-\frac{1}{2}$ for $0 \leq y \leq \pi$.

$$
y=\frac{2 \pi}{3} . \text { Therefore, } \cos ^{-1}\left(-\frac{1}{2}\right)=\frac{2 \pi}{3} .
$$

Find an angle $y$ on the interval $[0, \pi]$ such that $\cos y=-\frac{1}{2}$.


## Check Your Work

Perhaps the most common error when evaluating the inverse trigonometric functions is to fail to recognize the restrictions on the range. For instance, in Example 2(a), the result of the inverse cosine function must be an angle between 0 and $\pi$.
b. Let $y=\tan ^{-1} \sqrt{3}$.

Then $\tan y=\sqrt{3}$ for $-\frac{\pi}{2}<y<\frac{\pi}{2}$.
$y=\frac{\pi}{3}$. Therefore, $\tan ^{-1} \sqrt{3}=\frac{\pi}{3}$.
c. Let $y=\arctan (-1)$.

Find an angle $y$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y=\sqrt{3}$.


Find an angle $y$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ such that $\tan y=-1$.
Then $\tan y=-1$ for $-\frac{\pi}{2}<y<\frac{\pi}{2}$.
$y=-\frac{\pi}{4}$. Therefore, $\arctan (-1)=-\frac{\pi}{4}$.


## Apply the Skills

2. Find the exact values.
a. $\cos ^{-1}(-1)$
b. $\tan ^{-1}\left(-\frac{\sqrt{3}}{3}\right)$
c. $\arctan 0$

## Learn: Approximate Inverse Trigonometric Functions on a Calculator

On a calculator we press the kND followed by the sin key, cos key, or tan key to invoke $\sin ^{-1}, \cos ^{-1}$, or $\tan ^{-1}$. By definition, the values of the inverse trigonometric functions are in radians. However, we often use the inverse functions in applications where the degree measure of an angle is desired. In Example 3, we approximate the values of several inverse trigonometric functions in both radians and degrees.

## Example 3: Approximating Values of Inverse Functions

Use a calculator to approximate the function values in both radians and degrees.
a. $\tan ^{-1} 5.69$
b. $\cos ^{-1}\left(-\frac{3}{8}\right)$
c. $\arcsin (-0.6)$

## Solution:



## Apply the Skills

3. Use a calculator to approximate the function values in both radians and degrees.
a. $\tan ^{-1}(-7.92)$
b. $\arccos \frac{2}{7}$
c. $\sin ^{-1}(-0.81)$

We sometimes encounter applications in which we use the inverse trigonometric functions to find the value of an angle where the desired angle is not within the range of the inverse function. In such cases, we need to adjust the output value from the calculator to obtain an angle in the desired quadrant. This is demonstrated in Example 4.

## Example 4: Approximating Angles Based on Characteristics About the Angle

Use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle $\theta$ subject to the given conditions.
a. $\cos \theta=-\frac{8}{11}$ and $180^{\circ} \leq \theta \leq 270^{\circ}$
b. $\tan \theta=-\frac{9}{7}$ and $\frac{\pi}{2}<\theta<\pi$

## Solution:

Entering each of these into the calculator, we receive the following output:

a. $\cos ^{-1}\left(-\frac{8}{11}\right) \approx 136.7^{\circ}$

The calculator returns a second quadrant angle.

The reference angle is
$180^{\circ}-\cos ^{-1}\left(-\frac{8}{11}\right) \approx 43.3^{\circ}$.
The corresponding third quadrant angle is $\theta \approx 180^{\circ}+43.3^{\circ} \approx 223.3^{\circ}$.

b. $\tan ^{-1}\left(-\frac{9}{7}\right) \approx-0.9098$ (radians)

The calculator returns a negative fourth quadrant angle.

The reference angle is 0.9098 .
The corresponding angle in Quadrant II is $\theta \approx \pi-0.9098 \approx 2.2318$.


## Check Your Work

The results of Example 4 can be checked on a calculator. Evaluate $\cos \left(223.3^{\circ}\right)$ in degree mode and evaluate $\tan (2.2318)$ in radian mode. The results are approximately equal to $-8 / 11$ and $-9 / 7$, respectively.

|  |  |
| :---: | :---: |
| $\cos$ (223.3) |  |
| -811 | -. 7277727577 |
| tan(2.2318) - .an. |  |
|  |  |
| -9/7 |  |
|  | -1.285714285 |

## Apply the Skills

4. Use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle $\theta$ subject to the given conditions.
a. $\sin \theta=-\frac{3}{7}$ and $180^{\circ} \leq \theta \leq 270^{\circ}$
b. $\tan \theta=-\frac{8}{3}$ and $\frac{\pi}{2}<\theta<\pi$

## Learn: Compose Trigonometric Functions and Inverse Trigonometric Functions

Recall from Lesson 3.1 that inverse functions are defined such that

$$
\left(f \circ f^{-1}\right)(x)=f\left[f^{-1}(x)\right]=x \quad \text { and } \quad\left(f^{-1} \circ f\right)(x)=f^{-1}[f(x)]=x
$$

When composing a trigonometric function with its inverse and vice versa, particular care must be taken regarding their domains.

## Composing Trigonometric Functions and Their Inverses

$$
\begin{array}{rlrr}
\sin \left(\sin ^{-1} x\right)=x & \text { for } & -1 \leq x \leq 1 \\
\sin ^{-1}(\sin x)=x & \text { for } & -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\
\cos \left(\cos ^{-1} x\right)=x & \text { for } & -1 \leq x \leq 1 \\
\cos ^{-1}(\cos x)=x & \text { for } & 0 \leq x \leq \pi \\
\tan \left(\tan ^{-1} x\right)=x & \text { for } & x \in \mathbb{R} \\
\tan ^{-1}(\tan x)=x & \text { for } & -\frac{\pi}{2}<x<\frac{\pi}{2}
\end{array}
$$

## Example 5: Composing Inverse Trigonometric Functions

Find the exact values.
a. $\sin \left(\sin ^{-1} 1\right)$
b. $\sin ^{-1}\left(\sin \frac{7 \pi}{6}\right)$

## Solution:

a. $\quad \sin \left(\sin ^{-1} 1\right)=1$
b. $\sin ^{-1}\left(\sin \frac{7 \pi}{6}\right)$

$$
\begin{aligned}
& =\sin ^{-1}\left[\sin \left(-\frac{\pi}{6}\right)\right] \\
& =-\frac{\pi}{6}
\end{aligned}
$$

The value $x=1$ lies in the domain of the inverse sine function. Therefore, we can apply the inverse property $\sin \left(\sin ^{-1} x\right)=x$.
Since $\frac{7 \pi}{6}$ is not on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, it is not in the domain of the restricted sine function.
Therefore, we cannot conclude that $\sin ^{-1}(\sin x)=x$.
Rewrite $\sin \frac{7 \pi}{6}$ as an equivalent expression with an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
Apply the property $\sin ^{-1}(\sin x)=x$.

## Insights

As an alternative method, we can simplify the expressions from Example 5 by applying the order of operations.

$$
\sin \left[\sin ^{-1}(1)\right]=\sin \frac{\pi}{2}=1
$$

and

$$
\begin{aligned}
\sin ^{-1}\left(\sin \frac{7 \pi}{6}\right) & =\sin ^{-1}\left(-\frac{1}{2}\right) \\
& =-\frac{\pi}{6}
\end{aligned}
$$

## Apply the Skills

5. Find the exact values.
a. $\cos \left[\cos ^{-1}(-1)\right]$
b. $\cos ^{-1}\left(\cos \frac{4 \pi}{3}\right)$

## Example 6: Composing Trigonometric Functions and Inverse Trigonometric Functions

Find the exact value of $\cos \left[\tan ^{-1}\left(-\frac{8}{15}\right)\right]$.

## Solution:

Let $\theta=\tan ^{-1}\left(-\frac{8}{15}\right)$.
Since $-\frac{8}{15}<0$, angle $\theta$ is on the interval $\left(-\frac{\pi}{2}, 0\right)$.

Let $P(15,-8)$ be a point on the terminal side of $\theta$.
Then, $r=\sqrt{(15)^{2}+(-8)^{2}}=\sqrt{289}=17$.
$\cos \left[\tan ^{-1}\left(-\frac{8}{15}\right)\right]=\cos \theta=\frac{15}{17}$


## Check Your Work

You can confirm your answer from Example 6 on a calculator by applying the order of operations.


## Apply the Skills

6. Find the exact value of $\sin \left[\tan ^{-1}\left(\frac{12}{5}\right)\right]$.

## Example 7: Composing Trigonometric Functions and Inverse Trigonometric Functions

Find the exact value of $\sin \left[\cos ^{-1}\left(-\frac{3}{7}\right)\right]$.

## Solution:

Let $\theta=\cos ^{-1}\left(-\frac{3}{7}\right)$.
Since $-\frac{3}{7}<0$, angle $\theta$ is on the interval $\left(\frac{\pi}{2}, \pi\right)$.
Let $P(-3, y)$ be a point on the terminal side of $\theta$.
From Figure 4-63:

$$
\begin{aligned}
(-3)^{2}+y^{2} & =7^{2} \\
9+y^{2} & =49 \\
y^{2} & =40 \\
y & =\sqrt{40}=2 \sqrt{10}
\end{aligned}
$$



Figure 4-63


Therefore, $\sin \left[\cos ^{-1}\left(-\frac{3}{7}\right)\right]=\sin \theta=\frac{2 \sqrt{10}}{7}$.

## Apply the Skills

7. Find the exact value of $\cos \left[\sin ^{-1}\left(-\frac{2}{11}\right)\right]$.

In Example 8, we use a right triangle to visualize the composition of a trigonometric function and an inverse function, and then write an equivalent algebraic expression. This skill is used often in calculus.

## Example 8: Writing a Trigonometric Expression as an Algebraic Expression

Write the expression $\tan \left(\sin ^{-1} \frac{\sqrt{x^{2}-16}}{x}\right)$ as an algebraic expression for $x>4$.

## Solution:

Let $\theta=\sin ^{-1} \frac{\sqrt{x^{2}-16}}{x}$.
Since $x>4, \frac{\sqrt{x^{2}-16}}{x}>0$, and we know that $\theta$ is an acute angle.

We can set up a representative right triangle using the relationship
$\sin \theta=\frac{\mathrm{opp}}{\mathrm{hyp}}=\frac{\sqrt{x^{2}-16}}{x}$


To find an expression for the adjacent side $a$, apply the Pythagorean theorem.

$$
\begin{aligned}
a^{2}+\left(\sqrt{x^{2}-16}\right)^{2} & =x^{2} \\
a^{2}+x^{2}-16 & =x^{2} \\
a^{2} & =16 \\
a & =4
\end{aligned}
$$



Therefore, $\tan \left(\sin ^{-1} \frac{\sqrt{x^{2}-16}}{x}\right)=\tan \theta=\frac{\sqrt{x^{2}-16}}{4}$.

## Apply the Skills

8. Write the expression $\tan \left(\sin ^{-1} \frac{x}{\sqrt{x^{2}+9}}\right)$ as an algebraic expression for $x>0$.

## Learn: Apply Inverse Trigonometric Functions

We now revisit the scenario that we used to introduce this lesson. Example 9 shows how inverse trigonometric functions can help us find the measure of an unknown angle in an application.

## Example 9: Applying an Inverse Trigonometric Function

A yardstick casts a 4-ft shadow when the Sun is at an angle of elevation $\theta$. Determine the angle of elevation of the Sun to the nearest tenth of a degree.

## Solution:

The tip of the shadow and the points at the top and bottom of the yardstick form a right triangle with $\theta$ representing the angle of elevation of the Sun.

$\tan \theta=\frac{3}{4}$

$$
\begin{aligned}
& \theta=\tan ^{-1} \frac{3}{4} \quad \begin{array}{l}
\text { Use the inverse tangent to find the acute angle } \theta \\
\text { whose tangent is } \frac{3}{4} .
\end{array}
\end{aligned}
$$

$$
\theta \approx 36.9^{\circ}
$$

Be sure that your calculator is in degree mode.

## Apply the Skills

9. For the construction of a house, a $16-\mathrm{ft}$ by 6 - ft wooden frame is made. Find the angle that the diagonal beam makes with the base of the frame. Round to the nearest tenth of a degree.


## Learn: Evaluate the Inverse Secant, Cosecant, and Cotangent Functions

We need to establish intervals over which the secant, cosecant, and cotangent functions are one-to-one before we can define their corresponding inverse functions. These intervals are chosen to accommodate the entire range of function values and to make the restricted functions one-to-one. The graphs of the restricted secant, cosecant, and cotangent functions and their inverses are shown in Table 4-17.

| Inverse Function Relationship | Graphs |
| :---: | :---: |
| $y=\sec ^{-1} x \Leftrightarrow \sec y=x$ <br> where $\begin{array}{r} \|x\| \geq 1 \quad \text { and } \quad 0 \leq y<\frac{\pi}{2} \\ \quad \text { or } \quad \frac{\pi}{2}<y \leq \pi \end{array}$ |  |
| $y=\csc ^{-1} x \Leftrightarrow \csc y=x$ <br> where $\begin{array}{rlrl} \|x\| \geq 1 & \text { and } & -\frac{\pi}{2} & \leq y<0 \\ \text { or } & 0 & <y \leq \frac{\pi}{2} \end{array}$ |  |
| $y=\cot ^{-1} x \Leftrightarrow \cot y=x$ <br> where <br> $x$ is any real number and $0<y<\pi$ |   |

## Insights

It is important to note that there are infinitely many intervals over which the secant, cosecant, and cotangent functions can be restricted to define the inverse functions. These restrictions are not universally agreed upon.

A calculator does not have keys for inverse secant, cosecant, or cotangent. Therefore, to evaluate an inverse secant, cosecant, or cotangent on a calculator, we first rewrite the expression using inverse sine or cosine.

## Example 10: Approximating the Value of an Inverse Secant, Cosecant, or Cotangent

Approximate each expression in radians, rounded to 4 decimal places.
a. $\csc ^{-1} 3$
b. $\cot ^{-1}\left(-\frac{3}{4}\right)$

## Solution:

a. Let $\theta=\csc ^{-1} 3$.

Then $\csc \theta=3$ for $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \theta \neq 0$.
Since $y=\sin ^{-1} x$ has the same range as $y=\csc ^{-1} x$ for $x \neq 0$, we have:

$$
\csc \theta=3 \text { implies that } \frac{1}{\sin \theta}=3 .
$$

So, $\sin \theta=\frac{1}{3}$ and $\theta=\sin ^{-1} \frac{1}{3} \approx 0.3398$.
b. Let $\theta=\cot ^{-1}\left(-\frac{3}{4}\right)$.

Then $\cot \theta=-\frac{3}{4}$ for $0<\theta<\pi$.
Since the argument $-\frac{3}{4}$ is negative, $\theta=\cot ^{-1}\left(-\frac{3}{4}\right)$ must be a second quadrant angle.

The related expression using the inverse tangent function will not return a second quadrant angle but rather an angle on the interval $-\frac{\pi}{2}<\theta<0$. So, we will rewrite the expression $\cot \theta=-\frac{3}{4}$ using the cosine function.

From Figure 4-64, $\cos \theta=-\frac{3}{5}$.
Therefore, $\theta=\cos ^{-1}\left(-\frac{3}{5}\right) \approx 2.2143$.


Figure 4-64

## Apply the Skills

10. Approximate each expression in radians, rounded to 4 decimal places.
a. $\sec ^{-1}(-4)$
b. $\cot ^{-1}\left(-\frac{5}{12}\right)$

## Practice 4-7

## Practice Exercises

## Prerequisite Review

R.1. Determine if the relation defines $y$ as a one-to-one function of $x$.

R.2. Determine if the relation defines $y$ as a one-to-one function of $x$.

R.3. A one-to-one function is given. Write an equation for the inverse function.

$$
g(x)=\frac{3-x}{4}
$$

R.4. The graph of a function is given. Graph the inverse function.

## Concept Connections

1. A function must be $\qquad$ on its entire domain to have an inverse function.
2. If $\left(\frac{2 \pi}{3},-\frac{1}{2}\right)$ is on the graph of $y=\cos x$, what is the related point on $y=\cos ^{-1} x$ ?
3. The graph of $y=\tan ^{-1} x$ has two
$\qquad$ (horizontal/vertical) asymptotes represented by the equations
$\qquad$ and $\qquad$
4. The domain of $y=\arctan x$ is The output is a real number (or angle in radians) between $\qquad$ and
5. In interval notation, the domain of $y=\cos ^{-1} x$ is $\qquad$ The output is a real number (or angle in radians) between $\qquad$ and
$\qquad$ inclusive.
6. In interval notation, the domain of $y=\sin ^{-1} x$ is $\qquad$ The output is a real number (or angle in radians) between $\qquad$ and
$\qquad$ inclusive.

## Learn: Evaluate the Inverse Sine Function

For Exercises 7-12, find the exact value or state that the expression is undefined. (See Example 1, p. 568)
7. $\arcsin \frac{\sqrt{2}}{2}$
8. $\sin ^{-1}\left(-\frac{1}{2}\right)$
9. $\sin ^{-1} \pi$
10. $\sin ^{-1} \frac{3}{2}$
11. $\arcsin \left(-\frac{\sqrt{2}}{2}\right)$
12. $\arcsin \frac{\sqrt{3}}{2}$

For Exercises 13-16, find the exact value.
13. $\sin ^{-1} \frac{\sqrt{3}}{2}+\sin ^{-1} \frac{1}{2}$
14. $\sin ^{-1} \frac{\sqrt{2}}{2}-\sin ^{-1}(-1)$
15. $2 \sin ^{-1}\left(-\frac{\sqrt{2}}{2}\right)-\frac{\pi}{3}$
16. $\frac{\pi}{2}+3 \sin ^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

## Learn: Evaluate the Inverse Cosine and Tangent Functions

For Exercises 17-28, find the exact value or state that the expression is undefined. (See Example 2, p. 571)
17. $\arccos \left(-\frac{\sqrt{2}}{2}\right)$
18. $\tan ^{-1}(-\sqrt{3})$
19. $\cos ^{-1} 0$
20. $\tan ^{-1} 0$
21. $\cos ^{-1}(-2)$
22. $\arccos \frac{4}{3}$
23. $\arctan \frac{\sqrt{3}}{3}$
24. $\cos ^{-1} \frac{\sqrt{2}}{2}$
25. $\tan ^{-1} 1$
26. $\arctan \left(-\frac{\sqrt{3}}{3}\right)$
27. $\arccos \frac{\sqrt{3}}{2}$
28. $\cos ^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

For Exercises 29-32, find the exact value.
29. $\tan ^{-1}(-1)+\tan ^{-1} \sqrt{3}$
30. $\cos ^{-1} \frac{\sqrt{2}}{2}+\cos ^{-1} \frac{1}{2}$
31. $2 \cos ^{-1}\left(-\frac{\sqrt{3}}{2}\right)-\tan ^{-1} \frac{\sqrt{3}}{3}$
32. $3 \tan ^{-1} 1+\cos ^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

## Learn: Approximate Inverse Trigonometric Functions on a Calculator

For Exercises 33-36, use a calculator to approximate the function values in both radians and degrees. (See Example 3, p. 572)
33. a. $\cos ^{-1} \frac{3}{8}$
b. $\tan ^{-1} 25$
c. $\arcsin 0.05$
34. a. $\sin ^{-1} 0.93$
b. $\arccos 0.17$
c. $\arctan \frac{7}{4}$
35. a. $\tan ^{-1}(-28)$
b. $\arccos \frac{\sqrt{3}}{5}$
c. $\sin ^{-1}(-0.14)$

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36. a. $\cos ^{-1}(-0.75)$
b. $\tan ^{-1} \frac{8}{3}$
c. $\arcsin \frac{\pi}{7}$

For Exercises 37-46, use a calculator to approximate the degree measure (to 1 decimal place) or radian measure (to 4 decimal places) of the angle $\theta$ subject to the given conditions. (See Example 4, p. 573)
37. $\cos \theta=-\frac{5}{6}$ and $180^{\circ}<\theta<270^{\circ}$
38. $\sin \theta=-\frac{4}{5}$ and $180^{\circ}<\theta<270^{\circ}$
39. $\tan \theta=-\frac{12}{5}$ and $90^{\circ}<\theta<180^{\circ}$
40. $\cos \theta=-\frac{2}{13}$ and $180^{\circ}<\theta<270^{\circ}$
41. $\sin \theta=\frac{12}{19}$ and $90^{\circ}<\theta<180^{\circ}$
42. $\tan \theta=\frac{7}{15}$ and $180^{\circ}<\theta<270^{\circ}$
43. $\cos \theta=-\frac{5}{8}$ and $\pi<\theta<\frac{3 \pi}{2}$
44. $\tan \theta=-\frac{9}{5}$ and $\frac{\pi}{2}<\theta<\pi$
45. $\sin \theta=-\frac{17}{20}$ and $\pi<\theta<\frac{3 \pi}{2}$
46. $\cos \theta=\frac{1}{17}$ and $\frac{3 \pi}{2}<\theta<2 \pi$

## Learn: Compose Trigonometric Functions and Inverse Trigonometric Functions

For Exercises 47-58, find the exact values.
(See Example 5, p. 575)
47. $\sin \left(\sin ^{-1} \frac{\sqrt{2}}{2}\right)$
48. $\arcsin \left(\sin \frac{5 \pi}{3}\right)$
49. $\sin ^{-1}\left(\sin \frac{5 \pi}{4}\right)$
50. $\sin \left[\sin ^{-1}\left(-\frac{1}{2}\right)\right]$
51. $\cos \left(\cos ^{-1} \frac{2}{3}\right)$
52. $\arccos \left(\cos \frac{11 \pi}{6}\right)$
53. $\cos ^{-1}\left(\cos \frac{4 \pi}{3}\right)$
54. $\cos \left[\cos ^{-1}\left(-\frac{1}{2}\right)\right]$
55. $\tan ^{-1}\left(\tan \frac{2 \pi}{3}\right)$
56. $\tan \left(\tan ^{-1} 2\right)$
57. $\tan [\arctan (-\pi)]$
58. $\tan ^{-1}\left[\tan \left(-\frac{\pi}{6}\right)\right]$

For Exercises 59-70, find the exact values. (See Examples 6-7, pp. 575-576)
59. $\cos \left(\tan ^{-1} \frac{\sqrt{3}}{3}\right)$
60. $\sin \left(\cos ^{-1} \frac{1}{2}\right)$
61. $\tan \left[\sin ^{-1}\left(-\frac{2}{3}\right)\right]$
62. $\sin \left[\cos ^{-1}\left(-\frac{2}{3}\right)\right]$
63. $\sin \left(\cos ^{-1} \frac{3}{4}\right)$
64. $\sin \left(\tan ^{-1} \frac{4}{3}\right)$
65. $\sin \left[\tan ^{-1}(-1)\right]$
66. $\cos \left[\tan ^{-1}(-1)\right]$
67. $\cos \left[\sin ^{-1}\left(-\frac{2}{7}\right)\right]$
68. $\cos \left[\tan ^{-1}\left(-\frac{5}{12}\right)\right]$
69. $\tan \left[\cos ^{-1}\left(-\frac{5}{6}\right)\right]$
70. $\tan \left[\sin ^{-1}\left(-\frac{\sqrt{5}}{3}\right)\right]$

For Exercises 71-76, write the given expression as an algebraic expression. It is not necessary to rationalize the denominator. (See Example 8, p. 577)
71. $\cos \left(\sin ^{-1} \frac{x}{\sqrt{25+x^{2}}}\right)$ for $x>0$.
72. $\cot \left(\cos ^{-1} \frac{\sqrt{x^{2}-1}}{x}\right)$ for $x>1$.
73. $\sin \left(\tan ^{-1} x\right)$ for $x>0$.
74. $\tan \left(\sin ^{-1} x\right)$ for $|x|<1$.
75. $\tan \left(\cos ^{-1} \frac{3}{x}\right)$ for $x>3$.
76. $\sin \left(\cos ^{-1} \frac{\sqrt{x^{2}-25}}{x}\right)$ for $x>5$.

## Learn: Apply Inverse Trigonometric Functions

77. To meet the requirements of the Americans with Disabilities Act (ADA), a wheelchair ramp must have a slope of 1:12 or less. That is, for every 1 in. of "rise," there must be at least 12 in. of "run." (See Example 9, p. 578)
a. If a wheelchair ramp is constructed with the maximum slope, what angle does the ramp make with the ground? Round to the nearest tenth of a degree.
b. If the ramp is 22 ft long, how much elevation does the ramp provide? Round to the nearest tenth of a foot.
78. A student measures the length of the shadow of the Washington Monument to be 620 ft . If the Washington Monument is 555 ft tall, approximate the angle of elevation of the Sun to the nearest tenth of a degree.
79. A balloon advertising an open house is stabilized by two cables of lengths 20 ft and 40 ft tethered to the ground. If the perpendicular distance from the balloon to the ground is $10 \sqrt{3} \mathrm{ft}$, what is the degree measure of the angle each cable makes with the ground? Round to the nearest tenth of a degree if necessary.

80. A group of campers hikes down a steep path. One member of the group has an altimeter to measure altitude. If the path is 1250 yd and the amount of altitude lost is 480 yd , what is the angle of incline? Round to the nearest tenth of a degree.

81. Navajo Tube Hill, a snow tubing hill in Utah, is 550 ft long and has a 75 -ft vertical drop. Find the angle of incline of the hill. Round to the nearest tenth of a degree.
82. A ski run on Giant Steps Mountain in Utah is 1475 m long. The difference in altitude from the beginning to the end of the run is 350 m . Find the angle of the ski run. Round to the nearest tenth of a degree.
83. When granular material such as sand or gravel is poured onto a horizontal surface it forms a right circular cone. The angle that the surface of the cone makes with the horizontal is called the angle of repose. The angle of repose depends on a number of variables such as the shape of the particles and the amount of friction between them. "Stickier" particles have a greater angle of repose, and "slippery" particles have a smaller angle of repose. Find the angle of repose for the pile of dry sand. Round to the nearest degree.

84. For the construction of a bookcase, a 5-ft by 3 -ft wooden frame is made with a cross-brace in the back for stability. Find the angle that the diagonal brace makes with the base of the frame. Round to the nearest tenth of a degree.


Learn: Evaluate the Inverse Secant, Cosecant, and Cotangent Functions
85. Show that $\sec ^{-1} x=\cos ^{-1} \frac{1}{x}$ for $x \geq 1$.
86. Show that $\csc ^{-1} x=\sin ^{-1} \frac{1}{x}$ for $x \geq 1$.
87. Show that $\sec ^{-1} x+\csc ^{-1} x=\frac{\pi}{2}$ for $x \geq 1$.
88. Complete the table, giving the domain and range in interval notation for each inverse function.

| Inverse Function | Domain | Range |
| :--- | :--- | :--- |
| $y=\sin ^{-1} x$ |  |  |
| $y=\csc ^{-1} x$ |  |  |
| $y=\cos ^{-1} x$ |  |  |
| $y=\sec ^{-1} x$ |  |  |
| $y=\tan ^{-1} x$ |  |  |
| $y=\cot ^{-1} x$ |  |  |

For Exercises 89-94, find the exact values.
89. $\sec ^{-1} \frac{2 \sqrt{3}}{3}$
90. $\sec ^{-1}(-\sqrt{2})$
91. $\csc ^{-1}(-1)$
92. $\csc ^{-1}(2)$
93. $\cot ^{-1} \sqrt{3}$
94. $\cot ^{-1}(1)$

For Exercises 95-100, use a calculator to approximate each expression in radians, rounded to 4 decimal places. (See Example 10, p. 580)
95. $\mathrm{sec}^{-1} \frac{7}{4}$
96. $\csc ^{-1} \frac{6}{5}$
97. $\csc ^{-1}(-8)$
98. $\sec ^{-1}(-6)$
99. $\cot ^{-1}\left(-\frac{8}{15}\right)$
100. $\cot ^{-1}\left(-\frac{24}{7}\right)$

## Mixed Exercises

For Exercises 101-104, find the exact value if possible. Otherwise find an approximation to 4 decimal places or state that the expression is undefined.
101. a. $\sin \frac{\pi}{4}$
b. $\sin ^{-1} \frac{\pi}{4}$
c. $\sin ^{-1} \frac{\sqrt{2}}{2}$
102.a. $\cos \frac{2 \pi}{3}$
b. $\cos ^{-1} \frac{2 \pi}{3}$
c. $\cos ^{-1}\left(-\frac{1}{2}\right)$
103. a. $\cos ^{-1}\left(-\frac{\pi}{6}\right)$
b. $\cos \left(-\frac{\pi}{6}\right)$
c. $\cos ^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
104.a. $\sin ^{-1} \frac{7 \pi}{6}$
b. $\sin \frac{7 \pi}{6}$
c. $\sin ^{-1}\left(-\frac{1}{2}\right)$

For Exercises 105-108, find the inverse function and its domain and range.
105. $f(x)=3 \sin x+2$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
106. $g(x)=6 \cos x-4$ for $0 \leq x \leq \pi$
107. $h(x)=\frac{\pi}{4}+\tan x$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$
108. $k(x)=\pi+\sin x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$
109. A video camera located at ground level follows the liftoff of an Atlas V rocket from the Kennedy Space Center. Suppose that the camera is 1000 m from the launch pad.

a. Write the angle of elevation $\theta$ from the camera to the rocket as a function of the rocket's height, $h$.
b. Without the use of a calculator, will the angle of elevation be less than $45^{\circ}$ or greater than $45^{\circ}$ when the rocket is 400 m high?
c. Use a calculator to find $\theta$ to the nearest tenth of a degree when the rocket's height is $400 \mathrm{~m}, 1500 \mathrm{~m}$, and 3000 m .
110. The effective focal length $f$ of a camera is the distance required for the lens to converge light to a single focal point. The angle of view $\alpha$ of a camera describes the angular range (either horizontally, vertically, or diagonally) that is imaged by a camera.

a. Show that $\alpha=2 \arctan \frac{d}{2 f}$ where $d$ is the dimension of the image sensor or film.
b. A typical $35-\mathrm{mm}$ camera has image dimensions of 24 mm (vertically) by 36 mm (horizontally). If the focal length is 50 mm , find the vertical and horizontal viewing angles. Round to the nearest tenth of a degree.

For Exercises 111-114, use the relationship given in the right triangle and the inverse sine, cosine, and tangent functions to write $\theta$ as a function of $x$ in three different ways. It is not necessary to rationalize the denominator.
111.

112.

113.

114.


For Exercises 115-120, find the exact solution to each equation.
115. $-2 \sin ^{-1} x-\pi=0$
116. $3 \cos ^{-1} x-\pi=0$
117. $6 \cos ^{-1} x-3 \pi=0$
118. $4 \sin ^{-1} x+\pi=0$
119. $4 \tan ^{-1} 2 x=\pi$
120. $6 \tan ^{-1} 2 x=2 \pi$

## Write About It

121. Explain the difference between the reciprocal of a function and the inverse of a function.
122. Explain the flaw in the logic:
$\cos \left(-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$. Therefore, $\cos ^{-1} \frac{\sqrt{2}}{2}=-\frac{\pi}{4}$.
123. In terms of angles, explain what is meant when we find $\sin ^{-1}\left(-\frac{1}{2}\right)$.
124. Why must the domains of the sine, cosine, and tangent functions be restricted in order to define their inverse functions?

## Expanding Your Skills

125. In calculus, we can show that the area below the graph of $f(x)=\frac{1}{1+x^{2}}$, above the $x$-axis, and between the lines $x=a$ and $x=b$ for $a<b$, is given by $\tan ^{-1} b-\tan ^{-1} a$.

a. Find the area under the curve between $x=0$ and $x=1$.
b. Evaluate $f(0)$ and $f(1)$.
c. Find the area of the trapezoid defined by the points $(0,0),(1,0),[0, f(0)]$, and [1,f(1)] to confirm that your answer from part (a) is reasonable.

126. In calculus, we can show that the area below the graph of $f(x)=\frac{1}{\sqrt{1-x^{2}}}$, above the $x$-axis, and between the lines $x=a$ and $x=b$ for $a<b$, is given by $\sin ^{-1} b-\sin ^{-1} a$.

a. Find the area under the curve between $x=0$ and $x=0.5$.
b. Evaluate $f(0)$ and $f(0.5)$.
c. Find the area of the trapezoid defined by the points $(0,0),(1,0),[0, f(0)]$, and [ $0.5, f(0.5)$ ] to confirm that your answer from part (a) is reasonable.
127. The vertical viewing angle $\theta$ to a movie screen is the angle formed from the bottom of the screen to a viewer's eye to the top of the screen. Suppose that the viewer is sitting $x$ horizontal feet from an IMAX screen 53 ft high and that the bottom of the screen is 10 vertical feet above the viewer's eye level. Let $\alpha$ be the angle of elevation to the bottom of the screen.

a. Write an expression for $\tan \alpha$.
b. Write an expression for $\tan (\alpha+\theta)$.
c. Using the relationships found in parts
(a) and (b), show that
$\theta=\tan ^{-1} \frac{63}{x}-\tan ^{-1} \frac{10}{x}$.

## Technology Connections

128. Refer to the movie screen and observer in Exercise 127. The vertical viewing angle is given by $\theta=\tan ^{-1} \frac{63}{x}-\tan ^{-1} \frac{10}{x}$.
a. Find the vertical viewing angle (in radians) for an observer sitting 15, 25, and 35 ft away. Round to 2 decimal places.
b. Graph $y=\tan ^{-1} \frac{63}{x}-\tan ^{-1} \frac{10}{x}$ on a graphing utility on the window $[0,100,10]$ by $[0,1,0.1]$.
c. Use the Maximum feature on the graphing utility to estimate the distance that an observer should sit from the screen to produce the maximum viewing angle. Round to the nearest tenth of a foot.
129. a. Graph the functions $y=\tan ^{-1} x$ and $y=x-\frac{x^{3}}{3}+\frac{x^{5}}{5}$ on the window $\left[-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}\right]$ by $[-2,2,1]$.
b. Graph the functions $y=\tan ^{-1} x$ and $y=\frac{x}{1+\frac{x^{2}}{3-x^{2}}}$ on the window $\left[-\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}\right]$ by $[-2,2,1]$.
c. How do the functions in parts (a) and (b) compare for values of $x$ taken close to 0 ? The function values are close for $x$ taken near 0 .
Go online for more practice problems.

## Chapter 4 Review

## Chapter 4 Key Concepts

## Lesson 4-1 Angles and Their Measure Reference

\section*{| Key Concepts | Reference |
| :--- | :--- |}

The measure of an angle may be given in degrees, where $1^{\circ}$ is $\frac{1}{360}$ of a pp. 447-448 full rotation. A measure of $1^{\circ}$ can be further divided into 60 equal parts called minutes. Each minute can be divided into 60 equal parts called seconds.

The measure of an angle may be given in radians, where 1 radian is the measure of the central angle that intercepts an arc equal in length to the radius of the circle.

| $\theta=\frac{s}{r}$ gives the measure of a central angle $\theta$ (in radians), where $s$ is the <br> length of the arc intercepted by $\theta$ and $r$ is the radius of the circle. | p. 451 |
| :--- | :--- | :--- |
| The relationship $180^{\circ}=\pi$ radians provides the conversion factors to <br> convert from radians to degrees and vice versa. | p. 452 |
| Two angles with the same initial and terminal sides are called <br> coterminal angles. Coterminal angles differ in measure by an integer <br> multiple of $360^{\circ}$ or $2 \pi$ radians. | p. 453 |
| Given a circle of radius $r$, the length $s$ of an arc intercepted by a central <br> angle $\theta$ (in radians) is given by $s=r \theta$. | p. 456 |
| Points on the Earth are identified by their latitude and longitude. <br> Latitude is the angular measure of a central angle measuring north or <br> south from the equator. Lines of longitude, also called meridians, are <br> circles passing through both poles and running perpendicular to the <br> equator. | p. 457 |
| Suppose that a point on a circle of radius $r$ moves through an angle of $\theta$ <br> radians in time $t$. Then the angular and linear speeds of the point are <br> given by | p. 458 |
| angular speed: <br> linear speed:$\quad$$\quad v=\frac{s}{t}$ or $\quad v=\frac{r \theta}{t} \quad$ or $\quad v=r \omega$ |  |
| The area $A$ of a sector of a circle of radius $r$ with central angle $\theta$ <br> (in radians) is given by $A=\frac{1}{2} r^{2} \theta$. | p. 460 |

## Lesson 4-2 Trigonometric Functions Defined on the Unit Circle

| Key Concepts | Reference |
| :--- | :--- |

The unit circle is a circle of radius $r$ with center at the origin of a p. 470 rectangular coordinate system: $x^{2}+y^{2}=1$.
The six trigonometric functions can be defined as functions of real
p. 471 numbers by using the unit circle. Let $P(x, y)$ be the point associated with a real number $t$ measured along the circumference of the unit circle from the point $(1,0)$.

$$
\begin{array}{lll}
\sin t=y & \cos t=x & \tan t=\frac{y}{x}(x \neq 0) \\
\csc t=\frac{1}{y}(y \neq 0) & \sec t=\frac{1}{x}(x \neq 0) & \cot t=\frac{x}{y}(y \neq 0)
\end{array}
$$

The real number $t$ taken along the circumference of the unit circle
p. 473 gives the number of radians of the corresponding central angle $\theta$; that is, $\theta=t$ radians.

- The domain of both the sine function and cosine function is all real numbers.
- The domain of both the tangent function and secant function excludes real numbers $t$ that are odd multiples of $\frac{\pi}{2}$.
- The domain of both the cotangent function and cosecant function excludes real numbers $t$ that are multiples of $\pi$.
A function $f$ is periodic if $f(t+p)=f(t)$ for some constant $p$.
The smallest positive value $p$ for which $f$ is periodic is called the period of $f$.
- The period of the sine, cosine, secant, and cosecant functions is $2 \pi$.
- The period of the tangent and cotangent functions is $\pi$.
- The cosine and secant functions are even functions. $f(-x)=f(x)$.
- The sine, cosecant, tangent, and cotangent functions are odd functions. $f(-x)=-f(x)$.


## Lesson 4-3 Right Triangle Trigonometry

| Key Concepts | Reference |
| :---: | :---: |
| Each trigonometric function of an acute angle $\theta$ is one of the six possible ratios of the sides of a right triangle containing $\theta$. $\begin{aligned} & \sin \theta=\frac{\text { opp }}{\text { hyp }}, \cos \theta=\frac{\mathrm{adj}}{\text { hyp }}, \tan \theta=\frac{\mathrm{opp}}{\text { adj }}, \csc \theta=\frac{\text { hyp }}{\text { opp },} \\ & \sec \theta=\frac{\text { hyp }}{\text { adj } ;} \cot \theta=\frac{\text { adj }}{\text { opp }} \end{aligned}$ | p. 495 |
| Reciprocal Identities: $\csc \theta=\frac{1}{\sin \theta}, \sec \theta=\frac{1}{\cos \theta}, \cot \theta=\frac{1}{\tan \theta}$ <br> Quotient Identities: $\tan \theta=\frac{\sin \theta}{\cos \theta}, \cot \theta=\frac{\cos \theta}{\sin \theta}$ | p. 502 |
| Pythagorean Identities: $\sin ^{2} \theta+\cos ^{2} \theta=1, \tan ^{2} \theta+1=\sec ^{2} \theta, 1+\cot ^{2} \theta=\csc ^{2} \theta$ | p. 502 |
| The cofunction identities indicate that cofunctions of complementary angles are equal. | p. 503 |
| An angle of elevation or depression from an observer to an object is used in many applications of trigonometric functions. | p. 504 |

## Lesson 4-4 Trigonometric Functions of Any Angle

| Key Concepts | Reference |
| :---: | :---: |
| $\begin{array}{lll} \sin \theta=\frac{y}{r} & \csc \theta=\frac{r}{y}(y \neq 0) & \text { Let } P(x, y) \text { be a point on the } \\ \cos \theta=\frac{x}{r} & \sec \theta=\frac{r}{x}(x \neq 0) & \text { terminal side of angle } \theta \\ \tan \theta=\frac{y}{x}(x \neq 0) & \cot \theta=\frac{x}{y}(y \neq 0) & \text { drawn in standard position, } \\ & & \text { and } r \text { be the distance from } \\ P \text { to the origin. } \end{array}$ | p. 516 |
| Let $\theta$ be an nonquadrantal angle in standard position. The reference angle for $\theta$ is the acute angle $\theta^{\prime}$ formed by the terminal side of $\theta$ and the horizontal axis. | p. 517 |
| The value of each trigonometric function of $\theta$ is the same as the corresponding function of the reference angle $\theta^{\prime}$ except possibly for the sign. | p. 519 |

## Lesson 4-5 Graphs of Sine and Cosine Functions

| Key Concepts | Reference |
| :---: | :---: |
|  | p. 530 |
| Characteristics of $y=A \sin (B x-C)+D$ and $y=A \cos (B x-C)+D$ with $B>0$ : <br> 1. The amplitude is $\|A\|$. <br> 2. The period is $\frac{2 \pi}{B}$. <br> 3. The phase shift is $\frac{C}{B}$. <br> 4. The vertical shift is $D$. <br> 5. One full cycle is given on the interval $0 \leq B x-C \leq 2 \pi$. <br> 6. The domain is the set of real numbers. <br> 7. The range is $-\|A\|+D \leq y \leq\|A\|+D$. | p. 536 |

## Lesson 4-6 Graphs of Other Trigonometric Functions

## Key Concepts

The graph of $y=\csc x=\frac{1}{\sin x}$ has vertical asymptotes where $\sin x=0$.


Period: $2 \pi$
Amplitude: None
Domain: $\{x \mid x \neq n \pi$ for integers $n\}$
Range: $(-\infty,-1] \cup[1, \infty)$
Vertical asymptotes: $x=n \pi$
Symmetric to the origin (odd function)

The graph of $y=\sec x=\frac{1}{\cos x}$ has
pp. 553-554 vertical asymptotes where $\cos x=0$.


Period: $2 \pi$
Amplitude: None
Domain:
$\left\{x \left\lvert\, x \neq \frac{(2 n+1) \pi}{2}\right.\right.$ for integers $\left.n\right\}$
Range: $(-\infty,-1] \cup[1, \infty)$
Vertical asymptotes: $x=\frac{(2 n+1) \pi}{2}$
Symmetric to the $y$-axis (even function)
The graph of $y=\cot x=\frac{\cos x}{\sin x}$ has pp. 556-557 vertical asymptotes where $\sin x=0$.


Period: $\pi$
Amplitude: None
Domain: $\{x \mid x \neq n \pi$ for integers $n\}$
Range: $(-\infty, \infty)$
Vertical asymptotes: $x=n \pi$
Symmetric to the origin (odd function)

| To graph variations of $y=\csc x$ or $y=\sec x$, use the graph of the <br> related reciprocal function for reference. | p. 554 |
| :--- | :--- |
| To graph variations of $y=\tan x$ and $y=\cot x$, | p. 558 |
| 1. First graph two consecutive asymptotes. |  |
| 2. Plot an $x$-intercept halfway between the asymptotes. |  |
| 3. Sketch the general shape of the "parent" function between the |  |
| 4symptotes. |  |
| 4. Apply a vertical shift if applicable. |  |
| 5. Sketch additional cycles to the right or left as desired. |  |

## Lesson 4-7 Inverse Trigonometric Functions

## Key Concepts

Reference
p. 567
defined by
$y=\sin ^{-1} x \Leftrightarrow \sin y=x$ for $-1 \leq x \leq 1$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.
To compose the sine function and its inverse,
$\sin \left(\sin ^{-1} x\right)=x$ for $-1 \leq x \leq 1$ and $\sin ^{-1}(\sin x)=x$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$.
The inverse cosine function (or arccosine), denoted by $\cos ^{-1}$ or arccos
p. 570 is defined by
$y=\cos ^{-1} x \Leftrightarrow \cos y=x$ for $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.
To compose the cosine function and its inverse,
$\cos \left(\cos ^{-1} x\right)=x$ for $-1 \leq x \leq 1$ and $\cos ^{-1}(\cos x)=x$ for $0 \leq x \leq \pi$.
The inverse tangent function (or arctangent), denoted by $\tan ^{-1}$ or arctan, is defined by
$y=\tan ^{-1} x \Leftrightarrow \tan y=x$ for $x \in \mathbb{R}$ and $-\frac{\pi}{2}<y<\frac{\pi}{2}$.
The graph of the inverse tangent function has horizontal asymptotes
$y=-\frac{\pi}{2} \quad$ and $y=\frac{\pi}{2}$.
To compose the tangent function and its inverse,
$\tan \left(\tan ^{-1} x\right)=x$ for $x \in \mathbb{R}$ and $\tan ^{-1}(\tan x)=x$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$.

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## Chapter 4 Test

1. Convert $15.36^{\circ}$ to DMS (degree-minutesecond) form. Round to the nearest second if necessary.
2. Convert $130.3^{\circ}$ to radians. Round to 2 decimal places.
3. Find the exact length of the arc intercepted by a central angle of $27^{\circ}$ on a circle of radius 5 ft .
4. A skateboard designed for rough surfaces has wheels with diameter 60 mm . If the wheels turn at 2200 rpm , what is the linear speed in mm per minute?
5. A pulley is 20 in . in diameter. Through how many degrees should the pulley rotate to lift a load 3 ft ? Round to the nearest degree.
6. A circle has a radius of 9 yd . Find the area of a sector with a central angle of $120^{\circ}$.
7. For an acute angle $\theta$, if $\sin \theta=\frac{5}{6}$, evaluate $\cos \theta$ and $\tan \theta$.
8. Evaluate $\tan \frac{\pi}{6}-\cot \frac{\pi}{6}$ without the use of a calculator.
9. Given $\sin \theta=\frac{5}{13}$, use a Pythagorean identity to find $\cos \theta$.
10. Given $\sec 75^{\circ}=\sqrt{2}+\sqrt{6}$, find a cofunction of another angle with the same function value.
11. Use a calculator to approximate $\sin \frac{2 \pi}{11}$ to 4 decimal places.
12. A flagpole casts a shadow of 20 ft when the angle of elevation of the Sun is $40^{\circ}$. How tall is the flagpole? Round to the nearest foot.
13. A newly planted tree is anchored by a covered wire running from the top of the tree to a post in the ground 5 ft from the base of the tree. If the angle between the wire and the top of the tree is $20^{\circ}$, what is the length of the wire? Round to the nearest foot.

For Exercises 14-21, evaluate the function or state that the function is undefined at the given value.
14. $\sin \left(-\frac{3 \pi}{4}\right)$
15. $\tan 930^{\circ}$
16. $\sec \frac{11 \pi}{6}$
17. $\csc \left(-150^{\circ}\right)$
18. $\cot 20 \pi$
19. $\cos 690^{\circ}$
20. $\tan \frac{7 \pi}{3}$
21. $\sec \left(-630^{\circ}\right)$
22. Given $\sin \theta=\frac{5}{8}$ and $\cos \theta<0$, find $\tan \theta$.
23. Given $\sec \theta=\frac{4}{3}$ and $\sin \theta<0$, find $\csc \theta$.
24. Given $\tan \theta=-\frac{4}{3}$ and $\theta$ is in Quadrant IV, find $\sec \theta$.

For Exercises 25-28, select the trigonometric function, $f(t)=\sin t, g(t)=\cos t, h(t)=\tan t$ or $r(t)=\cot t$, with the given properties.
25. The function is odd, with period $2 \pi$, and domain of all real numbers.
26. The function is odd, with period $\pi$, and domain of all real numbers excluding odd multiples of $\frac{\pi}{2}$.
27. The function is even, with period $2 \pi$, and domain of all real numbers.
28. The function is odd, with period $\pi$, and domain of all real numbers excluding multiples of $\pi$.
29. Use the even-odd and periodic properties of the trigonometric functions to simplify $\cos (-\theta-2 \pi)-\sin (\theta-2 \pi) \cdot \cot (\theta+\pi)$
30. Suppose that a rectangle is bounded by the $x$-axis and the graph of $y=\sin x$ on the interval $[0, \pi]$.

a. Write a function that represents the area $A(x)$ of the rectangle for $0<x<\frac{\pi}{2}$.
b. Determine the area of the rectangle for $x=\frac{\pi}{6}$ and $x=\frac{\pi}{4}$.
For Exercises 31-32, determine the amplitude, period, phase shift, and vertical shift for each function.
31. $y=\frac{3}{4} \sin \left(2 \pi x-\frac{\pi}{6}\right)$
32. $y=-5 \cos (5 x+\pi)+7$

For Exercises 33-36, graph one period of the function.
33. $y=-2 \sin x$
34. $y=3 \cos 2 x$
35. $y=\sin \left(2 x-\frac{\pi}{4}\right)$
36. $y=3-2 \cos (2 \pi x-\pi)$
37. Write a function of the form $f(x)=A \cos (B x)$ for the given graph.

38. Identify each statement as true or false. If a statement is false, explain why.
a. The relative maxima of the graph of $y=\sin x$ correspond to the relative minima of the graph of $y=\csc x$.
b. The period of $y=\tan 2 x$ is $\pi$.
c. The amplitude of $y=2 \cot x$ is 2 .
d. The period of $y=\sec 2 x$ is $\pi$.
e. The vertical asymptotes of the graph of $y=\cot x$ occur where the graph of $y=\cos x$ has $x$-intercepts.

For Exercises 39-40, graph one period of the function.
39. $y=-4 \sec 2 x$
40. $y=\csc \left(2 x-\frac{\pi}{4}\right)$

For Exercises 41-42, graph two periods of the function.
41. $y=\tan 3 x$
42. $y=-4 \cot \left(x+\frac{\pi}{4}\right)$
43. Simplify each expression.
a. $\sin ^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\cos ^{-1} \frac{\sqrt{2}}{2}$
b. $\cos [\arctan (-\sqrt{3})$
44. Use a calculator to approximate the degree measure (to 1 decimal place) of the angle $\theta$ subject to the given conditions.

$$
\sin \theta=\frac{3}{8} \text { and } 90^{\circ} \leq \theta \leq 180^{\circ}
$$

45. Find the exact value of $\sin ^{-1}\left(\sin \frac{11 \pi}{6}\right)$.
46. Find the exact value of $\tan \left(\cos ^{-1} \frac{7}{8}\right)$.
47. Find the exact value of $\cos \left[\arcsin \left(-\frac{5}{6}\right)\right]$.
48. Write the expression $\tan \left(\cos ^{-1} \frac{x}{\sqrt{x^{2}+25}}\right)$ for $x \neq 0$ as an algebraic expression.
49. A radar station tracks a plane flying at a constant altitude of 6 mi on a path directly over the station. Let $\theta$ be the angle of elevation from the radar station to the plane.
a. Write $\theta$ as a function of the plane's ground distance $x>0$ from the station.

b. Without the use of a calculator, will the angle of elevation be less than $45^{\circ}$ or greater than $45^{\circ}$ when the plane's ground distance is 3.2 mi away?
c. Use a calculator to find $\theta$ to the nearest degree for $x=3.2,1.6$, and 0.5 mi .

Go online for more practice problems.

## Chapter 4 Cumulative Review

 ExercisesFor Exercises 1-9, solve the equations and inequalities. Write the solution set to inequalities in interval notation.

1. $|3 x-5|+3 \geq 6$
2. $(x-2)^{2}(x+5)<0$
3. $\frac{x+2}{x-3} \geq 2$
4. $3 x^{4}-x^{2}-10=0$
5. $\sqrt{x+7}=4-\sqrt{x-1}$
6. $2 x^{4}+5 x^{3}-29 x^{2}-17 x+15=0$
7. $e^{3 x}=2$
8. $\log (3 x+2)-\log x=3$
9. $\ln (x-1)+\ln x=\ln 2$
10. Given $x^{2}+y^{2}-4 x+6 y+9=0$
a. Write the equation of the circle in standard form.
b. Identify the center and radius.
c. Write the domain and range in interval notation.
11. Given $y=-x^{2}+2 x-4$
a. Identify the vertex.
b. Write the domain and range in interval notation.
12. Given $f(x)=x^{4}-2 x^{3}-4 x^{2}+8 x$
a. Find the $x$-intercepts of the graph of $f$.
b. Determine the end behavior of the graph of $f$.
13. a. Graph $y=\frac{3 x^{2}+x-5}{3 x-2}$.
b. Identify the asymptotes.
14. Given $y=-2 \sin (4 x-\pi)-6$, identify the
a. Domain and range in interval notation.
b. Amplitude.
c. Period.
d. Phase shift.
e. Vertical shift.
15. Given $f(x)=\tan x$ and $g(x)=\frac{x}{2}$, evaluate
a. $(f \circ g)\left(\frac{\pi}{2}\right)$
b. $(g \circ f)(\pi)$
16. Evaluate $\cos \left(-450^{\circ}\right)$.
17. Evaluate $\tan \left[\arcsin \left(-\frac{3}{8}\right)\right]$.
18. Write the logarithm as the sum or difference of logarithms. Simplify as much as possible. $\log \left(\frac{x^{2} y}{100 z}\right)$
19. Divide. Write the answer in standard form, $a+b i$.
$\frac{2-8 i}{3-5 i}$
20. Suppose that $y$ varies inversely as $x$ and directly as $z$. If $y$ is 12 when $x$ is 8 and $z$ is 3 , find the constant of variation $k$.

Go online for more practice problems.


