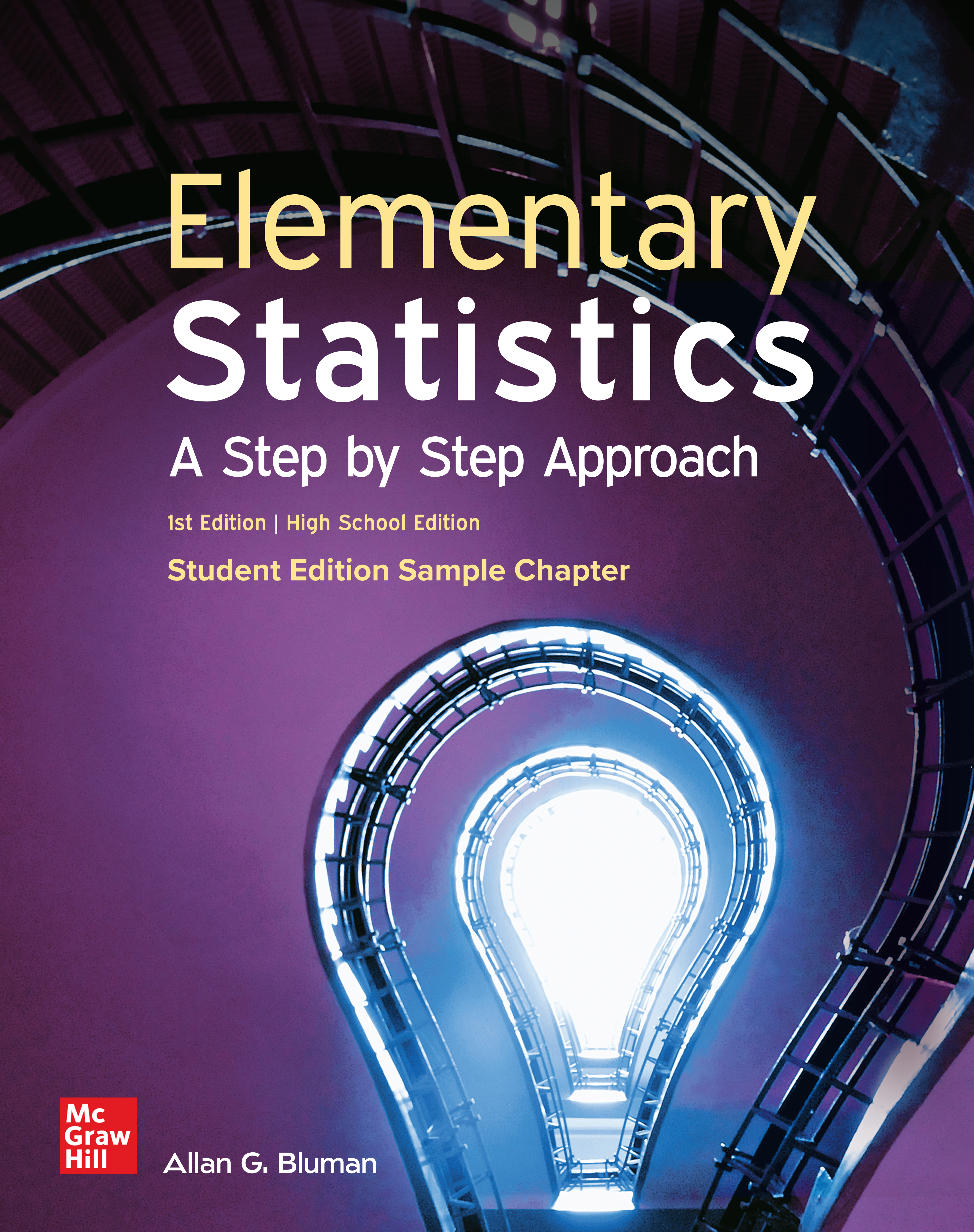

# Elementary Statistics

## A Step by Step Approach

1st Edition | High School Edition

**Student Edition Sample Chapter**

McGraw Hill

Allan G. Bluman

# Elementary Statistics

## A Step by Step Approach

**1st Edition | High School Edition**



McGraw Hill

**Allan G. Bluman**

Professor Emeritus

Community College of Allegheny County

mheducation.com/honorselectives

**ELEMENTARY STATISTICS: A STEP BY STEP APPROACH**

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# About the Author

## Allan G. Bluman


McGraw Hill

Allan G. Bluman is a professor emeritus at the Community College of Allegheny County, South Campus, near Pittsburgh, Pennsylvania. He has taught mathematics and statistics for over 35 years. He received an Apple for the Teacher award in recognition of his bringing excellence to the learning environment at South Campus. He has also taught statistics for Penn State University at the Greater Allegheny (McKeesport) Campus and at the Monroeville Center. He received his master's and doctor's degrees from the University of Pittsburgh.

He is also author of *Elementary Statistics: A Brief Version* and he was a coauthor of *Math in Our World*. In addition, he is the author of four mathematics books in the McGraw Hill DeMystified Series. They are *Pre-Algebra, Math Word Problems, Business Math*, and *Probability*.

He is married and has two sons, a granddaughter, and a grandson.

Dedication: To Betty Bluman, Earl McPeek, and Dr. G. Bradley Seager, Jr.

# Preface

## Approach

*Elementary Statistics: A Step by Step Approach* was written as an aid in the beginning statistics course to students whose mathematical background is limited to basic algebra. The book follows a nontheoretical approach without formal proofs, explaining concepts intuitively and supporting them with abundant examples. The applications span a broad range of topics certain to appeal to the interests of students of diverse backgrounds, and they include problems in business, sports, health, architecture, education, entertainment, political science, psychology, history, criminal justice, the environment, transportation, physical sciences, demographics, and travel and leisure.

## About This Book

### Reflects the Diverse World Around Us

We believe in unlocking the potential of every learner at every stage of life. To accomplish that, we are dedicated to creating products that reflect, and are accessible to, all the diverse, global customers we serve.

The High School Edition of this text received an extensive and thorough audit to ensure that the examples, applications, and topics referenced throughout support a welcoming and sensitive experience for all learners and uphold McGraw Hill's commitment to equity, inclusion, and diversity.

The learning system provides students with a useful framework to support learning, applying, and assessing concept mastery. Some of the features include the following:

- Abundant **exercises** are located at the end of major sections within each chapter.
- A **Data Bank** containing real data listing various attributes (e.g. educational level, cholesterol level, exercise habits, etc.) for 100 people and several additional data sets using real data are included and referenced in various exercises and online projects.
- An updated **Important Formulas** appendix containing the formulas and test tables is included with this textbook.
- End-of-chapter **Reviews**, **Vocabulary**, and **Key Formulas** give students a concise summary of the chapter topics and provide a good source for quiz or test preparation.
- **Review Exercises** are found at the end of each chapter.
- The **Appendices** provide students with extensive reference tables, formulas, and a glossary of key terms. The additional Online Appendices include algebra review, an outline for report writing, Bayes' theorem, and an alternative method for using the standard normal distribution. These can also be found in the online resources.
- The **Applying the Concepts** feature is included in all lessons and gives students an opportunity to think about the new concepts and apply them to examples and scenarios similar to those found in newspapers, magazines, and news programs.

# Acknowledgments

It is important to acknowledge the many people whose contributions have gone into this edition of *Elementary Statistics*. Very special thanks are due to Rachel Webb of Portland State University, who updated the Index of Applications, wrote some new exercises, made changes for inclusion and diversity, and updated the technology sections throughout, and to Yolanda Parker of Tarrant County College, who accuracy checked the manuscript. Finally, at McGraw Hill, thanks to Caroline Celano, Director, Mathematics; Megan Platt, Product Developer; Emily DiGiovanna, Marketing Manager; and Jane Mohr, Lead Content Project Manager.

—Allan G. Bluman

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## Online-Only Chapters

The topics in these chapters are outside the scope of most high school statistics courses.
These chapters are available online as part of the student eBook.

**Chapter 12** Other Chi-Square Tests

**Chapter 13** Analysis of Variance

**Chapter 14** Nonparametric Statistics

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## Chapter 12 Other Chi-Square Tests

**Statistics Around Us** Goodness of Fit: Eye Color Survey

- **12–1** Test for Goodness of Fit
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## Chapter 13 Analysis of Variance

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# Guided Tour - Chapter Opener

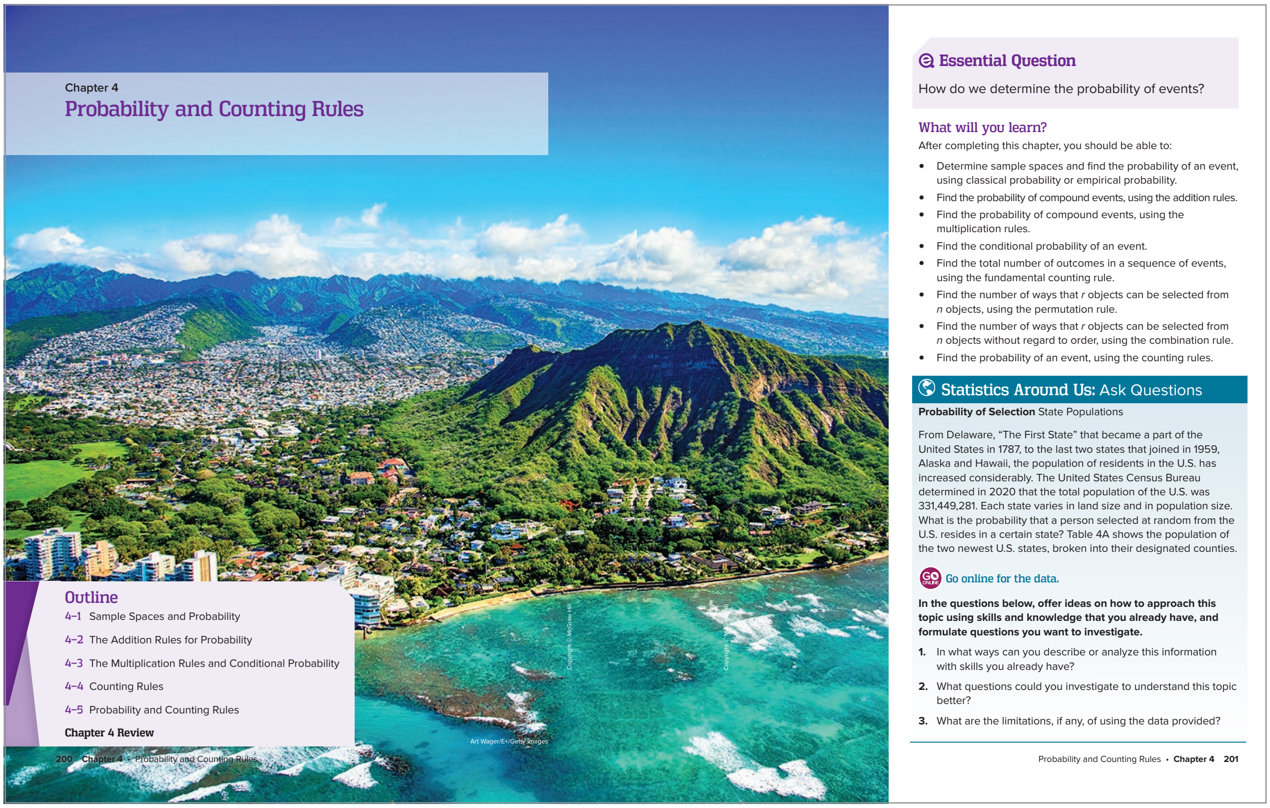
## Exclusively for High School Statistics

The quintessential text for students interested in a nontraditional math pathway, *Elementary Statistics: A Step by Step Approach* is crafted specifically for the high school classroom. Organized around clear objectives, with abundant examples, this text assumes only that students have experience with basic algebra. Examples and concepts are connected to a wide range of applied topics certain to appeal to students with diverse backgrounds and interests, including business, sports, health, architecture, education, entertainment, history, the environment, travel, and more.

## Stunning Visuals, Intuitive Layout

*Elementary Statistics* is designed to make statistics accessible and approachable for all students. This starts with its considerate, student-friendly design.

Each chapter opens with a bright visual related to the topic of the **Statistics Around Us** case study. The chapter openers are designed to be inviting, intriguing, and easy to navigate. The **Chapter Outline** provides a quick reference to what is covered in each chapter. The **Essential Question** and **What will you learn?** features are designed to remind the student of the important takeaway messages from each chapter.



Chapter 4

Probability and Counting Rules

Outline

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4–2 The Addition Rules for Probability

4–3 The Multiplication Rules and Conditional Probability

4–4 Counting Rules

4–5 Probability and Counting Rules

**Chapter 4 Review**

200 Chapter 4 • Probability and Counting Rules

Art Wager/E+/Getty Images

**Essential Question**

How do we determine the probability of events?

**What will you learn?**

After completing this chapter, you should be able to:

- Determine sample spaces and find the probability of an event, using classical probability or empirical probability.
- Find the probability of compound events, using the addition rules.
- Find the probability of compound events, using the multiplication rules.
- Find the conditional probability of an event.
- Find the total number of outcomes in a sequence of events, using the fundamental counting rule.
- Find the number of ways that *r* objects can be selected from *n* objects, using the permutation rule.
- Find the number of ways that *r* objects can be selected from *n* objects without regard to order, using the combination rule.
- Find the probability of an event, using the counting rules.

**Statistics Around Us:** Ask Questions

**Probability of Selection** State Populations

From Delaware, "The First State" that became a part of the United States in 1787, to the last two states that joined in 1959, Alaska and Hawaii, the population of residents in the U.S. has increased considerably. The United States Census Bureau determined in 2020 that the total population of the U.S. was 331,449,281. Each state varies in land size and in population size. What is the probability that a person selected at random from the U.S. resides in a certain state? Table 4A shows the population of the two newest U.S. states, broken into their designated counties.

Go online for the data.

**In the questions below, offer ideas on how to approach this topic using skills and knowledge that you already have, and formulate questions you want to investigate.**

1. In what ways can you describe or analyze this information with skills you already have?
2. What questions could you investigate to understand this topic better?
3. What are the limitations, if any, of using the data provided?

Probability and Counting Rules • **Chapter 4** 201

# Enhanced Pedagogy and Student Support

*Elementary Statistics* opens each chapter with features to capture student attention and guide their inquiry as they dive into the content.

Each chapter opens with an **Essential Question** that frames the learning in the context of the key takeaway and keeps students focused on the learning outcomes.

### Essential Question

How do we determine the probability of events?

### What will you learn?

After completing this chapter, you should be able to:

- Determine sample spaces and find the probability of an event, using classical probability or empirical probability.
- Find the probability of compound events, using the addition rules.
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3. What are the limitations, if any, of using the data provided?

Probability and Counting Rules • **Chapter 4** **201**

**What will you learn?** provides a more detailed view of the specific topics and concepts they will encounter as well as the skills and knowledge they will be developing throughout the chapter.

**Statistics Around Us** features open each chapter. These present real-life data and scenarios to connect the topic of the chapter to students' life experiences.

The **Go online** icon indicates that the lesson is extended online. In this case, a full data set is available to students in the digital resources.

# Guided Tour - Lesson Opener

## Learn, Practice, Apply

Lesson 4-1

### Sample Spaces and Probability

#### Learn: Probability, Outcomes, and Sample Spaces

**Probability** as a general concept can be defined as the chance of an event occurring. Many people are familiar with probability from observing or playing games of chance, such as card games, games using dice, or carnival games. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, and weather forecasting and in various other areas. As stated in Chapter 1, probability is the basis of inferential statistics. For example, predictions are based on probability, and hypotheses are tested by using probability.

The theory of probability grew out of the study of various games of chance using coins, dice, and cards. Since these devices lend themselves well to the application of concepts of probability, they will be used in this chapter as examples. This lesson begins by explaining some basic concepts of probability. Then the types of probability and probability rules are discussed.

Processes such as flipping a coin, rolling a die, or drawing a card from a deck are called *probability experiments.* A **probability experiment** is a chance process that leads to well-defined results called outcomes. An **outcome** is the result of a single trial of a probability experiment. A trial means flipping a coin once, rolling one die once, or the like. When a coin is tossed, there are two possible outcomes: head or tail. (*Note:* We exclude the possibility of a coin landing on its edge.) In the roll of a single die, there are six possible outcomes: 1, 2, 3, 4, 5, or 6. The set of all possible outcomes of a probability experiment is called the **sample space.**

Some sample spaces for various probability experiments are shown in Table 4–1.

TABLE 4–1 Examples of Sample Spaces for Probability Experiments

| Experiment | Sample Space |
|---|---|
| Toss one coin | Head, tail |
| Roll a die | 1, 2, 3, 4, 5, 6 |
| Answer a true/false question | True, false |
| Toss two coins | Head-head, tail-tail, head-tail, tail-head |

It is important to realize that when
outcomes, as shown in the fourth
up. Both coins could fall tails up. C
coin 1 could fall tails up and coin 2
H and T throughout this chapter.

Another assumption commonly
male births is equal to the number
being born is ½ and the probabilit
exactly true since biological sex is
and the exact split of males and fe

Chapter Lessons are divided into **Learn** sections, which focus on a particular topic or skill. Each Learn section starts with a Learn statement, and models a fully guided example for students to follow.

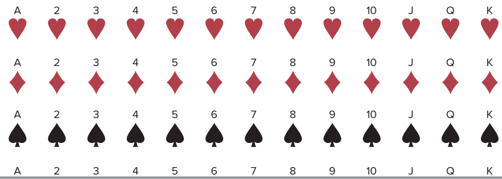
To support mathematical literacy, key **vocabulary** terms are bolded throughout the lesson. Terms are also summarized at the end of each chapter.

**Example 2:** Find the Sample Space—Drawing Cards

Find the sample space for drawing one card from an ordinary deck of cards.

**Solution:**

Since there are 4 suits (hearts, clubs, diamonds, and spades) and 13 cards for each suit (ace through king), there are 4 × 13 = 52 possible outcomes in the sample space. See Figure 4–1.




Each lesson contains **Examples** which are solved with step-by-step explanations, illustrations, and detailed solutions to provide solid guidance so students can approach and solve similar problems with the support they need and with confidence.

Important **formulas** and rules are highlighted in easy-to-find boxes for accessible student reference.

**Formula for Conditional Probability**

The probability that the second event *B* occurs given that the first event *A* has occurred can be found by dividing the probability that both events occurred by the probability that the first event has occurred.

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

## Applying the Concepts 4–1
### Tossing a Coin

Assume you are at a carnival and decide to play one of the games. You spot a table where a person is flipping a coin, and because you have an understanding of basic probability, you believe the odds of winning are in your favor. When you get to the table, you find out all you have to do is guess which side of the coin will be facing up after it is tossed. You are assured the coin is fair, meaning that each of the two sides has an equally likely chance of occurring. You think back about what you learned in your statistics class about probability before you decide what to guess. Answer the following questions about the coin-tossing game.

1. What is the sample space?
2. What are the possible outcomes?
3. What does the classical approach to probability say about computing probabilities for this type of problem?

You decide to guess heads, believing it has a 50% chance of coming up. A friend of yours, who had been playing the game for a while before you got there, tells you that heads has come up the last 9 times in a row. You remember the law of large numbers.

4. What is the law of large numbers, and does it change your thoughts a what will occur on the next toss?
5. What does the empirical approach to probability say about this probl and could you use it to solve this problem?
6. Can subjective probabilities be used to help solve this problem? Expl
7. Assume you could win $1 million if you could guess what the results c next toss will be. What would you guess? Why?

**Applying the Concepts** exercises appear at the end of each lesson and reinforce the concepts explained in the lesson. They give students an opportunity to think critically about the concepts and transfer their knowledge to a variety of scenarios.

## Using Technology: TI-84 Plus Step by Step

### Relative Frequency

TABLE TI4–1 Creating Relative Frequency Tables

| Desired Action | TI-84 Plus Instructions |
|---|---|
| Construct a relative frequency table | **1.** Enter the data values in $L_1$ and the frequencies in $L_2$.<br>**2.** Move the cursor to the top of the $L_3$ column so that $L_3$ is highlighted.<br>**3.** Type $\mathbf{L_2}$ divided by the sample size, then press **ENTER.** |

**TI-Example: Creating a Relative Frequency Table**

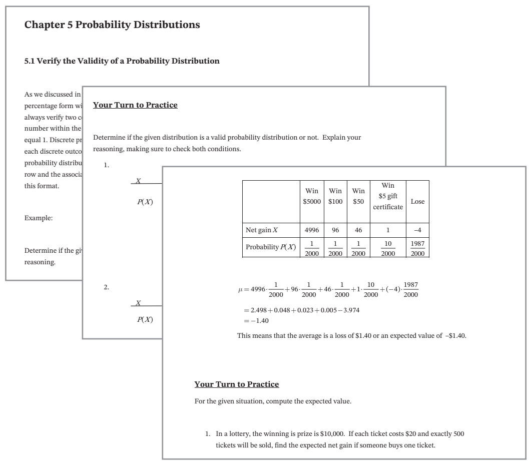
Construct a relative frequency table for the data in Table TI4–2 that represent the number of days patients stayed in the hospital following a knee replacement.

TABLE TI4–2 Number of Days in Hospital

| Days | Frequency |
|---|---|
| 3 | 15 |
| 4 | 32 |
| 5 | 56 |
| 6 | 19 |
| 7 | 5 |

**Using Technology** guides students through how to solve problems on their graphing calculators, or while using common statistical software packages. These directions are available in print for the calculator, and online as downloadable PDFs for the software.

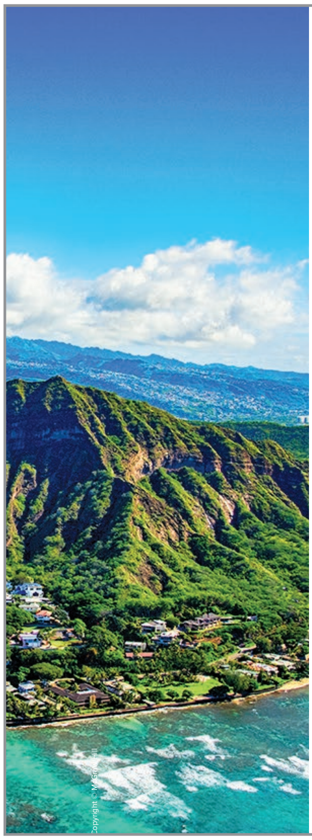
If students need additional practice with the quantitative skills required by the course, the **Corequesite Workbook** is also available in the digital resources. Available as either an interactive eBook or a printable PDF, the workbook is organized around the key math skills necessary for student success. Topics are revisited and summarized, and then students answer open-ended questions in the "Your Turn to Practice" section. Teachers can assign the workbook as needed, or let students use it as a self-guided study tool.



# Guided Tour - Lesson Features

## Real-World Connections

Statistics is an applied field that covers a variety of subject areas, many related to students' daily lives, which bring relevance and fascination to the learning experience.



### Essential Question

How do we determine the probability of events?

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### Statistics Around Us: Ask Questions

**Probability of Selection** State Populations

From Delaware, "The First State" that became a part of the United States in 1787, to the last two states that joined in 1959, Alaska and Hawaii, the population of residents in the U.S. has increased considerably. The United States Census Bureau determined in 2020 that the total population of the U.S. was 331,449,281. Each state varies in land size and in population size. What is the probability that a person selected at random from the U.S. resides in a certain state? Table 4A shows the population of the two newest U.S. states, broken into their designated counties.

Go online for the data.

**In the questions below, offer ideas on how to approach this topic using skills and knowledge that you already have, and formulate questions you want to investigate.**

1. In what ways can you describe or analyze this information with skills you already have?

Each case study opens with the **Statistics Around Us: Ask Questions** scenario, which presents a real-life scenario that students will be investigating as they apply the concepts and skills they are learning.

GO ONLINE

After reading the introduction, students are directed to access their digital resources, where they can view or download the data set for the case study. Students will return to the online activity after they complete the chapter.

Throughout the chapter, the **Statistics Around Us: Use Statistics** boxes direct students to revisit the case study and answer questions using the skills they have just mastered.

### Statistics Around Us: Use Statistics

Find the complement of each event using the population data of Alaska and Hawaii residents provided in Table 4A.

**a.** Selecting a person from Alaska

**b.** Selecting a person from a county with fewer than 1,000 residents

**c.** Selecting a person from a county that starts with the letter *K*

Go online for the data.

**Statistics Around Us: Analyze Results** boxes close each chapter. Here, students revisit the initial question they posed, review the results from the Use Statistics activities, and analyze their findings. This gives them a chance to reinforce their learning, check for understanding, and assess their knowledge. These questions can be completed and submitted online.

## Chapter 4 Review

### Statistics Around Us: Analyze Results

**Probability of Selection** State Populations

1. Review your results from the *Use Statistics* exercises. Have you answered the questions you posed at the start of the chapter?
2. Based on what you have learned in this chapter, are there new questions or values to consider?
3. What have you learned about this topic?

# Cross-Disciplinary Connections

Statistics is relevant to many other academic disiciplines, from science to history and beyond. Several features throughout every chapter connect the topic at hand to other disciplines.

### Historical Note

Venn diagrams were developed by mathematician John Venn (1834–1923) and are used in set theory and symbolic logic. They have been adapted to probability theory also. In set theory, the symbol ∪ represents the *union* of two sets, and $A \cup B$ corresponds to *A* or *B*. The symbol ∩ represents the *intersection* of two sets, and $A \cap B$ corresponds to *A* and *B*. Venn diagrams show only a general picture of the probability rules and do not portray all situations, such as $P(A) = 0$, accurately.

**228 Chapter 4** • Probability and Counting Rules

The **Historical Note** contextualizes the content of the chapter by connecting the topics to notable mathematicians, researchers, and historical events.

**SPEAKING OF STATISTICS** features discuss applied examples of the topics in a lesson across a variety of disciplines to help students make meaningful connections to statistics in their own lives. Examples covered include coin flips, video games, and more.

**SPEAKING OF STATISTICS Coins, Births, and Other Random (?) Events**

Examples of random events such as tossing coins are used in almost all books on probability. But is flipping a coin really a random event?

Tossing coins dates back to ancient Roman times when the coins usually consisted of the Emperor's head on one side (i.e., heads) and another icon such as a ship on the other side (i.e., tails). Tossing coins was used in both fortune telling and ancient Roman games.

A Chinese form of divination called the *I-Ching* (pronounced E-Ching) is thought to be at least 4000 years old. It consists of 64 hexagrams made up of six horizontal lines. Each line is either broken or unbroken, representing the yin and the yang. These 64 hexagrams are supposed to represent all possible situations in life. To consult the *I-Ching,* a question is asked, and then three coins are tossed six times. The way the coins fall, either heads up or heads down, determines whether the line is broken (yin) or unbroken (yang). Once the hexagram is determined, its meaning is consulted and interpreted to get the answer to the question. (*Note:* Another method used to determine the hexagram employs yarrow sticks.)

FIGURE 5–3 A Coin Toss
PNC/Stockbyte/Getty Images

## Abundant Practice and Assessment

Throughout each chapter, students encounter multiple opportunities to practice and apply the skills that they have learned.

### Applying the Concepts 4–1
Tossing a Coin

Assume you are at a carnival and decide to play one of the games. You spot a table where a person is flipping a coin, and because you have an understanding of basic probability, you believe the odds of winning are in your favor. When you get to the table, you find out all you have to do is guess which side of the coin will be facing up after it is tossed. You are assured the coin is fair, meaning that each of the two sides has an equally likely chance of occurring. You think back about what you learned in your statistics class about probability before you decide what to guess. Answer the following questions about the coin-tossing game.

1. What is the sample space?
2. What are the possible outcomes?
3. What does the classical approach to probability say about computing probabilities for this type of problem?

You decide to guess heads, believing it has a 50% chance of coming up. A friend of yours, who had been playing the game for a while before you got there, tells you that heads has come up the last 9 times in a row. You remember the law of large numbers.

4. What is the law of large numbers, and does it change your thoughts about what will occur on the next toss?
5. What does the empirical approach to probability say about this problem, and could you use it to solve this problem?
6. Can subjective probabilities be used to help solve this problem? Explain.
7. Assume you could win $1 million if you could guess what the results of the next toss will be. What would you guess? Why?

At the end of each lesson, students work through a problem similar to the guided examples in **Applying the Concepts.** Students can revisit the those stepped-out examples if they need additional, self-guided support.

Each lesson closes with **Practice Exercises**, which include short-answer and extended-response questions that review the content of the lesson.

Additional practice problems are also available in the digital resources. Premade assessments, versioned tests, and lesson-level quizzes are available.

### Practice 4–1
#### Practice Exercises

**In #1–7, answer questions related to probability and probability experiments.**

1. What is a probability experiment?
2. Define *sample space*.
3. What is the range of the values of the probability of an event?
4. What is the sum of the probabilities of all the outcomes in a sample space?
5. If the probability that it will rain tomorrow is 0.20, what is the probability that it won't rain tomorrow? Would you recommend taking an umbrella?
6. A probability experiment is conducted. Which of these cannot be considered a probability outcome?

**a.** $\frac{5}{8}$ **d.** 2.14 **g.** 1

**b.** 0.29 **e.** −0.63 **h.** 153%

**c.** $-\frac{1}{3}$ **f.** 0 **i.** 87%

**c.** Getting a number less than 7

**d.** Getting a prime number

9. If two dice are rolled one time, find the probability of getting these results:
   **a.** A sum less than 9
   **b.** A sum greater than or equal to 10
   **c.** A 3 on one die or on both dice
10. If a card is drawn from a deck, find the probability of getting these results:
    **a.** A 6 and a spade
    **b.** A black king
    **c.** A red card and a 7
    **d.** A diamond or a heart
    **e.** A black card
11. Human blood is grouped into four types. The percentages of Americans with each type are listed below.

O 43% A 40% B 12% AB 5%

Chapters end with a **Chapter Review**, which concludes and summarizes the key points of each chapter.

## Chapter 4 Review

### Statistics Around Us: Analyze Results

**Probability of Selection** State Populations

1. Review your results from the *Use Statistics* exercises. Have you answered the questions you posed at the start of the chapter?
2. Based on what you have learned in this chapter, are there new questions or values to consider?
3. What have you learned about this topic?

### What did you learn?

After completing this chapter, you should be able to:

- Determine sample spaces and find the probability of an event, using classical probability or empirical probability. Classical probability uses sample spaces and assumes that all outcomes in the sample space are equally likely. Empirical probability uses frequency distributions and is based on observation. (Lesson 4–1)
- Find the probability of compound events, using the addition rules. To find the probability of two mutually exclusive events occurring, add the probability of each event. To find the probability of two events when they are not mutually exclusive, add the possibilities of the individual events and then subtract the probability that both events occur at the same time. (Lesson 4–2)
- Find the probability of compound events, using the multiplication rules. To find the probability of two independent events occurring, multiply the probabilities of each event. To find the probability that two dependent events occur, multiply the probability that the first event occurs by the probability that the second event occurs, given that the first event has already occurred. (Lesson 4–3)
- Find the conditional probability of an event. Conditional probability is the probability that a second event occurs, given that the first event has already occurred. (Lesson 4–3)
- Find the total number of outcomes in a sequence of events, using the fundamental counting rule. The fundamental counting rule states that in a sequence of $n$ events in which the first one has $k_1$ possibilities and the second event has $k_2$ and the third has $k_3$ and so forth, the total number of possibilities of the sequence will be $k_1 \cdot k_2 \cdot k_3 \cdots k_n$. (Lesson 4–4)



Students revisit the **Statistics Around Us** case study by returning to their online worksheet to answer the questions they posed at the beginning of the chapter.

**What did you learn?** summarizes the learning objectives covered in each lesson for easy reference and reinforcement. A detailed summary is available in the digital resources.

### Chapter 4 Vocabulary

classical probability 207
combination 263
complement of an event 212
compound event 207
conditional probability 239
dependent events 239
disjoint events 223
empirical probability 214
equally likely events 207
event 207
fundamental counting rule 255
independent events 235
law of large numbers 218
mutually exclusive events 223
outcome 202
permutation 261
probability 202
probability experiment 202
sample space 202
simple event 207
subjective probability 218
tree diagram 204
Venn diagrams 213

**Vocabulary** terms and key formulas are listed in the chapter review for easy reference.

Finally, **Review Exercises** are organized by lesson. Additional review exercises and projects, such as **Data Projects** and **Critical Thinking Challenges,** are available in the digital resources.

## Chapter 4 Review

### Chapter 4 Review Exercises

**For #1–14, find the requested probabilities.**

#### Lesson 4–1

1. A twelve-sided die is rolled (containing the numbers 1, 2, 3, . . . , 10, 11, 12). Find the probability of each.
   a. Rolling an even number
   b. Rolling a multiple of 5
   c. Rolling a number less than 7
2. When a card is selected from a standard deck of 52 cards, find the probability of each.
   a. A face card
   b. A black card
   c. A 4 and a diamond
   d. A spade
3. Five popular burger chains had the following cheeseburger sales for one day.

| | |
|---|---|
| Burgers Deluxe | 1,455 |
| Midwest Burgers | 1,568 |
| Hometown Burgers | 2,177 |
| Beefy Burgers | 756 |
| The Burger Hut | 436 |

   Find each of the following for a single cheeseburger purchase.
   a. The probability that the cheeseburger was from Hometown Burgers
   b. The probability that the cheeseburger was from either Beefy Burgers or The Burger Hut
   c. The probability that the cheeseburger was not from Midwest Burgers
4. When three coins are flipped, find each of the following:
   a. The probability of flipping at least 1 tail
   b. The probability of flipping exactly 2 tails
   c. The probability of flipping no more than 2 heads

#### Lesson 4–2

5. In a recent school survey, 17 students preferred having salad at lunch, 32 student preferred sandwiches, and 26 preferred the hot lunch special. Calculate the probability that a randomly selected student, from the survey group, prefers the hot lunch special.







6. A nearby carnival game has large stuffed animals as prizes. There are 15 giraffes, 21 orangutans, 12 lions and 6 polar bears to choose from. Calculate each of the following:
   a. The probability that a prize awarded is a polar bear
   b. The probability that a prize awarded is not a giraffe
   c. The probability that a prize awarded is either a giraffe, orangutan, or polar bear
7. A local junior high gathered data about what students' activities are after school. The results showed that the probability that a student skateboards after school is 0.32. The probability that a student plays video games is 0.73 and the probability that they skateboard and play video games is 0.21. Calculate the probability that a randomly chosen student either skateboards or plays video games after school.
8. A high school counselor examined the senior enrollment in courses and found that the probability that a senior was in choir was 0.16. The probability that a senior was taking physics was 0.44. The probability that a senior was enrolled in both classes was 0.09. Calculate the probability that a senior was not enrolled in either class.

#### Lesson 4–3

9. A popular food chain is giving out superhero/villain trading cards with every purchase. 70% of the cards have a superhero on them, while 30% have a villain. Twenty percent of the superhero cards have a female character on them, while 5% of the villain cards have a female character on them. If cards are randomly distributed, what is the probability of getting a superhero card given that it is a female character?
10. Approximately 5% of the world's population are healthy carriers of a gene for sickle-cell disease. Choose 6 individuals at random. What is the probability that at least one of them carries the sickle-cell gene?
    *Source:* www.afro.who.int/health-topics/sickle-cell-disease
11. A small liberal arts college wanted to see if their exercise program was having an effect on students over their 4 years of attendance. The table below compares the exercise data collected for freshmen and seniors enrolled in the college.

| | Exercise 0-2 days/week | Exercise 3-5 days/week | Exercise 6-7 days/week |
|---|---|---|---|
| Freshmen | 335 | 215 | 50 |
| Seniors | 140 | 240 | 95 |

   If a student is selected at random, calculate the following:
   a. The probability that the student exercises 3–5 days/week given that they are a senior
   b. The probability that the student exercises less than 3 days/week
   c. Given that the student is a freshman, what is the probability that they exercise 0–2 days/week?

# Best in Class
# Digital Resources

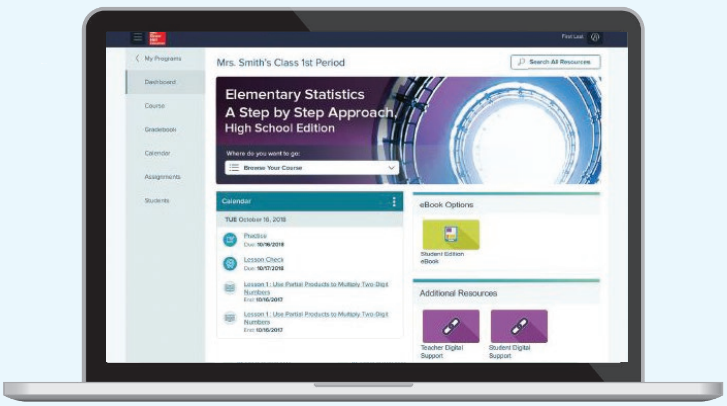
***Elementary Statistics: A Step by Step Approach*** **is enriched with multimedia content that enhance the teaching and learning experience both inside and outside of the classroom.**

Developed with the world's leading subject matter experts and organized by chapter level, the resources provide students with multiple opportunities to contextualize and apply their understanding. Teachers can save time, customize lessons, monitor student progress, and make data-driven decisions in the classroom with the flexible, easy-to-navigate instructional tools.

## Student Assignments

Resources are organized at the chapter level. To enhance the core content, teachers can add assignments, activities, and instructional aids to any lesson. The chapter landing page gives students access to:

- assigned activities;
- customizable, auto-graded assessments;
- interactive eBook;
- data sets, data projects, and classroom activities;
- and an interactive and printable Corequesite Workbook for additional math skills practice.




**Chapter landing page** links students to resources that support success.



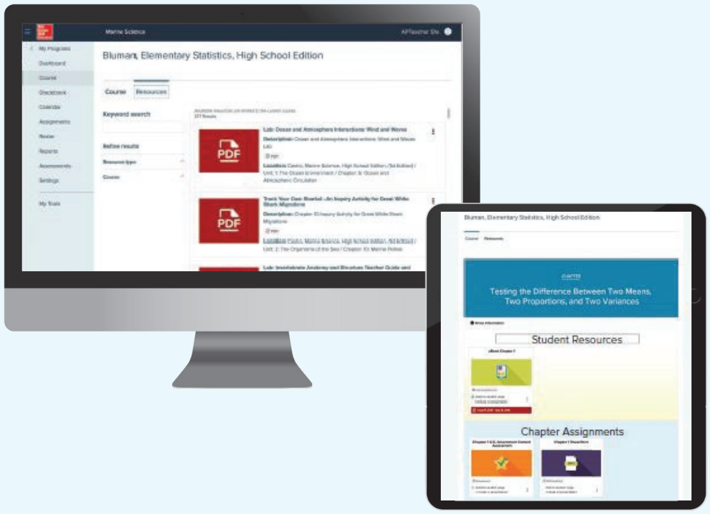
### Mobile Ready

Access to course content on-the-go is easier and more efficient than ever before with the *ReadAnywhere* mobile app.

## Teacher Resources

Teachers have access to the interactive eBook, plus a wealth of customizable chapter resources and powerful grade-book tools. Resources include:

- a solutions manual with answers to the end-of-chapter questions in the Student Edition;
- actionable reporting features that track student progress with data-driven insights;
- customizable PowerPoint presentations, labeled diagrams, visual aids, and additional ideas for lecture enrichment;
- customizable assignments and quiz banks that are automatically graded and populate easy-to-read reports.




**Searchable library** makes it easy to find and assign resources.

## Adaptive Learning with ALEKS

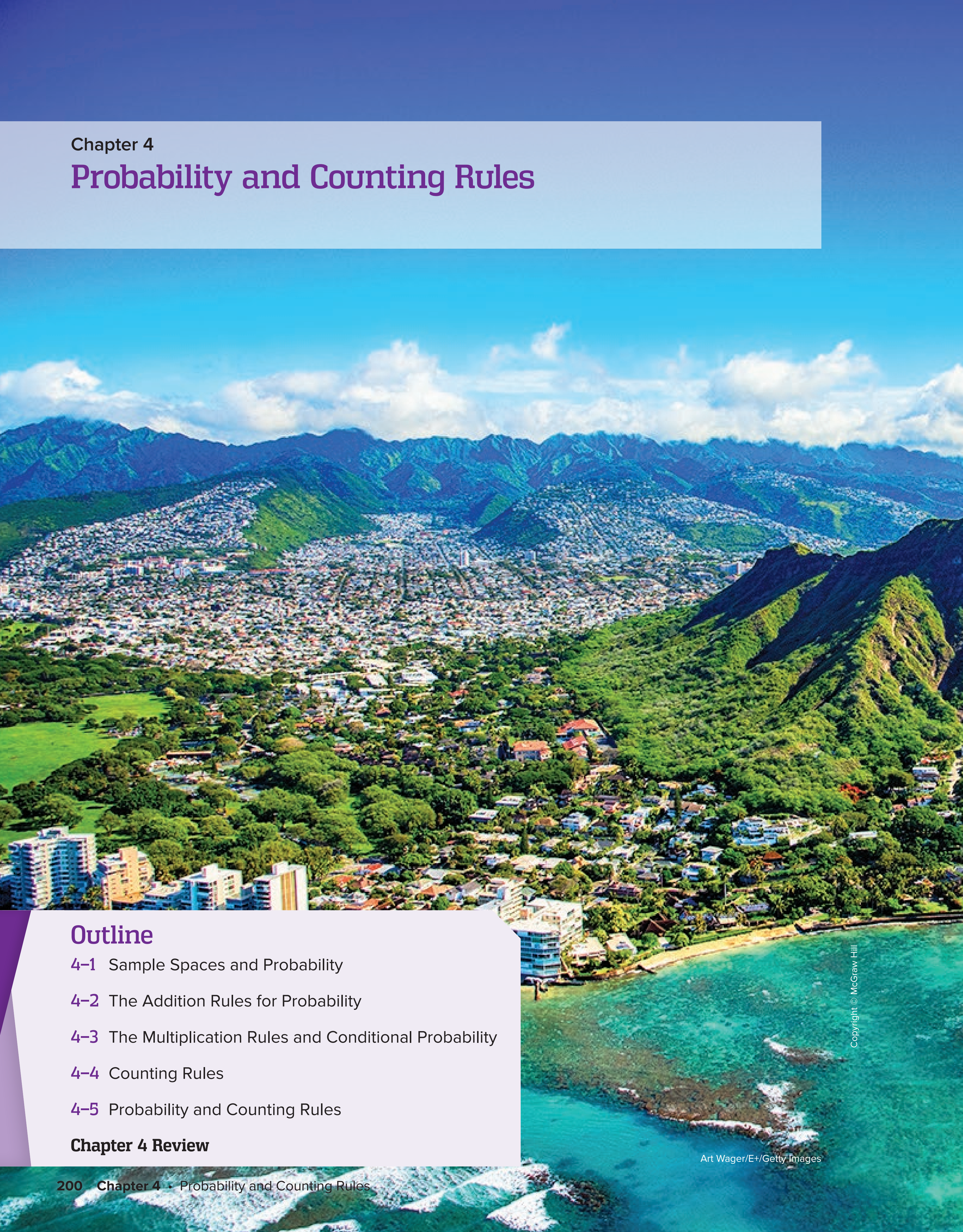
Available with the digital subscription as an add-on, ALEKS uses adaptive questioning to quickly and accurately determine exactly what math topics a student knows and doesn't know and instructs each student on the topics they are most ready to learn. ALEKS can:

- assess a student's proficiency and knowledge with an "Initial Knowledge Check;"
- track which topics have been mastered;
- identify areas that need more study;
- build student confidence with preparatory modules, video tutorials, and practice questions.


**ALEKS** is a personalized way for students to learn at their own pace.

Chapter 4

# Probability and Counting Rules



## Outline


4–1 Sample Spaces and Probability

4–2 The Addition Rules for Probability

4–3 The Multiplication Rules and Conditional Probability

4–4 Counting Rules

4–5 Probability and Counting Rules

**Chapter 4 Review**


Art Wager/E+/Getty Images



## Essential Question

How do we determine the probability of events?

### What will you learn?

After completing this chapter, you should be able to:

- Determine sample spaces and find the probability of an event, using classical probability or empirical probability.
- Find the probability of compound events, using the addition rules.
- Find the probability of compound events, using the multiplication rules.
- Find the conditional probability of an event.
- Find the total number of outcomes in a sequence of events, using the fundamental counting rule.
- Find the number of ways that *r* objects can be selected from *n* objects, using the permutation rule.
- Find the number of ways that *r* objects can be selected from *n* objects without regard to order, using the combination rule.
- Find the probability of an event, using the counting rules.

## Statistics Around Us: Ask Questions

**Probability of Selection** State Populations

From Delaware, "The First State" that became a part of the United States in 1787, to the last two states that joined in 1959, Alaska and Hawaii, the population of residents in the U.S. has increased considerably. The United States Census Bureau determined in 2020 that the total population of the U.S. was 331,449,281. Each state varies in land size and in population size. What is the probability that a person selected at random from the U.S. resides in a certain state? Table 4A shows the population of the two newest U.S. states, broken into their designated counties.

GO ONLINE Go online for the data.

**In the questions below, offer ideas on how to approach this topic using skills and knowledge that you already have, and formulate questions you want to investigate.**

1. In what ways can you describe or analyze this information with skills you already have?
2. What questions could you investigate to understand this topic better?
3. What are the limitations, if any, of using the data provided?

Lesson 4-1

# Sample Spaces and Probability

## Learn: Probability, Outcomes, and Sample Spaces

**Probability** as a general concept can be defined as the chance of an event occurring. Many people are familiar with probability from observing or playing games of chance, such as card games, games using dice, or carnival games. In addition to being used in games of chance, probability theory is used in the fields of insurance, investments, and weather forecasting and in various other areas. As stated in Chapter 1, probability is the basis of inferential statistics. For example, predictions are based on probability, and hypotheses are tested by using probability.

The theory of probability grew out of the study of various games of chance using coins, dice, and cards. Since these devices lend themselves well to the application of concepts of probability, they will be used in this chapter as examples. This lesson begins by explaining some basic concepts of probability. Then the types of probability and probability rules are discussed.

Processes such as flipping a coin, rolling a die, or drawing a card from a deck are called *probability experiments.* A **probability experiment** is a chance process that leads to well-defined results called outcomes. An **outcome** is the result of a single trial of a probability experiment. A trial means flipping a coin once, rolling one die once, or the like. When a coin is tossed, there are two possible outcomes: head or tail. (*Note:* We exclude the possibility of a coin landing on its edge.) In the roll of a single die, there are six possible outcomes: 1, 2, 3, 4, 5, or 6. The set of all possible outcomes of a probability experiment is called the **sample space.**

Some sample spaces for various probability experiments are shown in Table 4–1.

TABLE 4–1 Examples of Sample Spaces for Probability Experiments

| Experiment | Sample Space |
|---|---|
| Toss one coin | Head, tail |
| Roll a die | 1, 2, 3, 4, 5, 6 |
| Answer a true/false question | True, false |
| Toss two coins | Head-head, tail-tail, head-tail, tail-head |

It is important to realize that when two coins are tossed, there are *four* possible outcomes, as shown in the fourth experiment above. Both coins could fall heads up. Both coins could fall tails up. Coin 1 could fall heads up and coin 2 tails up. Or coin 1 could fall tails up and coin 2 heads up. Heads and tails will be abbreviated as H and T throughout this chapter.

Another assumption commonly made in probability theory is that the number of male births is equal to the number of female births and that the probability of a boy being born is ½ and the probability of a girl being born is ½. We know this is not exactly true since biological sex is not necessarily binary: it exists on a spectrum, and the exact split of males and females varies from location to location.

## **Example 1:** Find the Sample Space—Rolling Two Dice

Find the sample space for rolling two dice.

### Solution:

Each possible outcome of rolling two dice can be represented as an ordered pair. For example, the ordered pair (2,3) represents rolling a 2 on the first die and a 3 on the second die.

Since each die can land in six different ways, and two dice are rolled, the sample space can be presented by a rectangular array, as shown in Table 4–2. The sample space is the list of pairs of numbers in the chart.

TABLE 4–2 Possible Outcomes from Rolling Two Dice

| | 1 on Die 2 | 2 on Die 2 | 3 on Die 2 | 4 on Die 2 | 5 on Die 2 | 6 on Die 2 |
|---|---|---|---|---|---|---|
| 1 on Die 1 | (1,1) | (1,2) | (1,3) | (1,4) | (1,5) | (1,6) |
| 2 on Die 1 | (2,1) | (2,2) | (2,3) | (2,4) | (2,5) | (2,6) |
| 3 on Die 1 | (3,1) | (3,2) | (3,3) | (3,4) | (3,5) | (3,6) |
| 4 on Die 1 | (4,1) | (4,2) | (4,3) | (4,4) | (4,5) | (4,6) |
| 5 on Die 1 | (5,1) | (5,2) | (5,3) | (5,4) | (5,5) | (5,6) |
| 6 on Die 1 | (6,1) | (6,2) | (6,3) | (6,4) | (6,5) | (6,6) |

## **Example 2:** Find the Sample Space—Drawing Cards

Find the sample space for drawing one card from an ordinary deck of cards.

### Solution:

Since there are 4 suits (hearts, clubs, diamonds, and spades) and 13 cards for each suit (ace through king), there are $4 \times 13 = 52$ possible outcomes in the sample space. See Figure 4–1.

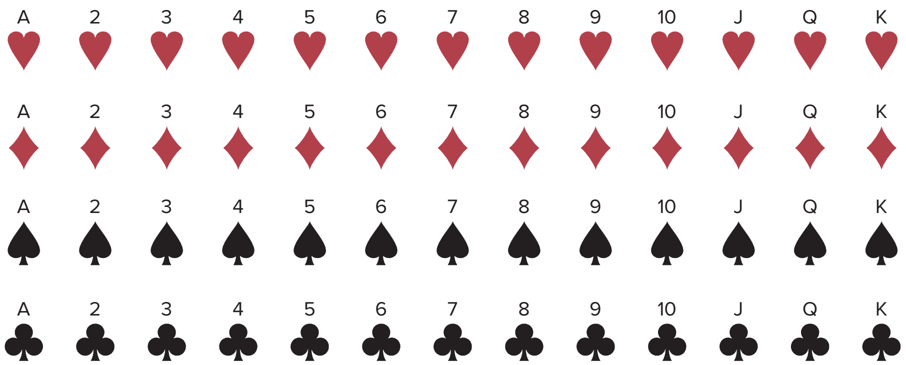



FIGURE 4–1 Sample Space for Drawing a Card
bundit jonwises/Shutterstock

## **Example 3:** Find the Sample Space—One Die, Rolled Three Times

A die is rolled three times, and the number of spots obtained is noted as either odd (1, 3, or 5) or even (2, 4, or 6). Find the sample space for the experiment.

### **Solution:**

To represent the possible outcomes, use O for rolling an odd number and E for rolling an even number. The outcomes of one die rolled multiple times can be represented by a series of letters representing the outcomes of each roll in order. So, EO represents rolling an even number on the first roll and an odd number on the second roll.

If a die is rolled one time, the sample space consists of two outcomes:

O E

If a die is rolled two times, the sample space consists of four outcomes:

OO OE EO EE

If a die is rolled three times, the sample space consists of eight outcomes:

OOO OOE OEO EOO EEO EOE OEE EEE

## Statistics Around Us: Use Statistics

Provided the population data for Alaska and Hawaii in Table 4A, find the number of outcomes in the sample space for selecting one person from these two states.

**Go online for the data.**

In Examples 1–3, the sample spaces were found by observation and reasoning; however, another way to find all possible outcomes of a probability experiment is to use a *tree diagram.*

# **Learn:** Using Tree Diagrams to Find Sample Spaces

A **tree diagram** is a device consisting of line segments emanating from a starting point and also from the outcome point. It is used to determine all possible outcomes of a probability experiment.

## **Example 4:** Use a Tree Diagram to Find a Sample Space

Use a tree diagram to find the sample space for the dominant hand of three children in a family.

### **Solution:**

To find the sample space of the dominant hand of three children in a family, represent possibilities as branching line segments. Then following each branch will reveal each possible outcome.

**Step 1** Since there are two possibilities (right-handed or left-handed) for the first child, draw two branches from a starting point, and label one R and the other L. See Figure 4–2.

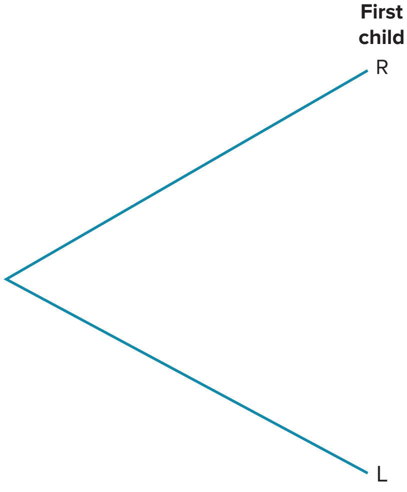



FIGURE 4–2 Tree Diagram for Dominant Hand of First Child

**Step 2** Then if the first child is right-handed, there are two possibilities for the second child (right-handed or left-handed), so draw two branches from R, and label one R and the other L. Do the same if the first child is a left-handed. See Figure 4–3.

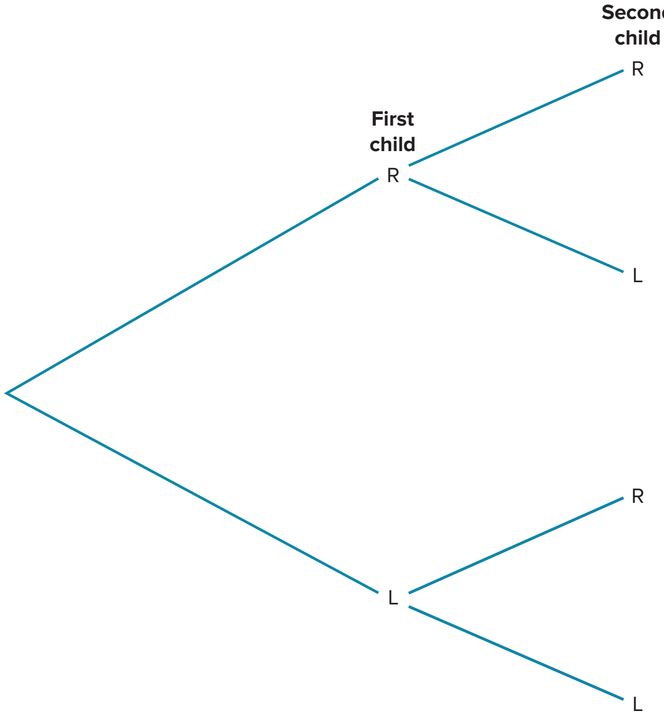



FIGURE 4–3 Tree Diagram for Dominant Hand of First Two Children

**Step 3** Follow the same procedure for the third child. See Figure 4–4.

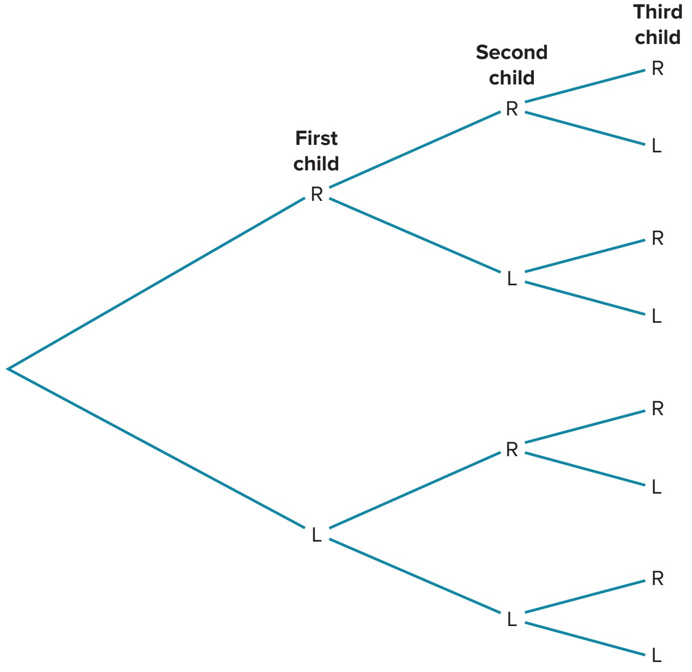



FIGURE 4–4 Tree Diagram for Dominant Hand of Three Children

**Step 4** To find the outcomes for the sample space, trace through all the possible branches, beginning at the starting point for each one, as shown in Figure 4–5.

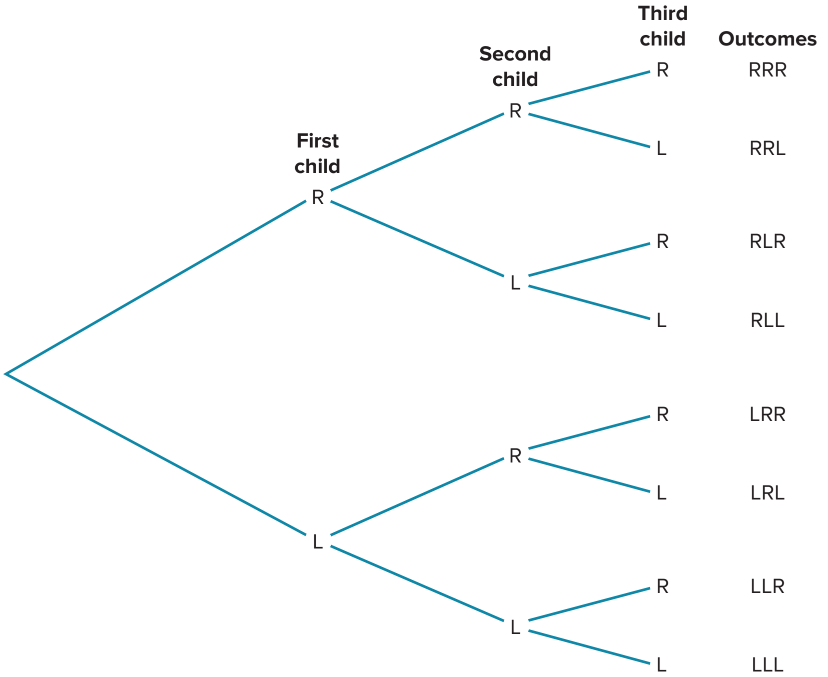



FIGURE 4–5 Tree Diagram and Outcomes for Dominant Hand of Three Children



## Historical Note

The famous Italian astronomer Galileo (1564–1642) found that sums of 10 and 11 occur more often than any other sum when three dice are tossed. Previously, it was thought that a sum of 9 occurred more often than any other sum.

## Learn: Simple and Compound Events

An outcome was defined previously as the result of a single trial of a probability experiment. In many problems, one must find the probability of two or more outcomes. For this reason, it is necessary to distinguish between an outcome and an event.

An **event** consists of a set of outcomes of a probability experiment. An event can be one outcome or more than one outcome. For example, if a die is rolled and a 6 shows, this result is called an *outcome* since it is a result of a single trial. An event with one outcome is called a **simple event.** The event of getting an odd number when a die is rolled is called a **compound event** since it consists of three outcomes (rolling a 1, 3, or 5) or three simple events. In general, a compound event consists of two or more outcomes or simple events.

## Learn: Classical Probability and Equally Likely Events

There are three basic interpretations of probability:

1. Classical probability
2. Empirical or relative frequency probability
3. Subjective probability

**Classical probability** uses sample spaces to determine the numerical probability that an event will happen. You do not actually have to perform the experiment to determine that probability. Classical probability is so named because it was the first type of probability studied formally by mathematicians in the 17th and 18th centuries.

*Classical probability assumes that all outcomes in the sample space are equally likely to occur.* For example, when a single die is rolled, each outcome has the same probability of occurring. Since there are six outcomes, each outcome has a probability of $\frac{1}{6}$. When a card is selected from an ordinary deck of 52 cards, you assume that the deck has been shuffled, and each card has the same probability of being selected. In this case, it is $\frac{1}{52}$.

**Equally likely events** are events that have the same probability of occurring.

### Formula for Classical Probability

The probability of any event $E$ is:

$$\frac{\text{Number of outcomes in } E}{\text{Total number of outcomes in the sample space}}$$

This probability is denoted by:

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ is the number of outcomes in $E$, and $n(S)$ is the number of outcomes in the sample space $S$.

Probabilities can be expressed as fractions, decimals, or—where appropriate—percentages. If you ask, "What is the probability of getting a head when a coin is tossed?", typical responses can be any of the following three:

"One-half."

"Point five."

"Fifty percent."

Note that, strictly speaking, a percent is not a probability. However, in everyday language, probabilities are often expressed as percents—for example, when we say, "There is a 60% chance of rain tomorrow." For this reason, some probabilities will be expressed as percents throughout this book.

The answers listed above are all equivalent. In most cases, the answers to examples and exercises given in this chapter are expressed as fractions or decimals, but percentages are used where appropriate.

### Rounding Rule for Probabilities

Probabilities should be expressed as reduced fractions or rounded to three decimal places. When the probability of an event is an extremely small decimal, it is permissible to round the decimal to the first nonzero digit after the point. For example, 0.0000587 would be 0.00006. When obtaining probabilities from one of the tables in Appendix A, use the number of decimal places given in the table. If decimals are converted to percentages to express probabilities, move the decimal point two places to the right and add a percent sign.

## Example 5: Calculate the Probability—Rolling Two Dice

When two dice are rolled, find the probability that one die will show a three and the other die will show a six.

### Solution:

The probability of an event $E$ is indicated by the formula:

$$P(E) = \frac{n(E)}{n(S)}$$

where $n(E)$ is the number of outcomes of $E$, and $n(S)$ is the total number of outcomes in the sample space. Here, $E$ is rolling two dice with one die showing a three and the other die showing a six.

**Step 1** As we learned in Example 1, there are 36 outcomes when two dice are rolled (see Table 4–1). So, the number of outcomes in the sample space, $n(S)$, is 36.

**Step 2** There are two ways to get a three on one die and a six on the other die—(3, 6) and (6, 3). So, the number of outcomes of the event, $n(E)$, is 2.

**Step 3** Using the probability formula, the probability of the is given by:

$$P(E) = \frac{n(E)}{n(S)} = \frac{2}{36} = \frac{1}{18}$$

## Example 6: Calculate the Probability—One Die, Rolled Three Times

A die is rolled three times, and the number of spots obtained is noted as either odd (1, 3, or 5) or even (2, 4, or 6). Find the probability that exactly two of the rolls results in an odd number.

### Solution:

**Step 1** We will use O for rolling an odd number and E for rolling an even number. As we learned in Example 3, the sample space is: OOO, OOE, OEO, EOO, EEO, EOE, OEE, EEE. Therefore, $n(S) = 8$.

**Step 2** Count the number of ways to get two odd rolls and one even roll in the sample space. There are three: OOE, OEO, and EOO. So, $n(E) = 3$.

**Step 3** Using the probability formula, the probability that exactly two of the rolls results in an odd number is:

$$P(E) = \frac{n(E)}{n(S)} = \frac{3}{8}$$

In probability theory, it is important to understand the meaning of the words *and* and *or.* For example, if you were asked to find the probability of getting a queen *and* a heart when you were drawing a single card from a deck, you would be looking for the queen of hearts. Here, the word *and* means "at the same time." The word *or* has two meanings. For example, if you were asked to find the probability of selecting a queen *or* a heart when one card is selected from a deck, you would be looking for one of the 4 queens or one of the 13 hearts. In this case, the queen of hearts would be included in both cases and counted twice. So, there would be $4 + 13 - 1 = 16$ possibilities.

On the other hand, if you were asked to find the probability of getting a queen *or* a king, you would be looking for one of the 4 queens or one of the 4 kings. In this case, there would be $4 + 4 = 8$ possibilities. In the first case, both events can occur at the same time; we say this is an example of the *inclusive or.* In the second case, both events cannot occur at the same time; we say this is an example of the *exclusive or.*

## Example 7: Calculate the Probability—Drawing a Specific Card

A card is drawn from an ordinary well-shuffled deck. Find the probability of getting:

**a.** A club.

**b.** A red card.

**c.** The 7 of spades.

**d.** A king.

**e.** A face card.



### Historical Note

Ancient Greeks and Romans made crude dice from animal bones, various stones, minerals, and ivory. When the dice were tested mathematically, some were found to be quite accurate.

**Solution:**

**a.** Refer to the sample space shown in Figure 4–1 on page 203. There are 13 clubs in the deck of 52 cards, so:

$$P(\clubsuit) = \frac{13}{52} = \frac{1}{4} = 0.25$$

**b.** Since there are 26 red cards in the deck, 13 hearts and 13 diamonds, the probability of getting a red card is:

$$P(\text{red card}) = \frac{26}{52} = \frac{1}{2}$$

**c.** Since there is only one seven of spades in the deck, the probability is:

$$P(7\spadesuit) = \frac{1}{52} \approx 0.019$$

**d.** There are 4 kings in a deck of 52 cards, so the probability is:

$$P(\text{king}) = \frac{4}{52} = \frac{1}{13} \approx 0.077$$

**e.** There are 12 face cards in an ordinary deck of cards—4 suits (diamonds, hearts, spades, and clubs) and 3 face cards of each suit (jack, queen, and king), so:

$$P(\text{face card}) = \frac{12}{52} = \frac{3}{13} \approx 0.231$$

## Statistics Around Us: Use Statistics

A new game show is selecting one person at random to participate. The game show is selecting a person from either Alaska or Hawaii. Provided the population data in Table 4A, find the probability of selecting:

**a.** A person who resides in Alaska.

**b.** A person who resides in Hawaii.

**c.** A person who resides in Kalawao County.

**d.** A person who resides in in a county with a population of greater than 50,000.

GO ONLINE Go online for the data.

## Learn: Probability Rules

There are four basic probability rules. These rules are helpful in solving probability problems, in understanding the nature of probability, and in deciding if your answers to the problems are correct.



### Historical Note

Paintings in tombs excavated in Egypt show that the Egyptians played games of chance. One game called *Hounds and Jackals* played in 1800 B.C. is similar to the present-day game of *Snakes and Ladders*.

### Probability Rules

1. The probability of any event $E$ is a number (either a fraction or decimal) between and including 0 and 1. This is denoted by $0 \leq P(E) \leq 1$.
2. The sum of the probabilities of all the outcomes in a sample space is 1.
3. If an event $E$ cannot occur (i.e., the event contains no members in the sample space), its probability is 0.
4. If an event $E$ is certain, the probability of $E$ is 1.

Rule 1 states that probability values range from 0 to 1. When the probability of an event is close to 0, its occurrence is highly unlikely. When the probability of an event is near 0.5, there is about a 50-50 chance the event will occur; and when the probability of an event is close to 1, the event is highly likely to occur. See Figure 4–6.

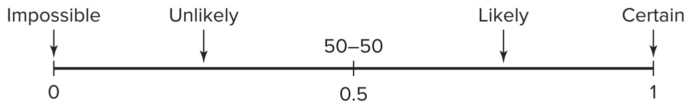



FIGURE 4–6 Range of Probability

Rule 2 can be illustrated by the example of rolling a single die. Each outcome in the sample space has a probability of $\frac{1}{6}$, and the sum of the probabilities of all the outcomes is 1, as shown in Table 4–3.

TABLE 4–3 Outcomes and Probabilities from Rolling One Die

| **Outcome** | 1 | | 2 | | 3 | | 4 | | 5 | | 6 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **Probability** | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ | | $\frac{1}{6}$ |

$$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$$

Rule 3 states that if an event cannot occur, then $P(E) = 0$. Rule 3 is illustrated in Example 8.

## Example 8: Calculate the Probability—Drawing a Specific Card

The black cards in an ordinary deck are removed; then the 26 red cards are well shuffled. One card is selected. Find the probability of getting a black card.



### Solution:

Since there are no black cards in the 26 red cards, it is impossible to get a black card, so:

$$P(\text{black card}) = \frac{0}{26} = 0$$

Rule 4 states that if $P(E) = 1$, the event $E$ is certain to occur. This rule is illustrated in Example 9.

### Example 9: Calculate the Probability—Drawing a Specific Card

The black cards are removed from an ordinary deck of cards. The 26 red cards are well shuffled, and one card in selected. Find the probability that it is a red card.

#### Solution:

Since there are 26 ways to get a red card, the probability of getting a red card is:

$$P(\text{red card}) = \frac{26}{26} = 1$$

### Statistics Around Us: Use Statistics

When selecting a contestant for the aforementioned game show, the residents of Hawaii are removed from the selection. Find the probably of selecting a person from the game show that resides in Alaska using the population data from Table 4A.

**Go online for the data.**

## Learn: The Complement of an Event

Another important concept in probability theory is that of *complementary events.* When a die is rolled, for instance, the sample space consists of the outcomes 1, 2, 3, 4, 5, and 6. The event $E$ of getting odd numbers consists of the outcomes 1, 3, and 5. The event of not getting an odd number is called the *complement* of event $E$, and it consists of the outcomes 2, 4, and 6.

The **complement of an event** $E$ is the set of outcomes in the sample space that are not included in the outcomes of event $E$. The complement of $E$ is denoted by $\overline{E}$ (read "$E$ bar").

Example 10 illustrates the concept of complementary events.

### Example 10: Finding Complements

Find the complement of each event:

**a.** Selecting a month that has 30 days

**b.** Selecting a day of the week that begins with the letter *S*

**c.** Rolling two dice and getting a number the sum 7

**d.** Selecting a letter of the alphabet (excluding *y*) that is a vowel

#### Solution:

**a.** Selecting a month that has 28 or 31 days: January, February, March, May, July, August, October, or December

**b.** Selecting a day of the week that does not begin with *S:* Monday, Tuesday, Wednesday, Thursday, or Friday

**c.** Rolling two dice and getting a sum of 2, 3, 4, 5, 6, 8, 9, 10, 11, or 12

**d.** Selecting a letter of the alphabet that is a consonant

## Statistics Around Us: Use Statistics

Find the complement of each event using the population data of Alaska and Hawaii residents provided in Table 4A.

**a.** Selecting a person from Alaska

**b.** Selecting a person from a county with fewer than 1,000 residents

**c.** Selecting a person from a county that starts with the letter *K*

GO ONLINE **Go online for the data.**

The outcomes of an event and the outcomes of the complement make up the entire sample space. For example, if two coins are tossed, the sample space is HH, HT, TH, and TT. The complement of “getting all heads” is not “getting all tails” since the event “all heads” is HH, and the complement of HH is HT, TH, and TT. So, the complement of the event “all heads” is the event “getting at least one tail.”

Since the event and its complement make up the entire sample space, it follows that the sum of the probability of the event and the probability of its complement will equal 1. That is, $P(E) + P(\bar{E}) = 1$. For example, let $E$ = all heads, or HH, and let $\bar{E}$ = at least one tail, or HT, TH, TT. Then $P(E) = \frac{1}{4}$ and $P(\bar{E}) = \frac{3}{4}$; so, $P(E) + P(\bar{E}) = \frac{1}{4} + \frac{3}{4} = 1$.

The rule for complementary events can be stated algebraically in three ways.

### Rule for Complementary Events

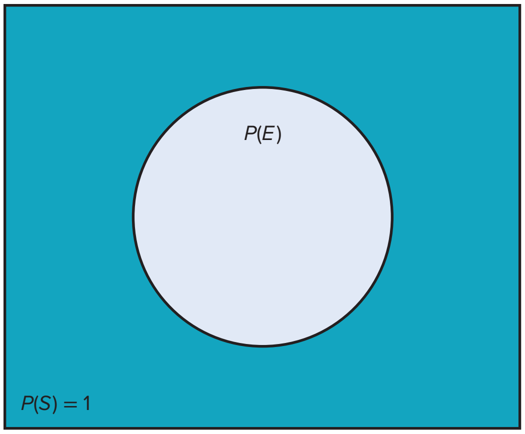
$P(\bar{E}) = 1 - P(E)$ or $P(E) = 1 - P(\bar{E})$ or $P(E) + P(\bar{E}) = 1$

Stated in words, the rule is: *If the probability of an event or the probability of its complement is known, the other can be found by subtracting the probability from 1.* This rule is important in probability theory because at times the best solution to a problem is to find the probability of the complement of an event and then subtract from 1 to get the probability of the event itself.

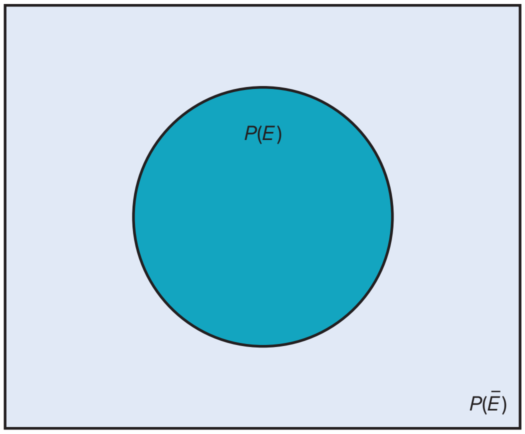
### Venn Diagrams

Probabilities can be represented pictorially by **Venn diagrams.** Figure 4–7(a) on page 214 shows the probability of a simple event *E*. The area inside the circle represents the probability of event *E*: $P(E)$. The area inside the rectangle represents the probability of all the events in the sample space $P(S)$.

The Venn diagram that represents the probability of the complement of an event $P(\bar{E})$ is shown in Figure 4–7(b). In this case, $P(\bar{E}) = 1 - P(E)$, which is the area inside the rectangle but outside the circle representing $P(E)$. Recall that $P(S) = 1$ and $P(E) = 1 - P(\bar{E})$. The reasoning is that $P(E)$ is represented by the area of the circle, and $P(\bar{E})$ is the probability of the events outside the circle.




**(a)** Simple probability




**(b)** $P(\bar{E}) = 1 - P(E)$

FIGURE 4–7 Venn Diagram for the Probability and Complement

## Example 11: Calculate the Probability Using the Complement

A study found that 51% of workers drive 10 miles or less to work, and the rest drive more than 10 miles. If a person is selected at random, find the probability that the person drives more than 10 miles to work.

*Source:* U.S. Department of Transportation

### Solution:

The event (the person drives more than 10 miles to work) and its complement (the person drives 10 miles or less to work) make up the entire sample space. Therefore, the sum of the probabilities of the event and its complement will equal 1.

$$P(\text{drives more than 10 miles to work}) + P(\text{person drives 10 miles or less to work}) = 1$$

So:

$$\begin{aligned} P(\text{drives more than 10 miles to work}) &= 1 - P(\text{person drives 10 miles or less to work}) \\ &= 1 - P(\text{person drives 10 miles or less to work}) \\ &= 1 - 0.51 = 0.49 = 49\% \end{aligned}$$

## Learn: Empirical Probability

The difference between classical and empirical probability is that classical probability assumes that certain outcomes are equally likely (such as the outcomes when a die is rolled), while **empirical probability** relies on actual experience to determine the likelihood of outcomes. In empirical probability, one might actually roll a given die 6000 times, observe the various frequencies, and use these frequencies to determine the probability of an outcome.

Suppose, for example, that a researcher for the American Automobile Association (AAA) asked 50 people who plan to travel over the Thanksgiving holiday how they will get to their destination. The results can be categorized in a frequency distribution, as shown in Table 4–4.

TABLE 4–4 Frequency Distribution of Travel Methods

| Method | Frequency |
|---|---|
| Drive | 41 |
| Fly | 6 |
| Train or bus | 3 |
| | 50 |

Now probabilities can be computed for various categories. For example, the probability of selecting a person who is driving is $\frac{41}{50}$ since 41 out of the 50 people said they were driving.

### Formula for Empirical Probability

Given a frequency distribution, the probability of an event being in a given class is:

$$P(E) = \frac{\text{frequency for the class}}{\text{total frequencies in the distribution}} = \frac{f}{n}$$

This probability is called empirical probability and is based on observation.

## **Example 12:** Calculate an Empirical Probability—Simple Event

In the travel survey just described, find the probability that the person will travel by plane over the Thanksgiving holiday.

### Solution:

The probability of an event being in the class "Fly" is indicated by the formula:

$$P(E) = \frac{\text{frequency for the class “Fly”}}{\text{total frequencies in the distribution}} = \frac{f}{n}$$

From the survey data, the total number of frequencies in the distribution sum to 50. Therefore, $n = 50$. The frequency for the class "Fly," one of the categories of the survey data, is 6, so $f = 6$. Therefore:

$$P(\text{plane}) = \frac{6}{50} = 0.12$$

## **Example 13:** Calculate Empirical Probabilities

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

**a.** A person has type O blood.

**b.** A person has type A or type B blood.

**c.** A person has neither type A nor type O blood.

**d.** A person does not have type AB blood.

*Source:* The American Red Cross

## Solution:

Construct a frequency table from the blood-type data given in the problem. See Table 4–5.

TABLE 4–5 Frequency Distribution of Blood Types

| Type | Frequency |
|---|---|
| A | 22 |
| B | 5 |
| AB | 2 |
| O | 21 |
| | Total: 50 |

For each part of the question, determine the values of $f$ and $n$ to use in the empirical probability formula.

**a.** $P(\text{O}) = \frac{f}{n} = \frac{21}{50}$

**b.** $P(\text{A or B}) = \frac{f}{n} = \frac{22 + 5}{50} = \frac{27}{50}$

**c.** $P(\text{neither A nor O}) = P(\text{B or AB}) = \frac{f}{n} = \frac{5 + 2}{50} = \frac{7}{50}$

**d.** We can find the probability of "not AB" by subtracting the probability of type AB from 1.

$$P(\text{AB}) = \frac{f}{n} = \frac{2}{50}$$

$$P(\text{not } AB) = 1 - P(\text{AB}) = 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25}$$

Another approach to finding $P$(not AB) involves determining the value of the equivalent expression $P$(A or B or O). This approach requires three values from the frequency table. It may be preferable, however, to use the complement of $P$(not AB) because it requires only one value from the frequency table.

## Example 14: Calculate Empirical Probabilities

A recent random survey found the following distribution for the sizes of families of students in a junior college. See Table 4–6.

TABLE 4–6 Frequency Distribution of Family Sizes

| Size | Frequency |
|---|---|
| 1 person | 14 |
| 2 persons | 18 |
| 3 persons | 27 |
| 4 persons | 35 |
| 5 or more persons | 6 |
| | 100 |

If a student is selected at random from the sample, find each of the following probabilities:

**a.** The student has a family size of 2 persons.

**b.** The student has a family size of fewer than 3 persons.

**c.** The student has a family size of at most 3 persons.

**d.** The student has a family size of at least 4 persons.

**Solution:**

**a.** $P(\text{2 persons}) = \frac{18}{100} = \frac{9}{50} = 0.18$

**b.** $P(\text{fewer than 3 persons}) = P(\text{1 or 2 persons}) = \frac{14 + 18}{100} = \frac{32}{100} = 0.32$

**c.** $P(\text{at most 3 persons}) = P(\text{1 or 2 or 3 persons}) = \frac{14 + 18 + 27}{100} = \frac{59}{100} = 0.59$

**d.** $P(\text{at least 4 persons}) = P(\text{4 or 5 or more persons}) = \frac{35 + 6}{100} = \frac{41}{100} = 0.41$

An alternative way to find $P$(at least 4 persons) would be to use the complement.

$$P(\text{at least 4 persons}) = 1 - P(\text{at most 3 persons}) = 1 - 0.59 = 0.41$$

## Statistics Around Us: Use Statistics

All residents in Alaska and Hawaii are completing a survey. If a survey participant is selected at random from the population sample provided in Table 4A, find each of the following probabilities:

**a.** The person selected is from Hawaii or Fairbanks North Star Borough.

**b.** The person selected is not from a county that starts with the letter *K*.

**c.** The person selected is neither from Alaska nor Maui County.

GO ONLINE Go online for the data.

Empirical probabilities can also be found by using a relative frequency distribution, as shown in Lesson 2–2. For example, the relative frequency distribution of the travel survey shown previously is provided in Table 4–6A.

TABLE 4–6A Frequency and Relative Frequency of Travel Methods

| Method | Frequency | Relative frequency |
|---|---|---|
| Drive | 41 | 0.82 |
| Fly | 6 | 0.12 |
| Train or bus | 3 | 0.06 |
| | 50 | 1.00 |

These frequencies are the same as the relative frequencies explained in Chapter 2.

## **Learn:** Law of Large Numbers

When a coin is tossed one time, it is common knowledge that the probability of getting a head is $\frac{1}{2}$. But what happens when the coin is tossed 50 times? Will it come up heads 25 times? Not all the time. You should expect about 25 heads if the coin is fair. But due to chance variation, 25 heads will not occur most of the time.

If the empirical probability of getting a head is computed by using a small number of trials, it is usually not exactly $\frac{1}{2}$. However, as the number of trials increases, the empirical probability of getting a head will approach the theoretical probability of $\frac{1}{2}$ if in fact the coin is fair (i.e., balanced). This phenomenon is an example of the **law of large numbers.**

You should be careful not to think that the number of heads and number of tails tend to "even out." As the number of trials increases, the proportion of heads to the total number of trials will approach $\frac{1}{2}$. This law holds for any game of chance—tossing dice, drawing a card, and so on. It should be pointed out that the probabilities that the proportions steadily approach may or may not agree with those theorized in the classical model. If not, it can have important implications, such as "the die is not fair."

## **Learn:** Subjective Probability

The third type of probability is called *subjective probability.* **Subjective probability** uses a probability value based on an educated guess or estimate, employing opinions and inexact information.

In subjective probability, a person or group makes an educated guess at the chance that an event will occur. This guess is based on the person's experience and evaluation of a solution. For example, a sportswriter may say there is a 70% probability that the Pirates will win the pennant next year. A physician might say that, on the basis of her diagnosis, there is a 30% chance the patient will need an operation. A seismologist might say there is an 80% probability that an earthquake will occur in a certain area. These are only a few examples of how subjective probability is used in everyday life.

All three types of probability (classical, empirical, and subjective) are used to solve a variety of problems in business, engineering, and other fields.

## **Learn:** Probability and Risk Taking

An area in which people sometimes misunderstand probability is risk taking. People may fear situations or events that have a relatively small probability of happening rather than those events that have a greater likelihood of occurring. For example, many people think the crime rate is increasing every year. However, in his book entitled *How Risk Affects Your Everyday Life,* author James Walsh states: "Despite widespread concern about the number of crimes committed in the United States, FBI and Justice Department statistics show that the national crime rate has remained fairly level for 20 years. It even dropped slightly in the early 1990s."

He further states, "Today most media coverage of risk to health and well-being focuses on shock and outrage." Shock and outrage make good stories and can scare us about the wrong dangers. For example, the author states that if a person is 20% overweight, the loss of life expectancy is 900 days (about 3 years), but loss of life expectancy from exposure to radiation emitted by nuclear power plants is 0.02 day. As you can see, being overweight is much more of a threat than being exposed to radioactive emission.

Many people gamble daily with their lives—for example, by making unhealthy choices. When people are asked to estimate the probabilities or frequencies of death from various causes, they tend to overestimate causes such as accidents, fires, and floods and to underestimate the probabilities of death from diseases (other than cancer), strokes, etc. For example, most people think their chances of dying from a heart attack are 1 in 20, when in fact they are almost 1 in 3; the chances of dying by pesticide poisoning are 1 in 200,000 (*True Odds* by James Walsh). The reason people think this way is that the news media sensationalize deaths resulting from catastrophic events and rarely mention deaths from disease.

When you are dealing with life-threatening catastrophes such as hurricanes, floods, automobile accidents, or texting while driving, it is important to get the facts. That is, get the actual numbers from accredited statistical agencies or reliable statistical studies, and then compute the probabilities and make decisions based on your knowledge of probability and statistics.

In summary, when you make a decision or plan a course of action based on probability, make sure you understand the true probability of the event occurring. Also, find out how the information was obtained (i.e., from a reliable source). Weigh the cost of the action and decide if it is worth it. Finally, look for other alternatives or courses of action with less risk involved.

## Applying the Concepts 4–1
### Tossing a Coin

Assume you are at a carnival and decide to play one of the games. You spot a table where a person is flipping a coin, and because you have an understanding of basic probability, you believe the odds of winning are in your favor. When you get to the table, you find out all you have to do is guess which side of the coin will be facing up after it is tossed. You are assured the coin is fair, meaning that each of the two sides has an equally likely chance of occurring. You think back about what you learned in your statistics class about probability before you decide what to guess. Answer the following questions about the coin-tossing game.

1. What is the sample space?
2. What are the possible outcomes?
3. What does the classical approach to probability say about computing probabilities for this type of problem?

You decide to guess heads, believing it has a 50% chance of coming up. A friend of yours, who had been playing the game for a while before you got there, tells you that heads has come up the last 9 times in a row. You remember the law of large numbers.

4. What is the law of large numbers, and does it change your thoughts about what will occur on the next toss?
5. What does the empirical approach to probability say about this problem, and could you use it to solve this problem?
6. Can subjective probabilities be used to help solve this problem? Explain.
7. Assume you could win $1 million if you could guess what the results of the next toss will be. What would you guess? Why?

# Practice 4–1

## Practice Exercises

**In #1–7, answer questions related to probability and probability experiments.**

1. What is a probability experiment?
2. Define *sample space*.
3. What is the range of the values of the probability of an event?
4. What is the sum of the probabilities of all the outcomes in a sample space?
5. If the probability that it will rain tomorrow is 0.20, what is the probability that it won't rain tomorrow? Would you recommend taking an umbrella?
6. A probability experiment is conducted. Which of these cannot be considered a probability outcome?

| | | |
|---|---|---|
| **a.** $\frac{5}{8}$ | **d.** 2.14 | **g.** 1 |
| **b.** 0.29 | **e.** −0.63 | **h.** 153% |
| **c.** $-\frac{1}{3}$ | **f.** 0 | **i.** 87% |

7. Classify each statement as an example of classical probability, empirical probability, or subjective probability.
   - **a.** The probability that a student will get a C or better in a statistics course is about 70%.
   - **b.** The probability that a new fast-food restaurant will be a success in Chicago is 35%.
   - **c.** The probability that interest rates will rise in the next 6 months is 0.50.
   - **d.** The probability that the unemployment rate will fall next month is 0.03.

**In #8–16, determine the probabilities of the events described.**

8. If a die is rolled one time, find these probabilities:
   - **a.** Getting a 9
   - **b.** Getting a number greater than 2
   - **c.** Getting a number less than 7
   - **d.** Getting a prime number
9. If two dice are rolled one time, find the probability of getting these results:
   - **a.** A sum less than 9
   - **b.** A sum greater than or equal to 10
   - **c.** A 3 on one die or on both dice
10. If a card is drawn from a deck, find the probability of getting these results:
    - **a.** A 6 and a spade
    - **b.** A black king
    - **c.** A red card and a 7
    - **d.** A diamond or a heart
    - **e.** A black card
11. Human blood is grouped into four types. The percentages of Americans with each type are listed below.

    O 43% A 40% B 12% AB 5%

    Choose one American at random. Find the probability that this person:
    - **a.** Has type B blood.
    - **b.** Has type AB or O blood.
    - **c.** Does not have type O blood.
12. The following survey consists of the percent of common oil-spill locations and where they occur.

| Factories/ buildings | Roads | Pipelines | Railroads | Other |
|---|---|---|---|---|
| 75% | 8% | 3% | 6% | 8% |

If an oil spill is selected at random, find the probability that:

- **a.** It occurred in an oil pipeline.
- **b.** It occurred in a factory or railroad.
- **c.** It did not occur in a road accident.

**13.** Rural speed limits for all 50 states are indicated below.

| 55 mph | 60 mph | 65 mph | 70 mph | 75 mph |
|---|---|---|---|---|
| 12 | 4 | 19 | 12 | 3 |

Choose one state at random. Find the probability that its speed limit is:

**a.** 60 or 70 miles per hour.

**b.** Greater than 65 miles per hour.

**c.** 70 miles per hour or less.

*Source:* https://en.wikipedia.org/wiki/Speed_limits_in_the_United_States

**14.** Elementary schools were classified by the number of computers they had.

| Computers | 1–10 | 11–20 | 21–50 | 51–100 | 100+ |
|---|---|---|---|---|---|
| Schools | 3170 | 4590 | 16,741 | 23,753 | 34,803 |

Choose one school at random. Find the probability that it has:

**a.** Between 11 and 20 computers.

**b.** More than 20 computers.

**c.** Fewer than 51 computers.

*Source: World Almanac*

**15.** According to *CarMax,* 22.25% of preowned automobiles sold are black, 19.34% are white, 17.63% are gray, and 14.64% are silver. If a preowned automobile is selected at random, find the probability that it is either gray or silver.

**16.** The source of federal government revenue for a specific year is:

50% from individual income taxes

32% from social insurance payroll taxes

10% from corporate income taxes

3% from excise taxes

5% other

If a revenue source is selected at random, what is the probability that it comes from individual or corporate income taxes?

*Source: New York Times Almanac*

**In #17–19, draw tree diagrams to represent all possible outcomes.**

**17.** A box contains a $1 bill, a $5 bill, a $10 bill, and a $20 bill. A bill is selected at random, and it is not replaced; then a second bill is selected at random. Draw a tree diagram and determine the sample space.

**18.** Draw a tree diagram and determine the sample space for tossing four coins.

**19.** A family special at a neighborhood restaurant offers dinner for four for $39.99. There are 3 appetizers, 4 entrees, and 3 desserts from which to choose. The special includes one of each. Represent the possible dinner combinations with a tree diagram.

**In #20–23, use probability to answer the questions.**

**20.** A person flipped a coin 100 times and obtained 73 heads. Can the person conclude that the coin was unbalanced?

**21.** A medical doctor stated that with a certain treatment, a patient has a 50% chance of recovering without surgery. In other words, "Either he will get well or he won't get well." Comment on this statement.

**22.** When you take an exam that uses multiple-choice questions with 5 answer selections for each question, is the probability of guessing a correct answer $\frac{1}{2}$ since the answer is either correct or incorrect?

**23.** Which is a better bet when flipping a coin 4 times: getting 2 heads (the mean) or getting either one or three heads?

GO ONLINE Go online for more practice problems.

Lesson 4–2

# The Addition Rules for Probability

## Learn: Mutually Exclusive Events

Many problems involve finding the probability of two or more events. For example, at a large political gathering, you might wish to know, for a person selected at random, the probability that the person is a female or is a Republican. In this case, there are three possibilities to consider:

1. The person is a female.
2. The person is a Republican.
3. The person is both a female and a Republican.

Consider another example. At the same gathering there are Republicans, Democrats, and Independents. If a person is selected at random, what is the probability that the person is a Democrat or an Independent? In this case, there are only two possibilities:

1. The person is a Democrat.
2. The person is an Independent.

The difference between the two examples is that in the first case, the person selected can be a female and a Republican at the same time. In the second case, the person selected cannot be both a Democrat and an Independent at the same time. In the second case, the two events are said to be *mutually exclusive;* in the first case, they are not mutually exclusive.

Two events are **mutually exclusive events** or **disjoint events** if they cannot occur at the same time (i.e., they have no outcomes in common).

In another situation, the events of getting a 4 and getting a 6 when a single card is drawn from a deck are mutually exclusive events since a single card cannot be both a 4 and a 6. On the other hand, the events of getting a 4 and getting a heart on a single draw are not mutually exclusive since you can select the 4 of hearts when drawing a single card from an ordinary deck.

### Example 1: Determining Mutually Exclusive Events

Determine whether the two events are mutually exclusive. Explain your answer.

**a.** Randomly selecting a female student

Randomly selecting a student who is a junior

**b.** Randomly selecting a person with type A blood

Randomly selecting a person with type O blood

**c.** Rolling a die and getting an odd number

Rolling a die and getting a number less than 3

**d.** Randomly selecting a person who is under 21 years of age

Randomly selecting a person who is over 30 years of age

**Solution:**

**a.** These events are not mutually exclusive since a student can be both female and a junior.

**b.** These events are mutually exclusive since a person cannot have type A blood and type O blood at the same time.

**c.** These events are not mutually exclusive since the number 1 is both an odd number and a number less than 3.

**d.** These events are mutually exclusive since a person cannot be both under 21 and over 30 years of age at the same time.

## Example 2: Determining Mutually Exclusive Events—Cards

Determine which events are mutually exclusive when a single card is drawn at random from a deck of cards.

**a.** Getting an ace and getting a queen

**b.** Getting a jack and getting a club

**c.** Getting a face card and getting a four

**d.** Getting a diamond and getting a red card

**Solution:**

**a.** These events are mutually exclusive since you cannot get an ace and a queen at the same time when only one card is selected.

**b.** These events are *not* mutually exclusive since you can get the jack of clubs when one card is drawn.

**c.** These events are mutually exclusive since you cannot get a face card and a four at the same time when one card is selected.

**d.** These events are *not* mutually exclusive since you can get a diamond and a red card at the same time.

## Statistics Around Us: Use Statistics

Determine whether the two events are mutually exclusive when a person is selected at random from the populations listed in Table 4A. Explain your answer.

**a.** Randomly selecting a person who lives in the U.S.

Randomly selecting a person who lives in Alaska

**b.** Randomly selecting a person who lives full time in Hawaii

Randomly selecting a person who lives full time in Alaska

**c.** Randomly selecting a person born in Cooper River Census Area

Randomly selecting a person born in Sitka City and Borough

GO ONLINE Go online for the data.

# **Learn:** Addition Rule for Mutually Exclusive Events

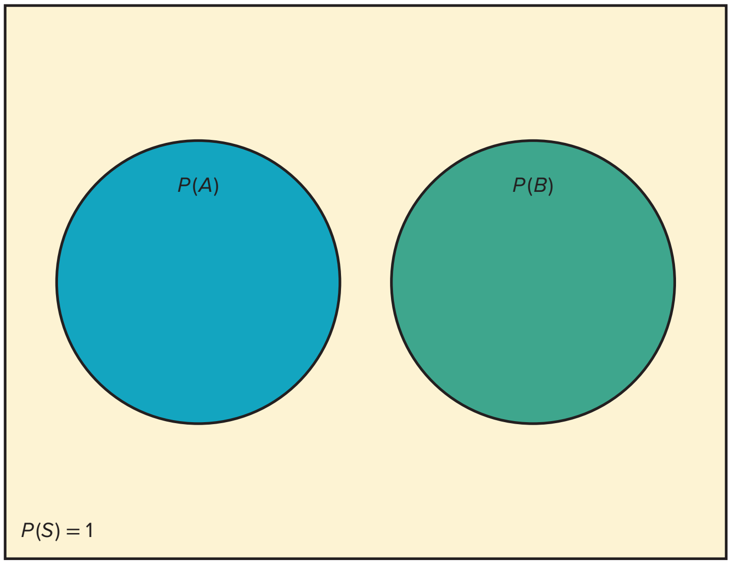
The probability of two or more events can be determined by the *addition rules.* The first addition rule is used when the events are mutually exclusive.

**Addition Rule 1**

When two events *A* and *B* are mutually exclusive, the probability that *A* or *B* will occur is:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B})$$

Figure 4–8 shows a Venn diagram that represents two mutually exclusive events *A* and *B*. In this case, $P(\text{A or B}) = P(\text{A}) + P(\text{B})$ since these events are mutually exclusive and do not overlap. In other words, the probability of occurrence of event *A* or event *B* is the sum of the areas of the two circles.




Mutually exclusive events
$P(A \text{ or } B) = P(A) + P(B)$

FIGURE 4–8 Venn Diagram for Addition Rule 1 When the Events Are Mutually Exclusive

The probability rules can be extended to three or more events. For three mutually exclusive events *A*, *B*, and *C*, $P(\text{A or B or C}) = P(\text{A}) + P(\text{B}) + P(\text{C})$.



## **Example 3:** Find the Probability of Mutually Exclusive Events

In the United States there are 59 different species of endangered mammals, 75 different species of endangered birds, and 68 species of endangered fish. If one animal is selected at random, find the probability that it is either a mammal or a fish.

*Source:* Based on information from the U.S. Fish and Wildlife Service

## Solution:

If events *A* and *B* are mutually exclusive, then addition rule 1 can be used.

$$P(\text{A or B}) = P(\text{A}) + P(\text{B})$$

**Step 1** Calculate the probability of each event. There are 59 species of mammals and 68 species of fish that are endangered and a total of 202 endangered species. Therefore:

$$P(\text{mammal}) = \frac{f}{n} = \frac{59}{202} \text{ and } P(\text{fish}) = \frac{f}{n} = \frac{68}{202}$$

**Step 2** Evaluate whether the events are mutually exclusive. A species can be a mammal or a fish, but not both. Therefore, the events *P*(mammal) and *P*(fish) are mutually exclusive.

**Step 3** Because the events are mutually exclusive, addition rule 1 can be applied.

$$P(\text{mammal or fish}) = P(\text{mammal}) + P(\text{fish}) = \frac{59}{202} + \frac{68}{202} = \frac{127}{202} = 0.629$$

## Example 4: Find the Probability of Mutually Exclusive Events

The number of employees in the corporate research and development centers for three local companies are listed in Table 4–7.

TABLE 4–7 Number of Corporate Research and Development Employees at Local Companies

| Company | Number of employees |
|---|---|
| U.S. Steel | 110 |
| Alcoa | 750 |
| Bayer Material Science | 250 |

If a research employee is selected at random, find the probability that the employee is employed by U.S. Steel or Alcoa.

*Source: Pittsburgh Tribune Review*

## Solution:

**Step 1** Calculate the individual probabilities. There are 110 employees of U.S. Steel and 750 employees of Alcoa and a total of 1110 employees. Therefore:

$$P(\text{U.S. Steel}) = \frac{110}{1110} \text{ and } P(\text{Alcoa}) = \frac{750}{1110}$$

**Step 2** Evaluate whether the events are mutually exclusive. An employee can work for U.S. Steel and Alcoa, but not both. Therefore, the events are mutually exclusive.

**Step 3** Because the events are mutually exclusive, use addition rule 1.

$$P(\text{U.S. Steel or Alcoa}) = P(\text{U.S. Steel}) + P(\text{Alcoa})$$

$$= \frac{110}{1110} + \frac{750}{1110} = \frac{860}{1110} = \frac{86}{111} = 0.775$$

## Example 5: Find the Probability of Mutually Exclusive Events Given Percents

A survey of tennis players found that 53% were white-collar workers (managers or supervisors), 24% were students, 12% were blue-collar workers, and 11% were other types of workers. If a tennis player is selected at random, find the probability that the player is either a student or a blue-collar worker.

*Source:* Tennis Industry Association

### Solution:

**Step 1** In the survey, 24% of tennis players were students and 12% were blue-collar workers. Therefore:

$$P(\text{student}) = 24\% = 0.24 \text{ and } P(\text{blue-collar}) = 12\% = 0.24$$

**Step 2** Evaluate whether the events are mutually exclusive. A tennis player can be a student or a blue-collar worker but not both, so the events are mutually exclusive.

**Step 3** Because the events are mutually exclusive, use addition rule 1.

$$P(\text{student or blue-collar}) = P(\text{student}) + P(\text{blue-collar}) = 0.24 + 0.12 = 0.36$$

## Statistics Around Us: Use Statistics

According to the 2020 U.S. Census, Alaska and Hawaii are among 34 U.S. states that saw less than 10% increase in population from 2010 to 2020, while 13 U.S. states saw an increase of 10% or more in population and 3 U.S. states declined in population from 2010 to 2020. If a state is selected at random from the United States, find the probability that the state saw an increase in population from the year 2010 to 2020.

## Learn: Addition Rule for Events That Are Not Mutually Exclusive

When events are not mutually exclusive, addition rule 2 can be used to find the probability of the events.

**Addition Rule 2**

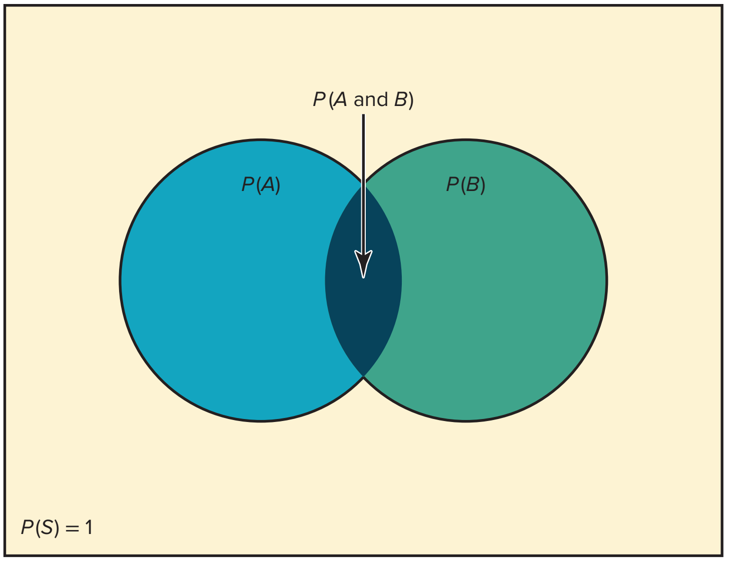
If A and B are *not* mutually exclusive, then:

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$



Note that addition rule 2 can also be used when the events are mutually exclusive since $P$(A and B) will equal 0 when A and B are mutually exclusive. However, it is important to make a distinction between the two situations.

Figure 4–9 on page 228 represents the probability of two events that are *not* mutually exclusive. In this case, $P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$. The area in the intersection or overlapping part of both circles corresponds to $P$(A and B); and when the area of circle *A* is added to the area of circle *B*, the overlapping part is counted twice. It must therefore be subtracted once to get the correct area or probability.




Nonmutually exclusive events
$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

FIGURE 4–9 Venn Diagram for Addition Rule 2 When Events Are Not Mutually Exclusive

## Example 6: Find the Probability of Events That Are Not Mutually Exclusive—Drawing Cards

A single card is drawn at random from an ordinary deck of cards. Find the probability that it is a seven or a black card.

### Solution:

If events *A* and *B* are not mutually exclusive, addition rule 2 can be used to find the probability that either *A* or *B* occur.

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

**Step 1** Calculate the probability of each event. There are four 7 cards and 26 black cards in a standard deck of 52 cards.

$$P(7) = \frac{4}{52} \text{ and } P(\text{a black card}) = \frac{26}{52}$$

**Step 2** Evaluate whether the events are mutually exclusive. Two cards in a standard deck can be both a 7 and black: a 7 of clubs and a 7 of spades. Therefore, these are not mutually exclusive events.

### Historical Note

Venn diagrams were developed by mathematician John Venn (1834–1923) and are used in set theory and symbolic logic. They have been adapted to probability theory also. In set theory, the symbol ∪ represents the *union* of two sets, and $A \cup B$ corresponds to *A* or *B*. The symbol ∩ represents the *intersection* of two sets, and $A \cap B$ corresponds to *A* and *B*. Venn diagrams show only a general picture of the probability rules and do not portray all situations, such as $P(A) = 0$, accurately.

**Step 3** Because the two events are not mutually exclusive, addition rule 2 should be used.

$$P(7 \text{ or a black card}) = P(7) + P(\text{black card}) - P(7\clubsuit \text{ or } 7\spadesuit)$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13} \approx 0.538$$

## Example 7: Find the Probability of Events That Are Not Mutually Exclusive

In an animal shelter, there are 8 dogs and 10 cats that are available for adoption. Six of the dogs are females, and 3 of the cats are females. If a dog or cat is adopted today, find the probability that it is a cat or a male.

### Solution:

**Step 1** Determine the sample space. If 6 of the 8 dogs are female, then 2 of the dogs are male. If 3 of the 10 cats are female, then 7 of the cats are male. Use this information to construct a table to arrange the data for the kind of pet (dog or cat) and the gender (male or female). See Table 4–8.

TABLE 4–8 Animal Shelter Data by Animal and Gender

| Animal | Females | Males | Total |
|---|---|---|---|
| Dog | 6 | 2 | 8 |
| Cat | 3 | 7 | 10 |
| Total: | 9 | 9 | 18 |

**Step 2** Calculate the probability of each event. Of the 18 animals, there are 9 males and 10 cats.

$$P(\text{male}) = \frac{9}{18} \text{ and } P(\text{cat}) = \frac{10}{18}$$

**Step 3** Evaluate whether the events are mutually exclusive. Of the 18 pets, 7 are both cats and males. Therefore, these are not mutually exclusive events. In other words:

$$P(\text{male cat}) = \frac{7}{18}$$

**Step 4** Because the two events are not mutually exclusive, addition rule 2 should be used.

$$P(\text{male or cat}) = P(\text{male}) + P(\text{cat}) - P(\text{male cat}) = \frac{9}{18} + \frac{10}{18} - \frac{7}{18} = \frac{12}{18} \approx 0.67$$

## Example 8: Find the Probability of Events That Are Not Mutually Exclusive

A grocery store beside a busy road also includes gasoline pumps. The probability that a person driving by stops for groceries is 0.19, the probability that a person driving by stops for gasoline is 0.31, and the probability that a person driving by stops for gasoline and groceries is 0.15. What is the probability that a person driving by stops for groceries or gasoline?

**Solution:**

**Step 1** Identify the probability of the events "stops for groceries" and "stops for gasoline."

$$P(\text{groceries}) = 0.19 \quad \text{and} \quad P(\text{gasoline}) = 0.31$$

**Step 2** Evaluate whether the events are mutually exclusive. Because people can stop for both groceries and gasoline, these are not mutually exclusive events.

$$P(\text{groceries and gasoline}) = 0.15$$

**Step 3** Because the two events are not mutually exclusive, use addition rule 2.

$$P(\text{groceries or gasoline}) = P(\text{groceries}) + P(\text{gasoline}) - P(\text{groceries and gasoline})$$

$$= 0.19 + 0.31 - 0.15 = 0.35$$

## Statistics Around Us: Use Statistics

One person from Hawaii will be selected at random to speak one-on-one with the President of the United States during his election campaign tour. Go online to find the probability that the person selected is aged 18 or over, or from Hawaii County, assuming population data is the same today.

**Go online for additional data.**

For three events that are *not* mutually exclusive,

$$P(\text{A or B or C}) = P(\text{A}) + P(\text{B}) + P(\text{C}) - P(\text{A and B}) - P(\text{A and C}) - P(\text{B and C}) + P(\text{A and B and C})$$

See Exercises 9 and 10 in this lesson.

In summary, when the two events are mutually exclusive, use addition rule 1. When the events are not mutually exclusive, use addition rule 2.

## Applying the Concepts 4–2

### Which Pain Reliever Is Best?

Assume that following an injury you received from playing your favorite sport, you obtain and read information on new pain medications. In that information you read of a study that was conducted to test the side effects of two new pain medications. Use the information in Table 4–9 to answer the questions and decide which, if any, of the two new pain medications you will use.

TABLE 4–9 Number and Type of Side Effects in 12-Week Clinical Trial

| Side Effect | Placebo $n = 192$ | Drug A $n = 186$ | Drug B $n = 188$ |
|---|---|---|---|
| Upper respiratory congestion | 10 | 32 | 19 |
| Sinus headache | 11 | 25 | 32 |
| Stomachache | 2 | 46 | 12 |
| Neurological headache | 34 | 55 | 72 |
| Cough | 22 | 18 | 31 |
| Lower respiratory congestion | 2 | 5 | 1 |

**1.** How many subjects were in the study?

**2.** How long was the study?

**3.** What were the variables under study?

**4.** What type of variables are they, and what level of measurement are they on?

**5.** Are the numbers in the table exact figures?

**6.** What is the probability that a randomly selected person was receiving a placebo?

**7.** What is the probability that a person was receiving a placebo or drug A? Are these mutually exclusive events? What is the complement to this event?

**8.** What is the probability that a randomly selected person was receiving a placebo or experienced a neurological headache?

**9.** What is the probability that a randomly selected person was not receiving a placebo or experienced a sinus headache?

# Practice 4–2

## Practice Exercises

**In #1–2, answer questions about mutually exclusive events.**

1. Explain briefly why addition rule 2 can be used when two events are mutually exclusive.

2. Determine whether these events are mutually exclusive.

   **a.** Roll two dice: Get a sum of 7 or get doubles.

   **b.** Select any course: It is a calculus course, and it is an English course.

   **c.** Select a registered voter: The voter is a Republican, and the voter is a Democrat.

**In #3–12, find the probabilities of the events described.**

3. The probability that John will drive to school is 0.37, the probability that he will ride with friends is 0.23, and the probability that his parents will take him is 0.4. He is not allowed to have passengers in the car when he is driving. What is the probability that John will have company on the way to school?

4. A survey of 60 randomly selected children asking what type of role model they admired showed the following results:

| Person | Number |
|---|---|
| Actors | 22 |
| Musicians | 16 |
| Athletes | 11 |
| Comedians | 8 |
| Politicians | 3 |

   If one child is selected at random, find each probability:

   **a.** The child selected an athlete or a musician.

   **b.** The child selected a comedian or politician.

   **c.** The child did not select an actor.

5. A pizza restaurant sold 24 cheese pizzas and 16 pizzas with one or more toppings. Twelve of the cheese pizzas were eaten at work, and 10 of the pizzas with one or more toppings were eaten at work. If a pizza was selected at random, find the probability of each:

   **a.** It was a cheese pizza eaten at work.

   **b.** It was a pizza with either one or more toppings, and it was not eaten at work.

   **c.** It was a cheese pizza, or it was a pizza eaten at work.

6. At a used-book sale, 100 books are adult books, and 160 are children's books. Of the adult books, 70 are nonfiction, while 60 of the children's books are nonfiction. If a book is selected at random, find the probability that it is:

   **a.** Fiction.

   **b.** Not a children's nonfiction book.

   **c.** An adult book or a children's nonfiction book.

7. A local postal carrier distributes first-class letters, advertisements, and magazines. For a certain day, she distributed the following numbers of each type of item.

| Delivered to | First-class letters | Ads | Magazines |
|---|---|---|---|
| Home | 325 | 406 | 203 |
| Business | 732 | 1021 | 97 |

   If an item of mail is selected at random, find these probabilities.

   **a.** The item went to a home.

   **b.** The item was an ad, or it went to a business.

   **c.** The item was a first-class letter, or it went to a home.

**8.** For a recent year, the population for a group of Midwestern states in millions was distributed as follows:

| Age group | Number |
|---|---|
| Under 5 years old | 5.1 |
| 5–17 years | 12.2 |
| 18–24 years | 6.8 |
| 25–44 years | 20.1 |
| 45–64 years | 14.2 |
| 65+ years | 9.1 |

*Source:* Based on the U.S. Census

If a person is selected at random from a Midwestern state, find the probability that the person is:

**a.** Either 5–17 years old or 25–44 years old.

**b.** Either 5–24 years old or 45–64 years old.

**c.** Either under 5 years old or over 64 years old.

**9.** If one card is drawn at random from an ordinary deck of shuffled cards, find the probability of getting:

**a.** A 2, 3, 4, or 5.

**b.** A club or an ace of diamonds.

**c.** A diamond or a 6.

**10.** Two dice are rolled. Find the probability of getting:

**a.** A sum of 2, 3, or 5.

**b.** Doubles or a sum of 8.

**c.** A sum greater than 10 or less than 4.

**d.** Based on the answers to *a, b, c,* which event is least likely to occur?

**11.** For a recent year, about 11 billion pounds of apples were harvested. About 4.4 billion pounds of apples were made into apple juice, about 1 billion pounds of apples were made into applesauce, and 1 billion pounds of apples were used for other commercial purposes. If 1 billion pounds of apples were selected at random, what is the probability that the apples were used for apple juice or applesauce?

*Source:* International Apple Institute

**12.** The probability that a customer selects a pizza with mushrooms or pepperoni is 0.55, and the probability that the customer selects only mushrooms is 0.32. If the probability that the customer selects only pepperoni is 0.17, find the probability of the customer selecting both items.

**In #13–14, answer questions about probabilities of events.**

**13.** Suppose that $P(A) = 0.42$, $P(B) = 0.38$, and $P(A \text{ or } B) = 0.70$. Are $A$ and $B$ mutually exclusive? Explain.

**14.** Events $A$ and $B$ are mutually exclusive with $P(A)$ equal to 0.392 and $P(A \text{ or } B)$ equal to 0.653. Find:

**a.** $P(B)$

**b.** $P(\text{not } A)$

**c.** $P(A \text{ and } B)$

GO ONLINE Go online for more practice problems.

# Using Technology: TI-84 Plus Step by Step

## Relative Frequency

TABLE TI4–1 Creating Relative Frequency Tables

| Desired Action | TI-84 Plus Instructions |
|---|---|
| Construct a relative frequency table | **1.** Enter the data values in $L_1$ and the frequencies in $L_2$.<br>**2.** Move the cursor to the top of the $L_3$ column so that $L_3$ is highlighted.<br>**3.** Type **$L_2$** divided by the sample size, then press **ENTER.** |

## TI-Example: **Creating a Relative Frequency Table**

Construct a relative frequency table for the data in Table TI4–2 that represent the number of days patients stayed in the hospital following a knee replacement.

TABLE TI4–2 Number of Days in Hospital

| Days | Frequency |
|---|---|
| 3 | 15 |
| 4 | 32 |
| 5 | 56 |
| 6 | 19 |
| 7 | 5 |

**1.** Follow step 1 in Table TI4–1 to enter the data. See Figure TI4–1.

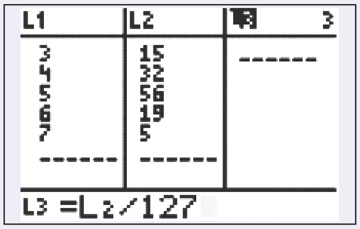



FIGURE TI4–1 Entering Data from Table TI4–2

**2.** Follow steps 2–3 in Table TI4–1 to complete the table. See Figure TI4–2.

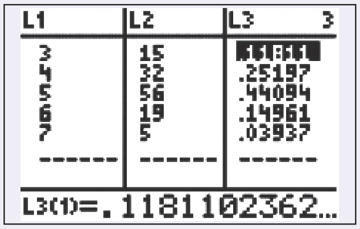



FIGURE TI4–2 Relative Frequency Output

**Using Technology:** Go online to learn how to use statistical software to answer questions.

Lesson 4–3

# The Multiplication Rules and Conditional Probability

Lesson 4–2 showed that the addition rules are used to compute probabilities for mutually exclusive and non-mutually exclusive events. This lesson introduces the multiplication rules.

## Learn: Multiplication Rules for Independent Events

The *multiplication rules* can be used to find the probability of two or more events that occur in sequence. For example, if you toss a coin and then roll a die, you can find the probability of getting a head on the coin *and* a 4 on the die. These two events are said to be *independent* since the outcome of the first event (tossing a coin) does not affect the probability outcome of the second event (rolling a die).

Two events A and B are **independent events** if the fact that A occurs does not affect the probability of B occurring.

Here are other examples of independent events:

Rolling a die and getting a 6, and then rolling a second die and getting a 3

Drawing a card from a deck and getting a queen, replacing it, and drawing a second card and getting a queen

To find the probability of two independent events that occur in sequence, you must find the probability of each event occurring separately and then multiply the answers. For example, if a coin is tossed twice, the probability of getting two heads is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. This result can be verified by looking at the sample space HH, HT, TH, TT. Then $P(\text{HH}) = \frac{1}{4}$.

**Multiplication Rule 1**

When two events are independent, the probability of both occurring is:

$$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B})$$



### Example 1: Find the Probability of Two Independent Events

A coin is flipped, and a die is rolled. Find the probability of getting a tail on the coin and a prime number (2, 3, or 5) on the die.

### Solution:

If events A and B are independent, Multiplication Rule 1 can be used to find the probability that both A or B occur.

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

**Step 1** Determine the sample space for the coin and for the die. The sample space for the coin is H, T, and the sample space for the die is 1, 2, 3, 4, 5, 6.

**Step 2** Determine the probability of each event separately. The probability of a tail, $P(\text{tail})$, is $\frac{1}{2}$, and the probability of a prime number, $P(\text{prime})$, is $\frac{3}{6}$.

**Step 3** Determine if the events are independent. The probability of flipping a tail does not affect the probability of rolling a prime, so the events are independent.

**Step 4** Because the events are independent, the probability of both occurring can be determined by Multiplication Rule 1.

$$P(\text{tail and prime number}) = P(\text{tail}) \cdot P(\text{prime number})$$

$$= \frac{1}{2} \cdot \frac{3}{6} = \frac{3}{12} = \frac{1}{4}$$

The problem in Example 1 can also be solved by using the sample space:

H1 H2 H3 H4 H5 H6 T1 T2 T3 T4 T5 T6

The solution is $\frac{1}{4}$ since there are 3 ways to get a tail and a prime number. They are T2, T3, and T5, and $\frac{3}{12} = \frac{1}{4}$.

## Example 2: Find the Probability of Drawing Cards with Replacement

A card is drawn from a deck and replaced, and then a second card is drawn. Find the probability of getting a club and then a queen.

### Solution:

**Step 1** Determine the sample space for a standard deck of cards. The sample space for drawing a card and noting its suit is: clubs, diamonds, hearts, and spades. There are thirteen cards of each suit in a deck. The sample space for drawing a card and noting its value is 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king, and ace. There are four of each value in a deck.

**Step 2** Determine the probability of each event separately. The probability of drawing a club, $P(\text{club})$, is $\frac{13}{52} = \frac{1}{4}$ and the probability of a queen, $P(\text{queen})$, is $\frac{4}{52} = \frac{1}{13}$.

**Step 3** Determine if the events are independent. Because the card is replaced, the probability of drawing a club on the first pick does not affect the probability of drawing a queen on the second pick. So, the events are independent.

**Step 4** Because the events are independent, the probability of both occurring can be determined by Multiplication Rule 1.

$$P(\text{club and a queen}) = \frac{1}{4} \cdot \frac{1}{13} = \frac{1}{52}$$

## Example 3: Find the Probability of Drawing Marbles with Replacement

An urn contains 2 red marbles, 5 blue marbles, and 3 white marbles. A marble is randomly selected, and its color is noted. Then it is replaced. Again, a marble is randomly selected, and its color is noted. Find the probability of each of these events.

**a.** Selecting 3 blue marbles

**b.** Selecting 1 white marble and then a red marble

**c.** Selecting 2 blue marbles and then one white marble

### Solution:

**Step 1** Because we will be finding the probability of a variety of events, it is helpful to determine the probability of each event separately. There are 10 marbles in all. For each selection, the probability of a red marble is $\frac{2}{10} = \frac{1}{5}$, the probability of a blue marble is $\frac{5}{10} = \frac{1}{2}$, and the probability of a white marble is $\frac{3}{10}$.

**Step 2** Determine if the events are independent. Because each marble selected is returned to the urn, the probability of picking any one color does not affect the probability of another further events. So, the events are independent.

**Step 3** The events are independent, so the probability of them occurring together can be determined by Multiplication Rule 1.

**a.** Selecting 3 blue marbles

$$P(\text{blue and blue and blue}) = P(\text{blue}) \cdot P(\text{blue}) \cdot P(\text{blue})$$
$$= \frac{5}{10} \cdot \frac{5}{10} \cdot \frac{5}{10} = \frac{125}{1000} = \frac{1}{8} = 0.125$$

**b.** Selecting 1 white marble and then a red marble

$$P(\text{white and red}) = P(\text{white}) \cdot P(\text{red})$$
$$= \frac{3}{10} \cdot \frac{2}{10} = \frac{6}{100} = \frac{3}{50} = 0.06$$

**c.** Selecting 2 blue marbles and then 1 white marble

$$P(\text{blue and blue and white}) = P(\text{blue}) \cdot P(\text{blue}) \cdot P(\text{white})$$
$$= \frac{5}{10} \cdot \frac{5}{10} \cdot \frac{3}{10} = \frac{75}{1000} = \frac{3}{40} = 0.075$$

Multiplication rule 1 can be extended to three or more independent events by using the formula:

$$P(\text{A and B and C and} \ldots \text{and K}) = P(\text{A}) \cdot P(\text{B}) \cdot P(\text{C}) \cdots P(\text{K})$$

When a small sample is selected from a large population and the subjects are not replaced, the probability of the event occurring changes so slightly that for the most part, it is considered to remain the same. Example 4 illustrates this concept.

## Example 4: Find the Probability of Three Independent Events

Approximately 9% of men have a type of color blindness that prevents them from distinguishing between red and green. If 3 men are selected at random, find the probability that all of them will have this type of red-green color blindness.

*Source: USA TODAY*

### Solution:

**Step 1** Determine the probability of each event separately. For each man selected, there is a 9% chance the man is color blind.

**Step 2** The probability of a randomly selected man being color blind does not affect the probability of another randomly selected man being color blind. So, the events are independent.

**Step 3** The events are independent, so the probability of them occurring together can be determined by Multiplication Rule 1.

Let *C* denote red-green color blindness. Then:

$$P(\text{C and C and C}) = P(\text{C}) \cdot P(\text{C}) \cdot P(\text{C})$$
$$= (0.09)(0.09)(0.09)$$
$$= 0.000729$$

So, the rounded probability is 0.0007. There is a 0.07% chance that all three men selected will have this type of red-green color blindness.

### Statistics Around Us: Use Statistics

According to the 2020 U.S. Census, 39,538,233 people live in California. Given that the total population in the U.S. in 2020 was 331,449,281, and assuming population data is the same today, find the probability that 2 people selected randomly from the United States both reside in California.

## Learn: Multiplication Rules for Dependent Events

In Examples 1–4, the events were independent of one another since the occurrence of the first event in no way affected the outcome of the second event. On the other hand, when the occurrence of the first event changes the probability of the occurrence of the second event, the two events are said to be *dependent.* For example, suppose a card is drawn from a deck and *not* replaced, and then a second card is drawn. What is the probability of selecting an ace on the first card and a king on the second card?

In this case, the events are dependent. The probability of selecting an ace on the first draw is $\frac{4}{52}$. If that card is *not* replaced, the probability of selecting a king on the second card is $\frac{4}{51}$ since there are 4 kings and 51 cards remaining. The outcome of the first draw has affected the outcome of the second draw.

When the outcome or occurrence of the first event affects the outcome or occurrence of the second event in such a way that the probability is changed, the events are said to be **dependent events.**

Here are some examples of dependent events:

Drawing a card from a deck, not replacing it, and then drawing a second card

Selecting a ball from an urn, not replacing it, and then selecting a second ball

Being a lifeguard and getting a suntan

Having high grades and getting a scholarship

Parking in a no-parking zone and getting a parking ticket

To find probabilities when events are dependent, use the multiplication rule with a modification in notation. For the problem just discussed, the probability of getting an ace on the first draw is $\frac{4}{52}$, and the probability of getting a king on the second draw is $\frac{4}{51}$. By the multiplication rule, the probability of both events occurring is:

$$\frac{4}{52} \cdot \frac{4}{51} = \frac{16}{2652} = \frac{4}{663} \approx 0.006$$

The event of getting a king on the second draw *given* that an ace was drawn the first time is called a *conditional probability.*

The **conditional probability** of an event B in relationship to an event A is the probability that event B occurs after event A has already occurred. The notation for conditional probability is $P(B|A)$. This notation does not mean that B is divided by A; rather, it means the probability that event B occurs given that event *A* has already occurred. In the card example, $P(B|A)$ is the probability that the second card is a king given that the first card is an ace, and it is equal to $\frac{4}{51}$ since the first card was *not* replaced.

**Multiplication Rule 2**

When two events are dependent, the probability of both occurring is:

$$P(\text{A and B}) = P(A) \cdot P(B|A)$$

## **Example 5:** Find the Probability of Two Dependent Events

For a specific year, 5.2% of U.S. workers were unemployed. During that time, 33% of those who were unemployed received unemployment benefits. If a person is selected at random, find the probability that the person received unemployment benefits if the person is unemployed.

*Source:* Bureau of Labor Statistics

## Solution:

If events *A* and *B* are dependent, the probability that both *A* and *B* occur is:

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$

**Step 1** Determine the probability of each event separately. For each person selected, there is a 5.2% chance they are unemployed, so *P*(unemployed) = 0.52. For each unemployed person, there is a 33% chance they receive unemployment benefits, so *P*(unemployment benefits | unemployed) = 0.33.

**Step 2** The occurrence of a randomly selected person being unemployed effects the occurrence of that person receiving unemployment benefits. Therefore, the events are dependent.

**Step 3** Since the events are dependent, the probability of them occurring together is:

$$P(\text{unemployment benefits and unemployed}) = P(U) \cdot P(B \mid U)$$
$$= (0.052)(0.33) = 0.017$$

There is a 0.017 probability that a person is unemployed and receiving unemployment benefits.

## Example 6: Find the Probability of Two Dependent Events

A recent survey found that 80% of randomly selected young adults live within 100 miles of where they grew up; of those, approximately 75% live within 10 miles of where they grew up. If a young adult is selected at random, find the probability that they live within 10 miles of where they grew up.

*Source*: https://www.census.gov/library/stories/2022/07/theres-no-place-like-home.html?utm_campaign=20220725msacos1ccstors&utm_medium=email&utm_source=govdelivery

## Solution:

**Step 1** Determine the probability of each event separately. For each young adult selected, there is an 80% chance they live within 100 miles of where they grew up, so *P*(within 100 miles) = 0.80. For each young adult living within 100 miles, there is a 75% chance they live within 10 miles of where they grew up, so *P*(within 10 miles | within 100 miles) = 0.75.

**Step 2** The occurrence of a randomly selected young adult living within 100 miles effects the occurrence of that person living within 10 miles. Therefore, the events are dependent.

**Step 3** Since the events are dependent, the probability of them occurring together is:

$$P(\text{within 100 miles and within 10 miles})$$
$$= P(\text{within 100 miles}) \cdot P(\text{within 10 miles} \mid \text{within 100 miles})$$
$$= (0.80) \cdot (0.75) = 0.60$$

There is 60% chance that a young adult selected at random lives within 10 miles of where they grew up.

This multiplication rule can be extended to three or more events, as shown in Example 7.

## Example 7: Find the Probability of Three Dependent Events—Drawing Cards Without Replacement

Three cards are drawn from an ordinary deck and not replaced. Find the probability of these events.

**a.** Getting 3 jacks

**b.** Getting an ace, a king, and a queen in order

**c.** Getting a club, a spade, and a heart in order

**d.** Getting 3 clubs

### Solution:

**a.** Getting 3 jacks

**Step 1** Identify the events and determine if each occurring event affects the probability the others that occur in sequence. There are four jacks in a deck of 52 cards. Each time a jack is selected, there is one less jack in the deck. Also, there is one less card in the deck. These events are dependent.

**Step 2** Determine the probability of each event: $P(\text{a jack}) = \frac{4}{52}$, $P(\text{a jack second}) = \frac{3}{51}$, and $P(\text{a jack third}) = \frac{2}{50}$.

**Step 3** Use Multiplication Rule 2 for dependent events.

$$P(3\text{ jacks}) = \frac{4}{52} \cdot \frac{3}{51} \cdot \frac{2}{50} = \frac{24}{132{,}600} = \frac{1}{5525} \approx 0.0002$$

**b.** Getting an ace, a king, and a queen in order

**Step 1** There are four aces, four kings, and four queens in a deck of 52 cards. Each time one of these is selected, it does not affect the number left in the deck of the other values. However, there is one less card in the deck. These events are dependent.

**Step 2** Determine the probability of each event: $P(\text{ace}) = \frac{4}{52}$, $P(\text{king second}) = \frac{4}{51}$, and $P(\text{queen third}) = \frac{4}{50}$.

**Step 3** Use Multiplication Rule 2 for dependent events.

$$P(\text{ace and king and queen}) = \frac{4}{52} \cdot \frac{4}{51} \cdot \frac{4}{50} = \frac{64}{132{,}600} = \frac{8}{16{,}575} \approx 0.0005$$

**c.** Getting a club, a spade, and a heart in order

**Step 1** There are 13 clubs, 13 spades, and 13 hearts in a deck of 52 cards. Each time one of these is selected, it does not affect the number left in the deck of the other suits. However, there is one less card in the deck. These events are dependent.

**Step 2** Determine the probability of each event: $P(\text{club}) = \frac{13}{52}$, $P(\text{spade}) = \frac{13}{51}$, and $P(\text{heart}) = \frac{13}{50}$.

**Step 3** Use Multiplication Rule 2 for dependent events.

$$P(\text{club and spade and heart}) = \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} = \frac{2197}{132{,}600} = \frac{169}{10{,}200} \approx 0.017$$

**d.** Getting 3 clubs

**Step 1** Identify the events and determine if each occurring event affects the probability the others occur in sequence. There are thirteen clubs in a deck of 52 cards. Each time a club is selected, there is one less club in the deck. Also, there is one less card in the deck. These events are dependent.

**Step 2** Determine the probability of each event: $P(\text{a club}) = \frac{13}{52}$, $P(\text{a club second}) = \frac{12}{51}$, and $P(\text{a club third}) = \frac{11}{50}$.

**Step 3** Use Multiplication Rule 2 for dependent events.

$$P(\text{3 clubs}) = \frac{13}{52} \cdot \frac{12}{51} \cdot \frac{11}{50} = \frac{1716}{132{,}600} = \frac{11}{850} \approx 0.013$$

## Statistics Around Us: Use Statistics

It's time to select the grand prize winners for the state of Alaska's giveaway! Once a person is selected, they are not replaced. Assume that any person could be chosen from the data provided in Table 4A, regardless of age. Find the probability of these events.

**a.** P(4 people from Anchorage Municipality)

**b.** P(1 person from Anchorage Municipality and 1 person from Matanuska-Susitna Borough)

GO ONLINE Go online for the data.

## Learn: Tree Diagrams and Probability

Tree diagrams can be used as an aid to finding the solution to probability problems when the events are sequential. Example 8 illustrates the use of tree diagrams.

### Example 8: Use a Tree Diagram to Calculate the Probability of Sequential Events

Box 1 contains 2 red marbles and 1 blue marble. Box 2 contains 3 blue marbles and 1 red marble. A coin is tossed. If it falls heads up, Box 1 is selected and a marble is drawn. If it falls tails up, Box 2 is selected and a marble is drawn. Find the probability of selecting a red marble.

**Solution:**

**Step 1** Construct a tree diagram to show all possible outcomes. The first two branches designate the selection of either Box 1 or Box 2. Then from Box 1, either a red marble or a blue marble can be selected. Likewise, a red marble or blue marble can be selected from Box 2. See Figure 4–10.

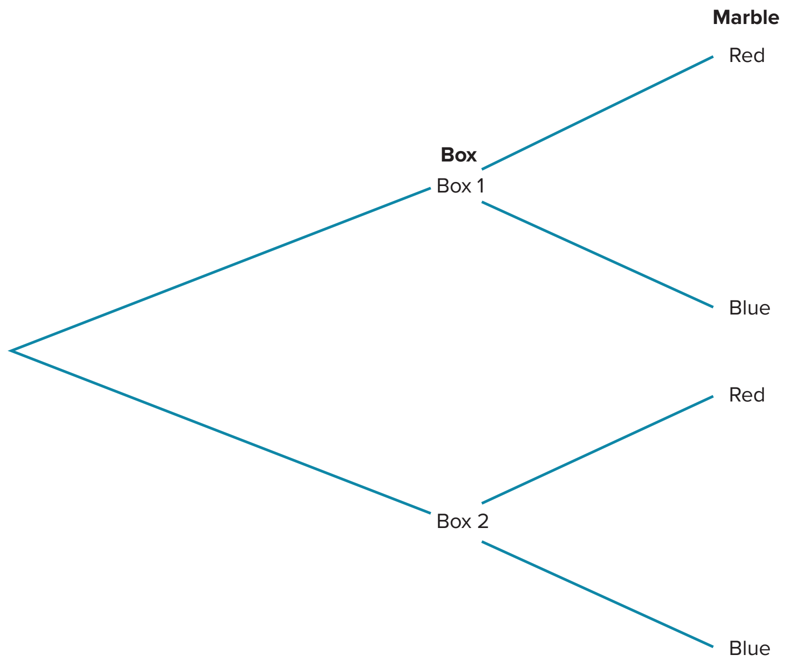



FIGURE 4–10 Tree Diagram of Outcomes for Selecting Marbles

**Step 2** Next, determine the probability for each branch and add it to the diagram. Since a coin is being tossed for the box selection, each branch has a probability of $\frac{1}{2}$—that is, heads for Box 1 or tails for Box 2. The probabilities for the second branches are found by using the basic probability rule. For example, if Box 1 is selected and there are 2 red marbles and 1 blue marble, the probability of selecting a red marble is $\frac{2}{3}$, and the probability of selecting a blue marble is $\frac{1}{3}$. If Box 2 is selected, and it contains 3 blue marbles and 1 red marble, the probability of selecting a red marble is $\frac{1}{4}$, and the probability of selecting a blue marble is $\frac{3}{4}$. See Figure 4–11.

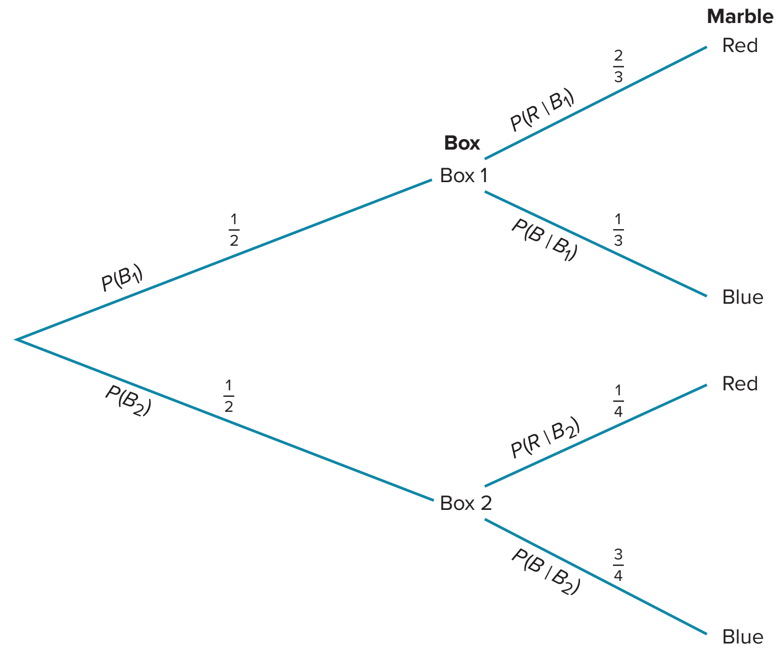



FIGURE 4–11 Tree Diagram with Probabilities for Selecting Marbles

**Step 3** To determine the probability for each outcome, follow each possible branch from left to right and multiply the associated probabilities using the rule $P(A \text{ and } B) = P(A) \cdot P(B|A)$. For example, the probability of selecting Box 1 and selecting a red marble is $\frac{1}{2} \cdot \frac{2}{3} = \frac{2}{6}$. The probability of selecting Box 1 and a blue marble is $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$. The probability of selecting Box 2 and selecting a red marble is $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$. The probability of selecting Box 2 and a blue marble is $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$. (Note that the sum of these probabilities is 1.) See Figure 4–12.

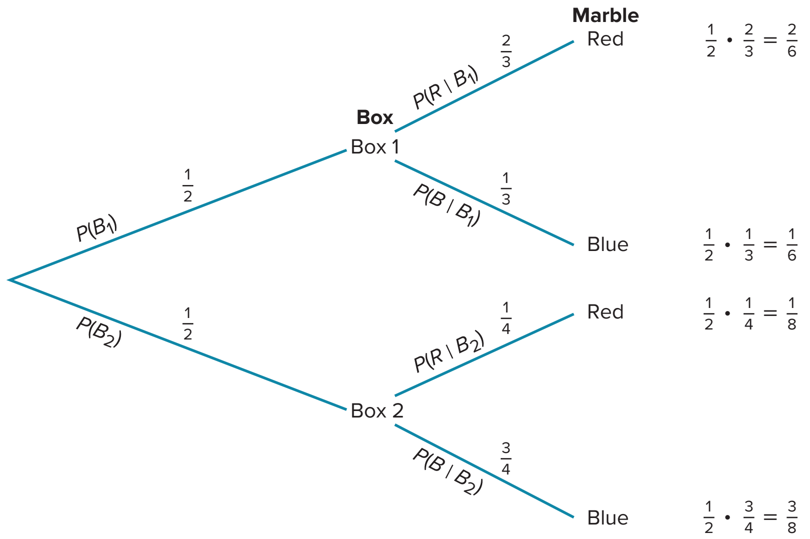



FIGURE 4–12 Calculating Probability Using a Tree Diagram

A red marble can be selected from either Box 1 or Box 2, so $P(\text{red}) = \frac{2}{6} + \frac{1}{8} = \frac{8}{24} + \frac{3}{24} = \frac{11}{24}$.

Tree diagrams can be used when the events are independent or dependent, and they can also be used for sequences of three or more events.

## Learn: Conditional Probability

The conditional probability of an event B in relationship to an event A was defined as the probability that event B occurs after event A has already occurred.

The conditional probability of an event can be found by dividing both sides of the equation for Multiplication Rule 2 by $P(A)$, as shown:

$$P(\text{A and B}) = P(\text{A}) \cdot P(\text{B}|\text{A})$$

$$\frac{P(A \text{ and } B)}{P(A)} = \frac{\cancel{P(A)} \cdot P(B|A)}{\cancel{P(A)}}$$

$$\frac{P(A \text{ and } B)}{P(A)} = P(B|A)$$

### Formula for Conditional Probability

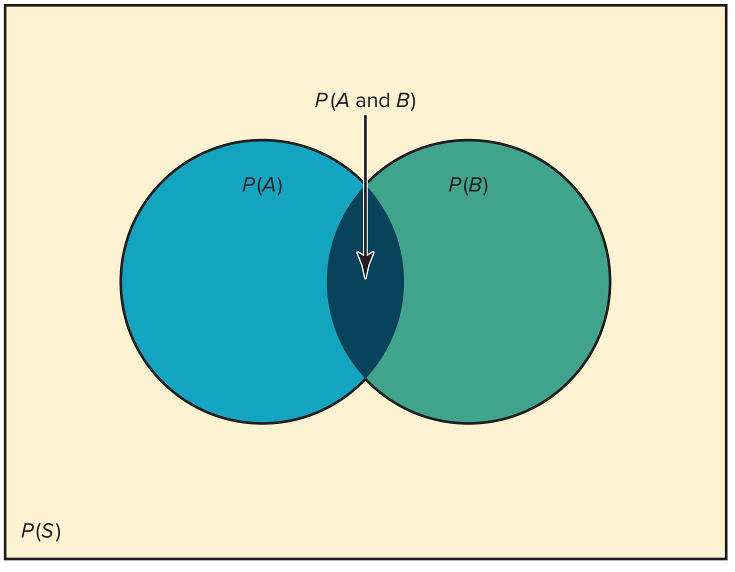
The probability that the second event *B* occurs given that the first event *A* has occurred can be found by dividing the probability that both events occurred by the probability that the first event has occurred.

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

The Venn diagram for conditional probability is shown in Figure 4–13. In this case:

$$P(B \mid A) = \frac{P(A \text{ and } B)}{P(A)}$$

which is represented by the area in the intersection or overlapping part of the circles A and *B*, divided by the area of circle A. The reasoning here is that if you assume A has occurred, then A becomes the sample space for the next calculation and is the denominator of the probability fraction *P*(A and B)/*P*(A). The numerator *P*(A and B) represents the probability of the part of *B* that is contained in A. So, *P*(A and B) becomes the numerator of the probability fraction *P*(A and B)/*P*(A). Imposing a condition reduces the sample space.




$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

FIGURE 4–13 Venn Diagram for Conditional Probability



Examples 9–10 illustrate the use of this rule.

## Example 9: Finding a Conditional Probability Given Probabilities

A box contains black chips and white chips. A person selects two chips without replacement. If the probability of selecting a black chip *and* a white chip is $\frac{15}{56}$, and the probability of selecting a black chip on the first draw is $\frac{3}{8}$, find the probability of selecting the white chip on the second draw, *given* that the first chip selected was a black chip.

### Solution:

For two events $A$ and $B$, the probability that the second event $B$ occurs given that the first event $A$ has occurred is $P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$.

**Step 1** Use the formula for conditional probability to express the probability of selecting the white chip on the second draw, given that the first chip selected was a black chip. Let $B$ = selecting a black chip and $W$ = selecting a white chip. Then $P(W|B) = \frac{P(B \text{ and } W)}{P(B)}$.

**Step 2** Identify the given probabilities. The probability of selecting a black chip and a white chip is $P(B \text{ and } W) = \frac{15}{56}$, and the probability of selecting a black chip on the first draw is $P(B) = \frac{3}{8}$.

**Step 3** Then:

$$P(W|B) = \frac{P(B \text{ and } W)}{P(B)} = \frac{15/56}{3/8}$$

$$= \frac{15}{56} \div \frac{3}{8} = \frac{15}{56} \cdot \frac{8}{3} = \frac{\overset{5}{\cancel{15}}}{\underset{7}{\cancel{56}}} \cdot \frac{\overset{1}{\cancel{8}}}{\underset{1}{\cancel{3}}} = \frac{5}{7} \approx 0.714$$

The probability of selecting a white chip on the second draw given that the first chip selected was black is $\frac{5}{7} \approx 0.714$.

## Example 10: Finding a Conditional Probability Given Probabilities

The probability that Sam parks in a no-parking zone *and* gets a parking ticket is 0.06, and the probability that Sam cannot find a legal parking space and has to park in the no-parking zone is 0.20. On Tuesday, Sam arrives at school and has to park in a no-parking zone. Find the probability that he will get a parking ticket.

### Solution:

**Step 1** Use the formula for conditional probability to express the probability of getting a parking ticket after having parked in a no-parking zone. Let $N$ = parking in a no-parking zone and $T$ = getting a ticket. Then $P(T|N) = \frac{P(N \text{ and } T)}{P(N)}$.

**Step 2** Identify the given probabilities. The probability of parking in a no-parking zone and getting a parking ticket is $P(N \text{ and } T) = 0.06$, and the probability of parking in the no-parking zone is $P(N) = 0.20$.

**Step 3** Then:

$$P(T|N) = \frac{P(N \text{ and } T)}{P(N)} = \frac{0.06}{0.20} = 0.30$$

Sam has a 0.30 probability or 30% chance of getting a parking ticket, given that he parked in a no-parking zone.

The conditional probability of events occurring can also be computed when the data are given in table form, as shown in Example 11.

## **Example 11:** Finding a Conditional Probability Given Frequencies

A survey asked 50 men and 50 women if they liked a recently released movie. The results of the survey are shown in Table 4–10.

TABLE 4–10 Responses on Movie Survey

| Gender | Yes | No | Total |
|---|---|---|---|
| Male | 32 | 18 | 50 |
| Female | 8 | 42 | 50 |
| Total: | 40 | 60 | 100 |

Use the formula for conditional probability to find the following probabilities.

**a.** The respondent was a female.

**b.** The respondent answered yes and was a female.

**c.** The respondent answered yes, given that the respondent was a female.

**d.** The respondent was a male, given that the respondent answered no.

### Solution:

Let:

M = respondent was a male Y = respondent answered yes

F = respondent was a female N = respondent answered no

**a.** The respondent was a female.

$$P(\text{F}) = \frac{\text{number of females}}{\text{number of respondents}} = \frac{50}{100} = 0.5$$

**b.** The respondent answered yes and was a female.

$$P(\text{F and Y}) = \frac{\text{number of females who responded yes}}{\text{number of respondents}} = \frac{8}{100} = 0.08$$

**c.** The respondent answered yes, given that the respondent was a female.

**Step 1** Use the formula for conditional probability to express the probability that the respondent answered yes, given that the respondent was a female.

Then $P(\text{Y}\,|\,\text{F}) = \dfrac{P(\text{F and Y})}{P(\text{F})}$.

**Step 2** Find the necessary probabilities.

The probability $P$(F and Y) was calculated in step *b*. The probability $P$(F) was calculated in step *a*.

$$P(\text{F and Y}) = \frac{8}{100} = 0.08 \qquad P(\text{F}) = \frac{50}{100} = 0.5$$

**Step 3** Find $P(\mathrm{Y} \mid \mathrm{F})$. Then:

$$P(Y \mid F) = \frac{P(F \text{ and } Y)}{P(F)} = \frac{8/100}{50/100}$$

$$= \frac{8}{100} \div \frac{50}{100} = \frac{\overset{4}{\cancel{8}}}{\underset{1}{\cancel{100}}} \cdot \frac{\overset{1}{\cancel{100}}}{\underset{25}{\cancel{50}}} = \frac{4}{25} = 0.16$$

or

$$P(\mathrm{Y} \mid \mathrm{F}) = \frac{P(\mathrm{F} \text{ and } \mathrm{Y})}{P(\mathrm{F})} = \frac{0.08}{0.5} = 0.16$$

**Alternative Solution:** Note that Table 4–11 provides a lot of information about the survey responses. If we had not already calculated $P(\mathrm{F} \text{ and } \mathrm{Y})$ and $P(\mathrm{F})$, it would have been more efficient to find $P(\mathrm{Y} \mid \mathrm{F})$ directly in the following way.

Since we are given the respondent is female, restrict to the row of values for females when using Table 4–11.

TABLE 4–11 Responses on Movie Survey

| Gender | Yes | No | Total |
|---|---|---|---|
| Male | 32 | 18 | 50 |
| Female | 8 | 42 | 50 |
| Total: | 40 | 60 | 100 |

$$P(\mathrm{Y} \mid \mathrm{F}) = \frac{\text{number of females who answered yes}}{\text{number of females}} = \frac{8}{50} = 0.16$$

**d.** The respondent was a male, given that the respondent answered no.

**Step 1** Determine the best approach for finding $P(M \mid N)$.

If we use the formula for conditional probability, $P(M \mid N) = \frac{P(N \text{ and } M)}{P(N)}$, we will need to first find $P(\mathrm{N} \text{ and } \mathrm{M})$ and $P(N)$. We can do this with the information given, but it will require more steps than if we can calculate it directly.

If we want to calculate $P(M \mid N)$ directly, we need to know the total number of individuals answering no and how many were men. Since Table 4–10 provides both pieces of information, it is more practical to compute $P(M \mid N)$ directly.

**Step 2** Find the probability.

Since we are given the respondent answered no, we restrict to the column of values for no responses in Table 4–12.

TABLE 4–12 Responses on Movie Survey

| Gender | Yes | No | Total |
|---|---|---|---|
| Male | 32 | 18 | 50 |
| Female | 8 | 42 | 50 |
| Total: | 40 | 60 | 100 |

$$P(M \mid N) = \frac{\text{number of men who answered no}}{\text{number of no's}} = \frac{18}{60} = 0.3$$

**Alternative Solution:** To use the formula for conditional probability, use the table to find $P(N \text{ and } M)$ and $P(N)$. $P(N \text{ and } M)$ is the number of people who responded no and who were male, divided by the total number of respondents: $P(N \text{ and } M) = \frac{18}{100} = 0.18$. $P(N)$ is the number of people who answered no, divided by the total number of respondents: $P(N) = \frac{60}{100} = 0.6$. Use the formula for conditional probability $P(M \mid N) = \frac{P(N \text{ and } M)}{P(N)} = \frac{0.18}{0.6} = 0.3$. As determined in Step 2, direct computation was more efficient.

## Statistics Around Us: Use Statistics

Hawaii is selecting people for a state-wide jury duty. The foreman will select only jurors that are aged 18 and over. The probability of selecting a juror that is aged 18 or over and from Maui County is 0.089. From the data in Table 4A, we can compute the probability that a randomly selected resident lives in Maui county is 0.113. Use the formula for conditional probability to find the probability that a Hawaii resident selected at random is 18 and over, given that the person resides in Maui County. Can you confirm your answer is correct by using additional data from the Census.gov site to calculate this conditional probability directly?

GO ONLINE Go online for additional data.

# Learn: Probabilities for "At Least"

The multiplication rules can be used with the complementary event rule (Lesson 4–1) to simplify solving probability problems involving "at least." Examples 12–14 illustrate how this is done.

## Example 12: Find the Probability of At Least One Card of a Given Suit

A person selects 3 cards from an ordinary deck and replaces each card after it is drawn. Find the probability that the person will get at least one heart.



### Solution:

It is much easier to find the probability that the person will not select a heart in three draws and subtract this value from 1. To do the problem directly, you would have to find the probability of selecting 1 heart, 2 hearts, and 3 hearts and then add the results.

Let:

$E$ = at least 1 heart is drawn and $\overline{E}$ = no hearts are drawn

$$P(\overline{E}) = \frac{39}{52} \cdot \frac{39}{52} \cdot \frac{39}{52} = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{3}{4} = \frac{27}{64}$$

$$P(E) = 1 - P(\overline{E})$$

$$= 1 - \frac{27}{64} = \frac{37}{64} \approx 0.578 = 57.8\%$$

So, a person will select at least one heart about 57.8% of the time.

## Example 13: Find the Probability of At Least One Specified Side of a Die

A single die is rolled 5 times; find the probability of getting at least one 3.

### Solution:

It is easier to find the probability of the complement of the event occurring, which is rolling no threes. Then subtract this probability from 1 to get the probability of rolling at least one 3.

**Step 1** Express the relationship between the probability of rolling no threes and the probability of rolling at least one three.

$$P(\text{at least one 3}) = 1 - P(\text{no 3's})$$

**Step 2** Calculate the probability of rolling no threes.

The probability of rolling no 3 with each roll is $\frac{5}{6}$. Because the results of each roll do not affect the results of subsequent rolls, the probability is the same for each roll. So, $P(\text{no threes}) = \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right) = \left(\frac{5}{6}\right)^5$.

**Step 3** Solve for $P$(at least one 3).

$$P(\text{at least one 3}) = 1 - P(\text{no 3}s)$$

$$= 1 - \left(\frac{5}{6}\right)^5 = 1 - \frac{3125}{7776} = \frac{4651}{7776} \approx 0.598$$

There is a 59.8% of getting at least one 3 when a die is rolled 5 times.

## Example 14: Find the Probability of At Least One, Given Percents

The Neckware Association of America reported that 3% of ties sold in the United States are bow ties. If 4 customers who purchased a tie are randomly selected, find the probability that at least 1 purchased a bow tie.

**Solution:**

**Step 1** Let $E$ = at least 1 bow tie is purchased and $\bar{E}$ = no bow ties are purchased. The relationship between the probability that at least one bow tie is purchased and the probability that no bow ties are purchased can be expressed as $P(E) = 1 - P(\bar{E})$.

**Step 2** Since $P(E) = 0.03$, $P(\bar{E}) = 1 - 0.03 = 0.97$. Because each customer's purchase does not affect the next purchase, the probability is the same for each customer. So, for four customers:

$P(\bar{E})$ = $P$(customer 1 does not buy a bow tie) · $P$(customer 2 does not buy a bow tie) · $P$(customer 3 does not buy a bow tie) · $P$(customer 4 does not buy a bow tie)

$= (0.97)(0.97)(0.97)(0.97) \approx 0.885$

**Step 3** Solve for $P(E)$.

$$P(\text{at least one bow tie is purchased}) = 1 - 0.885 = 0.115$$

There is an 11.5% chance of a person purchasing at least one bow tie.

## Applying the Concepts 4–3
### Life on Other Planets?

Intelligent life as we know it has never been detected outside of Earth. Because the locations involved are so distant, direct observations of other life in our galaxy are not currently possible. Scientists have attempted to assign probabilities to the occurrence of certain features that they suspect would be required for life in our galaxy. For example, scientists have tried to determine which percentage of stars are like our Sun, which of these stars have planets, and so on. Many of the criteria to include in such calculations and their values are still being debated, but the list in Table 4–13 is an example of what such a set of probabilities might look like.

TABLE 4–13 Assumed Probabilities for Features Required for Life

| Characteristic | Assumed probability |
|---|---|
| A star is like our sun | 1 in 5 |
| A sun-like star has planets | 2 in 5 |
| A planet has the conditions for life | 2 in 9 |
| Life has evolved on such a planet | 1 in 9 |
| Life has sufficient intelligence to communicate | 1 in 100 |
| People on Earth can detect this communication | 1 in 10 |

1. Compute the probability that a randomly chosen star in our galaxy has a planet with life whose communications we can detect.
2. Would you use the addition or multiplication rule? Why?
3. Are the characteristics independent or dependent?
4. Are the computations affected by the assumption of independence or dependence?
5. Intelligent life appeared on Earth only relatively recently in its long history. Assuming this happens elsewhere as well, how might this affect calculations like these?
6. A survey of U.S. adults asked whether they believed that intelligent life exists on other planets. By age group, 76% of respondents aged 18–29, 69% of respondents ages 30–49, and 57% of respondents aged 50+ believed that intelligent life probably exists on other planets. Assuming 100 people were surveyed in each age group, construct a two-way table for this information.
   *Source:* Pew Research
7. Based on your two-way table, what is the probability that a randomly selected person is aged 18–29 given that they do not believe intelligent life exists on other planets? What is the probability that a randomly selected person is aged 50+ given that they believe intelligent life exists on other planets?
8. How might you reconcile the beliefs of U.S. adults and the probabilities scientists developed about the existence of life on other planets?

# Practice 4–3

## Practice Exercises

**In #1, determine whether events are independent or dependent.**

1. State which events are independent and which are dependent.

   **a.** Tossing a coin and drawing a card from a deck

   **b.** Drawing a ball from an urn, not replacing it, and then drawing a second ball

   **c.** Getting a raise in salary and purchasing a new car

   **d.** Driving on ice and having an accident

**In #2–20, compute probabilities to complete the exercises and answer questions.**

2. Sixteen percent of wildfires are caused by lightning. Select 4 wildfires at random and find the probability that:

   **a.** None is caused by lightning.

   **b.** All four are caused by lightning.

   *Source:* National Academy of Sciences

3. Sixty-three percent of professionals prefer a promotion without a raise rather than a raise with no promotion. If 3 professionals are selected at random, find the probability that all will select a promotion with no raise. Is this a likely or unlikely event?

   *Source:* Korn Ferry Survey

4. Sixty percent of adults say that a smile from a stranger improves their day. If 6 people are randomly selected, find the probability that none will improve their day when smiled at by a stranger.

   *Source:* Dignity Health/Qualtrics Survey

5. A flashlight has 6 batteries, 2 of which are defective. If 2 are selected at random without replacement, find the probability that both are defective.

6. Four cards are drawn from an ordinary deck of cards without replacement. Find these probabilities.

   **a.** All cards are kings.

   **b.** All cards are red cards.

   **c.** All cards are spades.

7. In a box of 12 batteries, 2 are dead. If 2 batteries are selected at random for a flashlight, find the probability that both are dead. Would you consider this event likely or unlikely?

8. Urn 1 contains 4 red marbles and 2 black marbles. Urn 2 contains 1 red marble and 3 black marbles. Urn 3 contains 3 red marbles and 3 black marbles. If an urn is selected at random and a marble is drawn, find the probability that it will be black.

9. A production process produces an item. On average, 15% of all items produced are defective. Each item is inspected before being shipped, and the inspector misclassifies an item 10% of the time. What proportion of the items will be "classified as good"? What is the probability that an item is defective given that it was classified as good?

10. Roll two standard dice and add the numbers. What is the probability of getting a number larger than 9 for the first time on the third roll?

11. At the Avonlea Country Club, 73% of the members play bridge and swim, and 82% play bridge. If a member is selected at random, find the probability that the member swims, given that the member plays bridge.

12. In a pizza restaurant, 95% of the customers order pizza. If 65% of the customers order pizza and a salad, find the probability that a customer who orders pizza will also order a salad.

13. Below are listed the numbers of doctors in various specialties by gender.

| | Pathology | Pediatrics | Psychiatry |
|---|---|---|---|
| Male | 12,575 | 33,020 | 27,803 |
| Female | 5,604 | 33,351 | 12,292 |

Choose one doctor at random.

a. Find $P(\text{male} \mid \text{pediatrician})$.

b. Find $P(\text{pathologist} \mid \text{female})$.

c. Are the characteristics "female" and "pathologist" independent? Explain.

*Source: World Almanac*

14. Only 27% of U.S. adults get enough leisure time exercise to achieve cardiovascular fitness. Choose 3 adults at random. Find the probability that

a. All 3 get enough daily exercise.

b. At least 1 of the 3 gets enough exercise.

*Source:* www.infoplease.com

15. In a department store there are 120 customers, 90 of whom will buy at least 1 item. If 5 customers are selected at random, one by one, find the probability that all will buy at least 1 item.

16. At the dentist's office, 93% of the patients arrive on time. If 5 patients are selected at random, find the probability that at least one patient was not on time.

17. If 5 cards are drawn at random from a deck of 52 cards and are not replaced, find the probability of getting at least one diamond.

18. If 3 letters of the alphabet are selected at random, find the probability of getting at least 1 letter *x*. Letters can be used more than once. Would you consider this event likely to happen? Explain your answer.

19. A die is rolled twice. Find the probability of getting at least one 6.

20. In a large vase, there are 8 roses, 5 daisies, 12 lilies, and 9 orchids. If 4 flowers are selected at random, and not replaced, find the probability that at least 1 of the flowers is a rose. Would you consider this event likely to occur? Explain your answer.

**In #21–23, answer questions about independence of events and conditional probability.**

21. Let *A* and *B* be two mutually exclusive events. Are *A* and *B* independent events? Explain your answer.

22. An admissions director knows that the probability a student will enroll after a campus visit is 0.55, or $P(E) = 0.55$. While students are on campus visits, interviews with professors are arranged. The admissions director computes these conditional probabilities for students enrolling after visiting three professors, DW, LP, and MH.

$P(E \mid DW) = 0.95$ $\quad$ $P(E \mid LP) = 0.55$
$P(E \mid MH) = 0.15$

What do you observe about the results?

23. A sample space has events *A* and *B*, such that $P(A) = 0.342$, $P(B) = 0.279$, and $P(A \text{ or } B) = 0.601$. Are *A* and *B* mutually exclusive? Are *A* and *B* independent? Find $P(A \mid B)$, $P(\text{not } B)$, and $P(A \text{ and } B)$.

GO ONLINE Go online for more practice problems.

Lesson 4–4

# Counting Rules

Many times a person must know the number of all possible outcomes for a sequence of events. To determine this number, three rules can be used: the *fundamental counting rule,* the *permutation rule,* and the *combination rule.* These rules are explained here, and they will be used in Lesson 4–5 to find probabilities of events.

## Learn: The Fundamental Counting Rule

**Fundamental Counting Rule**

In a sequence of $n$ events in which the first one has $k_1$ possibilities and the second event has $k_2$ and the third has $k_3$, and so forth, the total number of possibilities of the sequence will be:

$$k_1 \cdot k_2 \cdot k_3 \cdots k_n$$

*Note:* In this case, *and* means to multiply.

Examples 1–2 illustrate the fundamental counting rule.

### Example 1: Find the Number of Outcomes—The Fundamental Counting Rule or a Tree

A coin is tossed, and a die is rolled. Find the number of outcomes for the sequence of events.

**Solution:**

By the fundamental counting rule, in a sequence of two events in which the first one has $k_1$ possibilities and the second event has $k_2$, the total number of possibilities of the sequence will be $k_1 \cdot k_2$. Since the coin can land either heads up or tails up, and since the die can land with any one of six numbers showing face up, there are $2 \cdot 6 = 12$ possibilities.

A tree diagram can also be drawn for the sequence of events.

**Step 1** Represent the number of possibilities for the first event. A coin can land either heads up or tails up. See Figure 4–14.

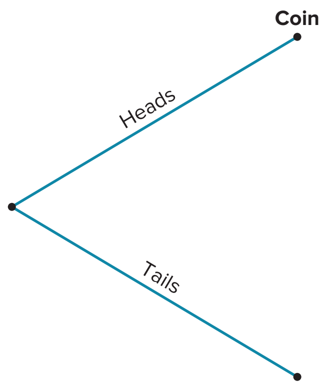



FIGURE 4–14 Tree Diagram for Flipping Coin

**Step 2** Represent the number of possibilities for the second event. The die can land with any one of six numbers showing face up. See Figure 4–15.

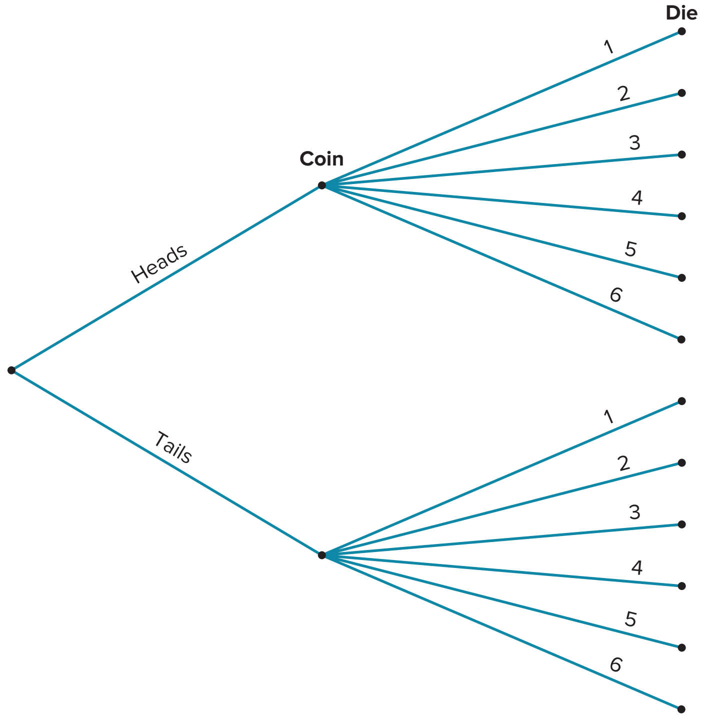



FIGURE 4–15 Tree Diagram for Flipping Coin and then Rolling Die

**Step 3** Follow each possible branch from left to right to determine the outcomes, as shown in Figure 4–16. Counting the list of outcomes, we again obtain the result of 12 possible outcomes.

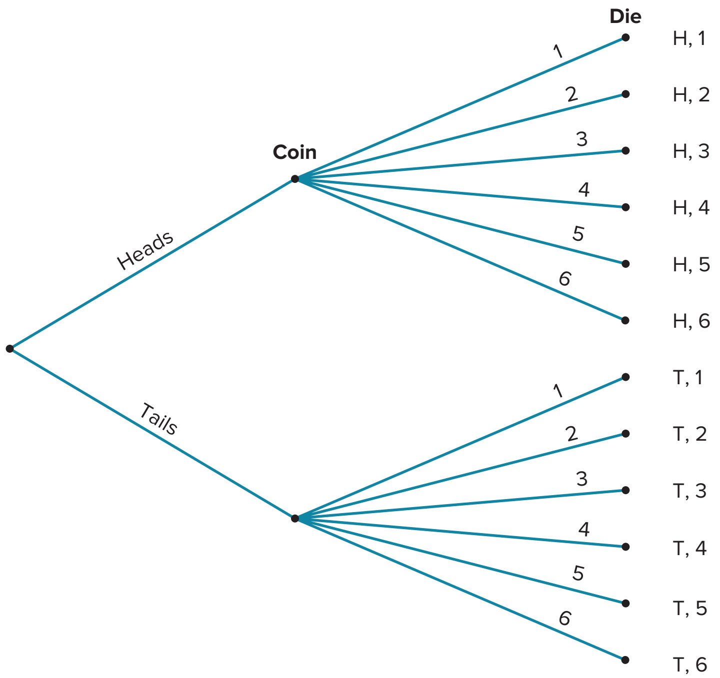



FIGURE 4–16 Tree Diagram and List of Outcomes for Flipping Coin, Rolling Die

## Statistics Around Us: Use Statistics

Hawaii wants to classify their residents based on the county the live in, their voting affiliation, and their gender. In addition to the five counties in Hawaii, the voting affiliations are classified as Constitution, Democratic, Green, Libertarian, Natural Law, Republican, or Socialist, while gender can be listed as male, female, or undisclosed. How many different ways can a resident be labeled?

## Example 2: Find the Number of Outcomes—The Fundamental Counting Rule or a Tree

There are four blood types, A, B, AB, and O. Each blood type can have one of two factors: Rh+ and Rh–. Finally, a blood donor is classified as either male or female. How many different ways can a donor's blood be labeled?

### Solution:

Since there are 4 possibilities for blood type, 2 possibilities for Rh factor, and 2 possibilities for the gender of the donor, there are $4 \cdot 2 \cdot 2$, or 16, different classification categories by the fundamental counting rule.

A tree diagram can also be drawn for the sequence of events.

**Step 1** Represent the number of possibilities for the first event. There are four blood types: A, B, AB, and O. See Figure 4–17.

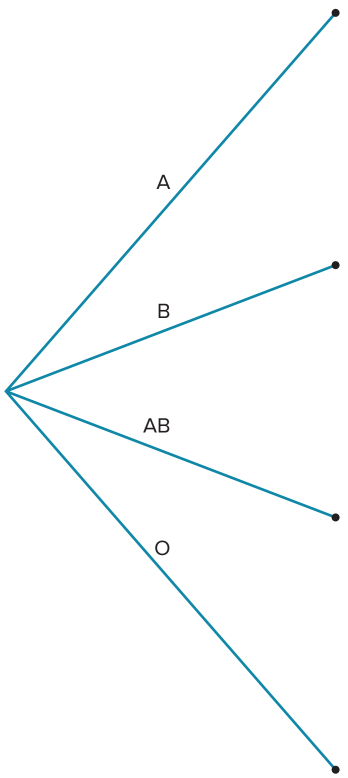



FIGURE 4–17 Tree Diagram of Blood Types

**Step 2** Represent the number of possibilities for the second event. Each blood type can be Rh+ and Rh–. See Figure 4–18.

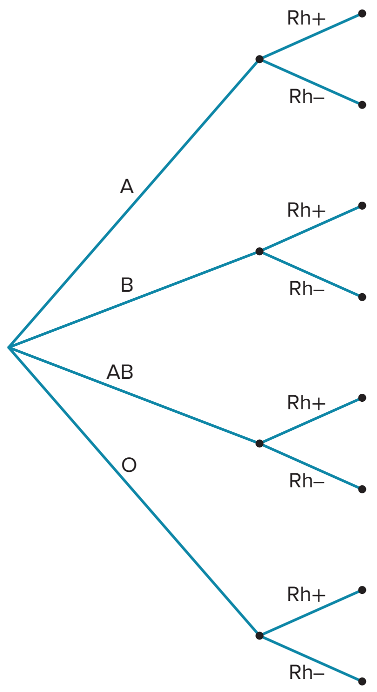



FIGURE 4–18 Tree Diagram of Blood Types and then Rh Factors

**Step 3** Represent the number of possibilities for the third event. Each donor is classified as male or female. See Figure 4–19.

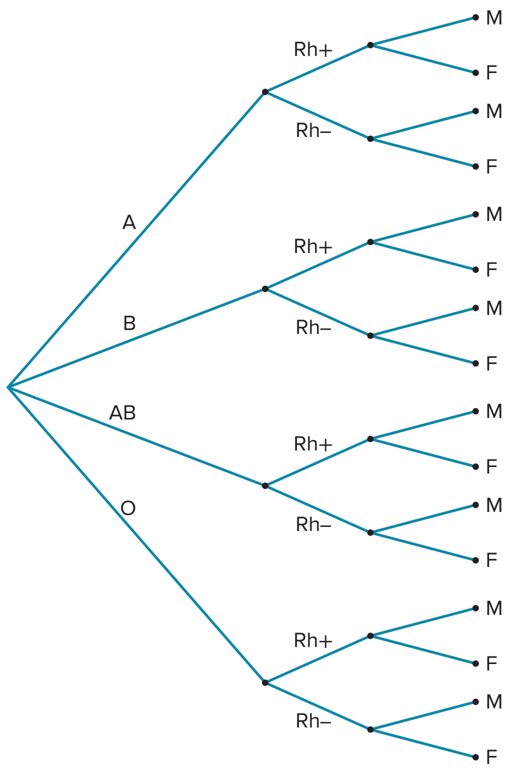



FIGURE 4–19 Tree Diagram of Blood Types, Rh Factors, and then Gender

**Step 4** Follow each possible branch from left to right to determine the possible outcomes. There are 16 outcomes, as shown in Figure 4–20.

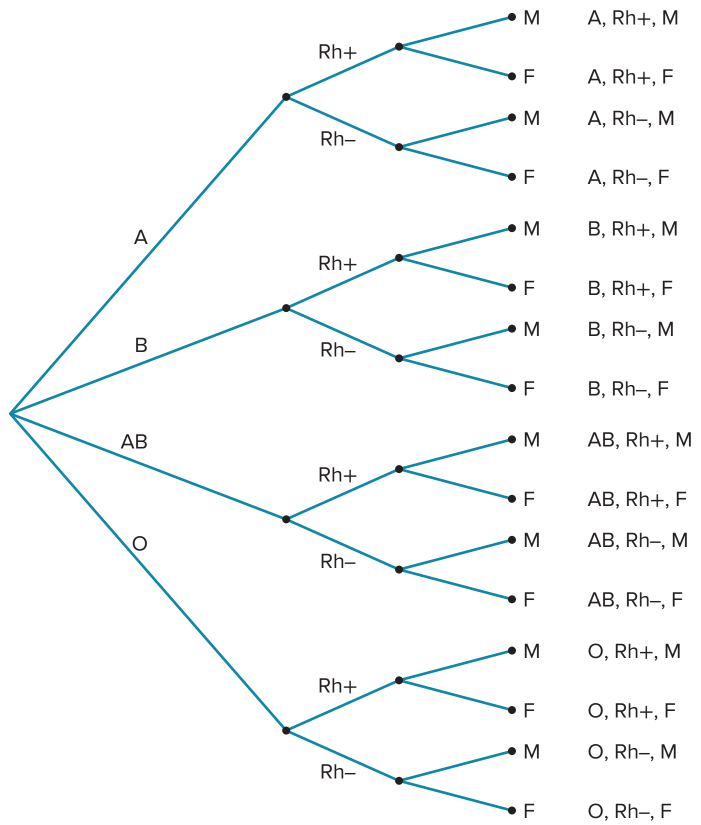



FIGURE 4–20 Tree Diagram and List of Outcomes for Blood Types, Rh Factors, and then Gender

## Example 3: Find the Number of Outcomes—The Fundamental Counting Rule

Students wishing to plan their education after high school can select from these categories:

Type of school: University, four-year college, two-year college

Major field: Business, education, psychology, health care, computer science

Living accommodations: On-campus, off-campus

Work: Full-time, part-time, not working

How many different situations can students select for their educational experience?



### Solution:

The person can choose one selection from each category. Using the fundamental counting rule, the person has $3 \cdot 5 \cdot 2 \cdot 3 = 90$ ways to select their higher education experience.

A tree diagram could be constructed to represent these four events in sequence. However, such a diagram would have 90 branches and therefore would not be an efficient method to use for this scenario.

When determining the number of different possibilities of a sequence of events, you must know whether repetitions are permissible.

### Example 4: Find the Number of Possible Outcomes—The Fundamental Counting Rule

The first year the state of Pennsylvania issued railroad memorial license plates, the plates had a picture of a steam engine followed by four digits. Assuming that repetitions are allowed, how many railroad memorial plates could be issued?

#### Solution:

Since there are four spaces to fill for each space, the total number of plates that can be issued is $10 \cdot 10 \cdot 10 \cdot 10 = 10{,}000$. *Note:* Actually there was such a demand for the plates, Pennsylvania had to use letters also.

Now if repetitions are not permitted, the first digit in the plate in Example 4 could be selected in 10 ways, the second digit in 9 ways, etc. So, the total number of plates that could be issued is $10 \cdot 9 \cdot 8 \cdot 7 = 5040$.

The same situation occurs when one is drawing marbles from an urn or cards from a deck. If the marble or card is replaced before the next one is selected, repetitions are permitted since the same one can be selected again. But if the selected marble or card is not replaced, repetitions are not permitted since the same ball or card cannot be selected the second time.

These examples illustrate the fundamental counting rule. In summary: *If repetitions are permitted, the numbers stay the same going from left to right. If repetitions are not permitted, the numbers decrease by 1 for each place left to right.*

## Learn: Permutations

Two other rules that can be used to determine the total number of possibilities of a sequence of events are the permutation rule and the combination rule.

These rules use *factorial notation.* The factorial notation uses the exclamation point.

$$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

To use the formulas in the permutation and combination rules, a special definition of 0! is needed: $0! = 1$.

#### Factorial Formulas

For an integer $n \geq 1$:

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$

We further define the value of 0! as $0! = 1$.



#### Historical Note

In 1808, Christian Kramp first used factorial notation.

A **permutation** is an arrangement of *n* objects in a specific order. Examples 5 and 6 illustrate permutations.

### Example 5: Find the Number of Possible Outcomes When Order Matters

A family has 4 children. How many different ways can they line up in a row for a photograph?

#### Solution:

There are 4 choices for the first person, 3 choices for the second person, 2 choices for the third person, and 1 way to select for the fourth person. So, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. There are 24 different ways for four children to line up for a photograph.

In Example 5, all objects were used up. But what happens when not all objects are used up? The answer to this question is given in Example 6.

### Example 6: Find the Number of Possible Outcomes When Order Matters

A business owner wishes to rank the top 3 locations selected from 5 locations for a business. How many different ways can she rank them?

#### Solution:

Using the fundamental counting rule, she can select any one of the 5 for first choice, then any one of the remaining 4 locations for her second choice, and finally, any one of the remaining locations for her third choice, as shown.

$$5 \cdot 4 \cdot 3 = 60$$

The solutions in Examples 5 and 6 are permutations.

## Learn: Permutation Rule for Distinct Objects

**Permutation Rule 1**

The arrangement of *n* objects in a specific order using *r* objects at a time is called a *permutation of* n *objects taking* r *objects at a time*. It is written as ${}_nP_r$, and the formula is:

$${}_nP_r = \frac{n!}{(n - r)!}$$



Although Examples 5 and 6 were solved by the multiplication rule, they can now be solved by the permutation rule.

In Example 5, four family members were arranged in order. So, both $n = 4$ and $r = 4$, and the number of arrangements is:

$$_4P_4 = \frac{4!}{(4 - 4!)} = \frac{4!}{0!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{1} = 24$$

(Recall that 0! = 1.)

In Example 6, three locations were selected from 5 locations, so $n = 5$ and $r = 3$. The number of arrangements is:

$$_5P_3 = \frac{5!}{(5 - 3)!} = \frac{5!}{2!} = \frac{5 \cdot 4 \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{\cancel{2} \cdot \cancel{1}} = 60$$

Examples 7 and 8 illustrate the permutation rule.

## **Example 7:** Find the Number of Possible Outcomes When Order Matters

A radio talk show host can select 3 of 6 special guests for the program. The order of appearance of the guests is important. How many different ways can the guests be selected?

### Solution:

Since the order of appearance on the show is important, there are $_6P_3$ ways to select the guests.

$$_6P_3 = \frac{6!}{(6 - 3)!} = \frac{6!}{3!} = \frac{6 \cdot 5 \cdot 4 \cdot 3!}{3!} = 120$$

There can be 120 different ways to select 3 guests and present them on the program in a specific order.

## **Example 8:** Find the Number of Possible Outcomes When Order Matters

A school musical director can select 2 musical plays to present next year. One play will be presented in the fall, and one play will be presented in the spring. If the music director has 9 plays to pick from, how many different possibilities are there?

### Solution:

Order is important since one play can be presented in the fall and the other play in the spring.

$$_9P_2 = \frac{9!}{(9 - 2)!} = \frac{9!}{7!} = \frac{9 \cdot 8 \cdot 7!}{7!} = 72$$

There are 72 different possibilities.



# **Learn:** Permutation Rule for Collections with Identical Objects

In the previous examples, all items involving permutations were different, but when some of the items are identical, a second permutation rule can be used.

### Permutation Rule 2

The number of permutations of $n$ objects when $r_1$ objects are identical, $r_2$ objects are identical, . . . , $r_p$ objects are identical is:

$$\frac{n!}{r_1!r_2!\cdots r_p!}$$

where $r_1 + r_2 + \cdots + r_p = n$.

## Example 9: Find the Number of Rearrangements—Some Identical Objects

How many permutations of the letters can be made from the word *STATISTICS*?

**Solution:**

In the word *STATISTICS*, there are 3 S's, 3 T's, 2 I's, 1 A, and 1 C.

$$\frac{10!}{3!3!2!1!1!} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{3\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1\cdot 1\cdot 1} = 50{,}400$$

There are 50,400 permutations that can be made from the word *STATISTICS*.

## Learn: Combinations

FIGURE 4–21 Selections of Colorful Bolts of Fabric
The McGraw Hill Companies, Inc./Evelyn Jo Hebert, photographer

Suppose a dress designer wishes to select two colors of material to design a new dress. The designer has on hand four colors to choose from. How many different possibilities can there be in this situation? This type of problem differs from previous ones in that the order of selection is not important. That is, if the designer selects yellow and red, this selection is the same as the selection red and yellow. This type of selection is called a *combination.*



A selection of distinct objects without regard to order is called a **combination.**

The difference between a permutation and a combination is that in a combination, the order or arrangement of the objects is not important; by contrast, order *is* important in a permutation. Example 9, in which we rearranged letters in the word *STATISTICS*, illustrates this difference. In Example 10, we will compare the combinations and permutations for the same set of objects.

## **Example 10:** Find the Number of Permutations and Combinations

Find the different way to take 2-letter groupings from the letters A, B, C, and D.

**a.** List all of the 2-letter permutations of the letters A, B, C, and D.

**b.** List all of the 2-letter combinations of the letters A, B, C, and D.

### Solution:

**a.** The permutations for the letters A, B, C, and D are:

| | | | |
|---|---|---|---|
| AB | BA | CA | DA |
| AC | BC | CB | DB |
| AD | BD | CD | DC |

**b.** In permutations, AB is different from BA. But in combinations, AB is the same as BA since the order of the objects does not matter in combinations. Therefore, if duplicates are removed from a list of permutations, what is left is a list of combinations, as shown.

| | | | |
|---|---|---|---|
| AB | ~~BA~~ | ~~CA~~ | ~~DA~~ |
| AC | BC | ~~CB~~ | ~~DB~~ |
| AD | BD | CD | ~~DC~~ |

So, the combinations of A, B, C, and D are AB, AC, AD, BC, BD, and CD. (Alternatively, BA could be listed and AB crossed out, etc.) The combinations have been listed alphabetically for convenience, but this is not a requirement.

*Combinations are used when the order or arrangement is not important, as in the selecting process.* Suppose a committee of 5 students is to be selected from 25 students. The 5 selected students represent a combination since it does not matter who is selected first, second, etc.

## **Learn:** Combination Rule

### Combination Rule

The number of combinations of *r* objects selected from *n* objects is denoted by ${}_nC_r$ and is given by the formula:

$${}_nC_r = \frac{n!}{(n-r)!r!}$$

*Note:* ${}_nC_n = 1$.

## **Example 11:** Find the Number of Combinations—Combination Rule

How many combinations of 4 distinct objects are there, taken 2 at a time?

### Solution:

Since this is a combination problem, the answer is:

$$_4C_2 = \frac{4!}{(4-2)!2!} = \frac{4!}{2!2!} = \frac{\overset{2}{\cancel{4}} \cdot 3 \cdot \cancel{2!}}{\cancel{2} \cdot 1 \cdot \cancel{2!}} = 6$$

This agrees with our result in part *b* of Example 10, where we found there were 6 2-letter combinations of the letters A, B, C, and D.

## Statistics Around Us: Use Statistics

How many combinations of counties in Alaska are there, taken 20 at a time?

**Go online for the data.**

Notice that the expression for $_nC_r$ is:

$$\frac{n!}{(n-r)!r!}$$

which is the formula for permutations with $r!$ in the denominator. In other words:

$$_nC_r = \frac{_nP_r}{r!}$$

This $r!$ divides out the duplicates from the number of permutations. For each two letters, there are two permutations but only one combination. So, dividing the number of permutations by $r!$ eliminates the duplicates. This result can be verified for other values of *n* and *r*.

## **Example 12:** Find the Number of Combinations—Combination Rule

The director of Movies at the Park must select 4 movies from a total of 10 movies to show on Movie Night at the Park. How many different ways can the selections be made, if the order in which the movies are shown is not important?

### Solution:

$$_{10}C_4 = \frac{10!}{(10-4)!4!} = \frac{10!}{6!4!} = \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1} = 210$$

The director has 210 different ways to select four movies from 10 movies.



## **Example 13:** Find the Number of Combinations—Committee Selection

In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

## Solution:

Solving this problem requires calculating two sets of combinations.

**Step 1** The committee will include 3 women selected from 7 women, which can be done in ${}_7C_3$, or 35, ways.

**Step 2** The committee will include 2 men selected from 5 men, which can be done in ${}_5C_2$, or 10, ways.

**Step 3** Finally, by the Fundamental Counting Rule, the total number of different ways is $35 \cdot 10 = 350$, since you are choosing both men and women. Using the formula gives:

$$ {}_7C_3 \cdot {}_5C_2 = \frac{7!}{(7-3)!3!} \cdot \frac{5!}{(5-2)!2!} = 350 $$

### Summary of Counting Rules

**Fundamental counting rule:** In a sequence of $n$ events in which the first one has $k_1$ possibilities, the second event has $k_2$ possibilities, the third has $k_3$ possibilities, etc., the total number of possibilities of the sequence will be:

$$ k_1 \cdot k_2 \cdot k_3 \cdots k_1 $$

**Permutation rule 1:** The number of permutations of $n$ objects taking $r$ objects at a time when order is important is:

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

**Permutation rule 2:** The number of permutations of $n$ objects when $r_1$ objects are identical, $r_2$ objects are identical, . . . , $r_p$ objects are identical is:

$$ \frac{n!}{r_1!r_2! \cdots r_p!} $$

**Combination rule:** The number of combinations of $r$ objects taken from $n$ objects when order is not important is:

$$ {}_nC_r = \frac{n!}{(n-r)!r!} $$

## Applying the Concepts 4–4
### Garage Door Openers

Garage door openers originally had a series of four on/off switches so that homeowners could personalize the frequencies that opened their garage doors. If all garage door openers were set at the same frequency, anyone with a garage door opener could open anyone else's garage door.

1. Use a tree diagram to show how many different positions 4 consecutive on/off switches could be in.

After garage door openers became more popular, another set of 4 on/off switches was added to the systems.

2. Find a pattern of how many different positions are possible with the addition of each on/off switch.
3. How many different positions are possible with 8 consecutive on/off switches?
4. Is it reasonable to assume, if you owned a garage door opener with 8 switches, that someone could use your garage door opener to open your garage door by trying all the different possible positions?

For a specific year it was reported that the ignition keys for Dodge Caravans were made from a single blank that had five cuts on it. Each cut was made at one out of five possible levels. For that year assume there were 420,000 Dodge Caravans sold in the United States.

5. How many different possible keys can be made from the same key blank?
6. How many different Dodge Caravans could any one key start?

Look at the ignition key for your car and count the number of cuts on it. Assume that the cuts are made at one of any of five possible levels. Most car companies use one key blank for all their makes and models of cars.

7. Conjecture how many cars your car company sold over recent years and then figure out how many other cars your car key could start. What could you do to decrease the odds of someone being able to open another vehicle with their own key?

# Practice 4–4

## Practice Exercises

**In #1–7, determine the number of items or outcomes that can be created as described.**

1. How many 5-digit zip codes are possible if digits can be repeated? If there cannot be repetitions?
2. List all the permutations of the letters in the word *MATH*.
3. A movie theater plans to show 7 different movies, one per night, over 7 days. How many different ways could the manager select the order of the films the theater will show?
4. Four eight-sided dice are rolled. How many different outcomes are there?
5. The call letters of a radio station must have 4 letters. The first letter must be a K or a W. How many different station call letters can be made if repetitions are not allowed? If repetitions are allowed?
6. A baseball team has nine players. How many different batting orders are there? If the pitcher always bats last, how many different batting orders are there?
7. A high school librarian received a grant to purchase one book from each of these categories. There are 5 biographies, 8 mysteries, and 3 science fiction. How many different ways can a book be selected?

**In #8, evaluate the factorials and permutations.**

8. Evaluate each expression.

| | | |
|---|---|---|
| **a.** 0! | **e.** $_7P_3$ | **i.** $_9P_5$ |
| **b.** 8! | **f.** $_{11}P_9$ | **j.** $_{11}P_6$ |
| **c.** 1! | **g.** $_5P_5$ | |
| **d.** 5! | **h.** $_6P_0$ | |

**In #9–15, determine the number of choices or permutations that can be made as described.**

9. An artist can select 4 colors for a sign. If the artist has 6 colors to choose from, how many different ways can the artist paint the sign?
10. How many different ways can a city health department inspector visit 5 restaurants in a city with 10 restaurants?
11. How many different 4-letter permutations can be written from the word *hexagon*?
12. A particular cell phone company offers 4 models of phones, each in 6 different colors and each available with any one of 5 calling plans. How many combinations are possible?
13. The Foreign Language Club is showing a four-movie marathon of subtitled movies. How many ways can they choose 4 from the 11 available?
14. How many permutations can be made using all the letters in the word *MASSACHUSETTS*?
15. How many different ways can 6 identical hardback books, 3 identical paperback books, and 3 identical boxed books be arranged on a shelf in a bookstore?

**In #16, calculate the combinations.**

16. Evaluate each expression.

| | |
|---|---|
| **a.** $_{10}C_5$ | **d.** $_{10}C_9$ |
| **b.** $_4C_4$ | **e.** $_5C_4$ |
| **c.** $_6C_0$ | |

**In #17–31, determine the number of choices or outcomes that can be created as described.**

17. How many ways can 5 soccer players and 7 football players be selected from 10 soccer players and 10 football players?
18. An instructor wants to program a test correction machine to correct a 10-question true/false test. How many different ways are possible for the answer sheet? If the instructor decides to have 5 questions with answers that are true and five questions with answers that are false, how many different answer sheets are possible?

19. In how many ways can you choose 3 kinds of ice cream and 2 toppings from a dessert buffet with 10 kinds of ice cream and 6 kinds of toppings?

20. Six students are performing one song each in a jazz vocal recital. Two students have repertoires of five numbers, and the others have four songs each prepared. How many different programs are possible without regard to order? Assume that the repertory selections are all unique.

21. There are 7 women and 5 men in a department. How many ways can a committee of 4 people be selected? How many ways can this committee be selected if there must be 2 men and 2 women on the committee? How many ways can this committee be selected if there must be at least 2 women on the committee?

22. How many ways can you pick 4 students from 10 students (6 men, 4 women) if you must have an equal number of each gender or all of the same gender?

23. There are 16 seniors and 15 juniors in a particular social organization. In how many ways can 4 seniors and 2 juniors be chosen to participate in a charity event?

24. An advertising manager decides to have an ad campaign in which 8 special calculators will be hidden at various locations in a shopping mall. If he has 17 locations from which to pick, how many different possible combinations can he choose?

25. A buyer decides to stock 8 different posters. How many ways can she select these 8 if there are 20 from which to choose?

26. Anderson Research Company decides to test market a product in 6 areas. How many different ways can 3 areas be selected in a certain order for the first test?

27. How many ways can a dinner patron select 3 appetizers and 2 vegetables if there are 6 appetizers and 5 vegetables on the menu?

28. The Environmental Protection Agency must investigate 9 mills for complaints of air pollution. How many different ways can a representative select 5 of these to investigate this week?

29. How many different ways can you select one or more coins if you have 2 nickels, 1 dime, and 1 half-dollar?

30. In how many ways can 3 people be seated in a circle? 4? *n*? (*Hint:* Think of them standing in a line before they sit down and/or draw diagrams.)

31. A game of concentration (memory) is played with a standard 52-card deck. How many potential two-card matches are there (i.e., one jack “matches” any other jack)?

GO ONLINE Go online for more practice problems.

# Using Technology: TI-84 Plus Step by Step

## Factorials, Permutations, and Combinations

TABLE TI4–3 Calculating Factorials, Permutations, and Combinations

<table>
<tr><th>Desired Action</th><th>TI-84 Plus Instructions</th></tr>
<tr><td>Calculate factorials n!</td><td>

**1.** Type the value of *n*.

**2.** Press **MATH** and move the cursor to PRB, then press **4** for !. See Figure TI4–3.

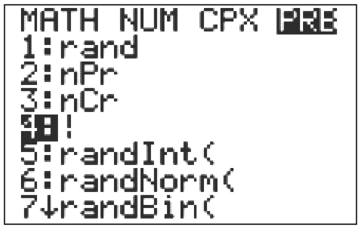



FIGURE TI4–3 Selecting Factorial Calculation

**3.** Press **ENTER.**

</td></tr>
<tr><td>Calculate permutations $_nP_r$</td><td>

**1.** Type the value of *n*.

**2.** Press **MATH** and move the cursor to PRB, then press **2** for $_nP_r$. See Figure TI4–4.

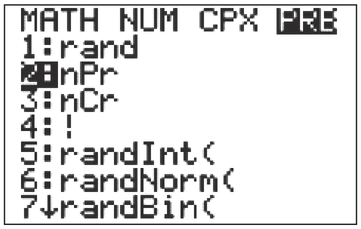



FIGURE TI4–4 Selecting Permutation Calculation

**3.** Type the value of *r*.

**4.** Press **ENTER.**

</td></tr>
<tr><td>Calculate combinations $_nC_r$</td><td>

**1.** Type the value of *n*.

**2.** Press **MATH** and move the cursor to PRB, then press **3** for $_nC_r$. See Figure TI4–5.

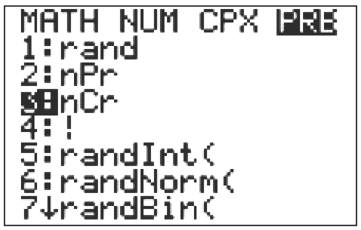



FIGURE TI4–5 Selecting Combination Calculation

**3.** Type the value of *r*.

**4.** Press **ENTER.**

</td></tr>
</table>

## TI Example

**a.** Calculate 5!

Locate “Calculate factorials *n*!” in Table TI4–3 and follow steps 1–3, using $n = 5$.

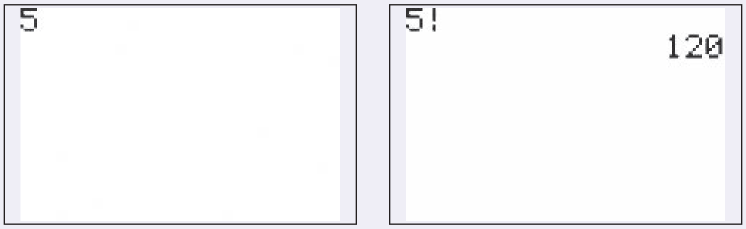



FIGURE TI4–6 Input and Output Screens for 5! Calculation

**b.** Calculate ${}_6P_3$.

Locate “Calculate permutations ${}_nP_r$” in Table TI4–3 and follow steps 1–4, using $n = 6$ and $r = 3$.


FIGURE TI4–7 Input and Output Screens for 6P3 Calculation

**c.** Calculate ${}_{10}C_3$.

Locate “Calculate combinations ${}_nC_r$” in Table TI4–3 and follow steps 1–4, using $n = 10$ and $r = 3$.


FIGURE TI4–8 Input and Output Screens for 10C3 Calculation



**Using Technology:** Go online to learn how to use statistical software to answer questions.

Lesson 4–5

# Probability and Counting Rules

## Learn: Using Counting Rules and Probability Rules Together

The counting rules can be combined with the probability rules in this chapter to solve many types of probability problems. By using the fundamental counting rule, the permutation rules, and the combination rule, you can compute the probability of outcomes of many experiments, such as getting a certain hand when 5 cards are dealt or selecting a committee of 3 women and 2 men from a club consisting of 10 women and 10 men.

### Example 1: Find Probability Using Counting Rules

In a lottery for charity, a person selects a three-digit number, and repetitions are permitted. If a winning number is selected, find the probability that it will have all three digits the same.

**Solution:**

From Lesson 4–1, we know:

$$P(\text{three-digit number with the same digits}) = \frac{\text{\# of three-digit numbers with the same digits}}{\text{\# of three-digit numbers}}$$

Since there are 10 different digits, there are $10 \cdot 10 \cdot 10 = 1000$ ways to select a three-digit number, by the Fundamental Counting Rule.

Counting all three-digit numbers in which all digits are the same (that is, 000, 111, 222, . . . , 999), there are 10 possibilities. So the probability of selecting a winning number that has 3 identical digits is:

$$P(\text{three-digit number with the same digits}) = \frac{\text{\# of three-digit numbers with the same digits}}{\text{\# of three-digit numbers}}$$

$$= \frac{10}{1000} = \frac{1}{100}$$

### Example 2: Find Probability Using Counting Rules

There are 8 pairs of mixed doubles teams (each made up of one man and one woman) in a tennis club. If 1 man and 1 woman are selected at random to plan the summer tournament, find the probability that they are a mixed doubles team.

**Solution:**

We will find the probability by using the formula:

$$P(\text{two people selected are a team}) = \frac{\text{\# of mixed doubles teams}}{\text{\# of ways to select 1 man and 1 woman}}$$

Since there are 8 ways to select the first person and 8 ways to select the partner, there are 8 · 8, or 64, ways to select the couple. Since there are 8 mixed doubles teams, the solution is:

$$P(\text{two people selected are a team}) = \frac{\text{\# of mixed doubles teams}}{\text{\# of ways to select 1 man and 1 woman}}$$

$$= \frac{8}{64} = \frac{1}{8}$$

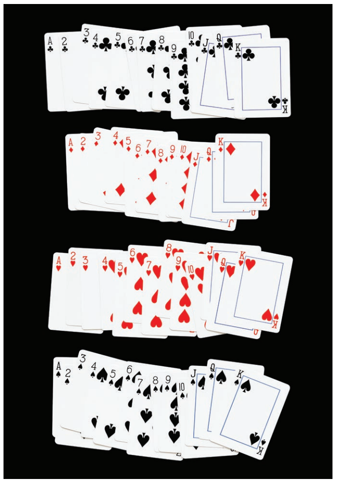


FIGURE 4–22 Deck of 52 Cards Organized by Suit
bundit jonwises/Shutterstock

## Example 3: Find Probability Using Combinations—Drawing Cards

Find the probability of getting 4 aces when 5 cards are drawn from an ordinary deck of cards.

### Solution:

We will find the probability by using the formula:

$$P(\text{4 aces}) = \frac{\text{\# of ways to draw 5 cards with 4 aces}}{\text{\# of ways to draw 5 cards from a deck}}$$

There are ${}_{52}C_5$ ways to draw 5 cards from a deck. There is only 1 way to get 4 aces (that is, ${}_4C_4$). There are 48 possibilities for the fifth card because 4 aces drawn from a deck of 52 cards leaves 48 remaining cards. Therefore, there are 48 ways to get 4 aces and 1 other card from the remaining deck.

$$P(\text{4 aces}) = \frac{\text{\# of ways to draw 5 cards with 4 aces}}{\text{\# of ways to draw 5 cards from a deck}}$$

$$= \frac{(\text{\# of ways to draw 4 aces from a deck}) \cdot (\text{\# of ways to draw a fifth card})}{\text{\# of ways to draw 5 cards from a deck}}$$

$$= \frac{{}_4C_4 \cdot 48}{{}_{52}C_5} = \frac{1 \cdot 48}{2{,}598{,}960} = \frac{48}{2{,}598{,}960} = \frac{1}{54{,}145}$$

## Example 4: Find a Probability Using Combinations

A student needs to select two topics to write two term papers for a course. There are 8 topics in economics and 11 topics in science. Find the probability that the student selects one topic in economics and one topic in science to complete the assignment.

### Solution:

We will find the probability by using the formula:

$$P(\text{economics and science}) = \frac{\text{\# of ways to select the two topics}}{\text{\# of ways to select 2 topics from all 19 possible}}$$

There are ${}_8C_1$ ways to choose 1 economics topic from the 8 available and ${}_{11}C_1$ ways to choose 1 science topic from the 11 available. So, there are ${}_8C_1 \cdot {}_{11}C_1$ ways to select the two topics. There are ${}_{19}C_2$ ways to select 2 topics from the 19 total topics available.

$$P(\text{economics and science}) = \frac{\text{\# of ways to select the two topics}}{\text{\# of ways to select 2 topics from all 19 possible}}$$

$$= \frac{{}_8C_1 \cdot {}_{11}C_1}{{}_{19}C_2} = \frac{8 \cdot 11}{19 \cdot 18} = \frac{88}{171} \approx 0.515 \text{ or } 51.5\%$$

So, there is about a 51.5% probability that a person will select one topic from economics and one topic from science.

## Example 5: Find Probability Using Combinations—Defective Items

A box contains 24 integrated circuits, 4 of which are defective. If 4 are sold at random, find the following probabilities.

**a.** Exactly 2 are defective.

**b.** None is defective.

**c.** All are defective.

**d.** At least 1 is defective.

### Solution:

In each case, we will need to use the number of ways to sell 4 integrated circuits from a box of 24. There are ${}_{24}C_4 = 10{,}626$ ways.

**a.** We will find the probability by using the formula:

$$P(\text{exactly 2 defectives}) = \frac{\text{\# of ways to select exactly 2 defective}}{\text{\# of ways to select 4 circuits from 24 total circuits}}$$

There are ${}_4C_2$ ways to choose 2 defective integrated circuits from the 4 available and ${}_{20}C_2$ ways to choose two non-defective ones from the 20 available. So, there are ${}_4C_2 \cdot {}_{20}C_2$ ways to select exactly 2 defective (and 2 non-defective) circuits.

$$P(\text{exactly 2 defectives}) = \frac{\text{\# of ways to select exactly 2 defective}}{\text{\# of ways to select 4 circuits from 24 total circuits}}$$

$$= \frac{{}_4C_2 \cdot {}_{20}C_2}{{}_{24}C_4} = \frac{1140}{10{,}626} = \frac{190}{1771}$$

**b.** We will find the probability by using the formula:

$$P(\text{no defectives}) = \frac{\text{\# of ways to select 4 non-defective from 20 possible}}{\text{\# of ways to select 4 circuits from 24 total circuits}}$$

The number of ways to choose no defective circuits is to select all 4 circuits from the 20 non-defective circuits, ${}_{20}C_4$.

$$P(\text{no defectives}) = \frac{\text{\# of ways to select 4 non-defective from 20 possible}}{\text{\# of ways to select 4 circuits from 24 total circuits}}$$

$$= \frac{{}_{20}C_4}{{}_{24}C_4} = \frac{4845}{10{,}626} = \frac{1615}{3542}$$

**c.** We will find the probability by using the formula:

$$P(\text{all defective}) = \frac{\text{\# of ways to select 4 defective from 4 possible}}{\text{\# of ways to select 4 circuits from 24 total circuits}}$$

The number of ways to choose 4 defective circuits from the 4 defective circuits, ${}_4C_4 = 1$.

$$P(\text{all defective}) = \frac{\text{\# of ways to select 4 defective from 4 possible}}{\text{\# of ways to select 4 circuits from 24 total circuits}}$$

$$= \frac{{}_4C_4}{{}_{24}C_4} = \frac{1}{10{,}626}$$

**d.** To find the probability of at least 1 defective transistor, find the probability that there are no defective integrated circuits and then subtract that probability from 1.

$$P(\text{at least 1 defective}) = 1 - P(\text{no defectives})$$

$$= 1 - \frac{(\text{\# of ways to select 4 nondefective from 20 possible})}{(\text{\# of ways to select 4 circuits from 24 total circuits})}$$

$$= 1 - \frac{{}_{20}C_4}{{}_{24}C_4} = 1 - \frac{1615}{3542} = \frac{1927}{3542}$$

## Statistics Around Us: Use Statistics

The United States is choosing 10 states to participate in a contest to create a statue of their state bird. Previous census data found that 3 U.S. states declined in population from 2010 to 2020. If 10 states are chosen at random using this census data, find the following probabilities.

**a.** Exactly three chosen states had a decline in population from 2010 to 2020.

**b.** None of the states chosen had a decline in population from 2010 to 2020.

**c.** At least 1 state chosen had a decline in population from 2010 to 2020.

As indicated at the beginning of this lesson, the counting rules and the probability rules can be used to solve a large variety of probability problems. These include those found in business, economics, biology, and other fields.

## Applying the Concepts 4–5
### Counting Rules and Probability

One of the biggest problems for students when doing probability problems is to decide which formula or formulas to use. Another problem is to decide whether two events are independent or dependent. Use the following problems to help develop a better understanding of these concepts.

Assume you are given a five-question multiple-choice quiz. Each question has 5 possible answers: A, B, C, D, and E.

**1.** How many events are there?

**2.** Are the events independent or dependent?

**3.** If you guess at each question, what is the probability that you get all of them correct?

**4.** What is the probability that a person guesses answer A for each question?

Assume that you are given a five-question matching test in which you are to match the correct answers in the right column with the questions in the left column. You can use each answer only once.

**5.** How many events are there?

**6.** Are the events independent or dependent?

**7.** What is the probability of getting them all correct if you are guessing?

**8.** What is the difference between the two problems?

# Practice 4–5

## Practice Exercises

**In #1–8, find the probabilities of the events described.**

**1.** Find the probability of getting 2 face cards (king, queen, or jack) when 2 cards are drawn from a deck without replacement.

**2.** Six men and seven women apply for two identical jobs. If the jobs are filled at random, find the following:

- **a.** The probability that both are filled by men.
- **b.** The probability that both are filled by women.
- **c.** The probability that one man and one woman are hired.
- **d.** The probability that the one man and one woman who are twins are hired.

**3.** A package contains 12 resistors, 3 of which are defective. If 4 are selected, find the probability of getting:

- **a.** 0 defective resistors.
- **b.** 1 defective resistor.
- **c.** 3 defective resistors.

**4.** Find the probability of getting 3 cards of one denomination and 2 of another when 5 cards are dealt from an ordinary deck.

**5.** A drawer contains 11 identical red socks and 8 identical black socks. Suppose that you choose 2 socks at random in the dark.

- **a.** What is the probability that you get a pair of red socks?
- **b.** What is the probability that you get a pair of black socks?
- **c.** What is the probability that you get 2 unmatched socks?

**6.** Find the probability of selecting 3 science books and 4 math books from 8 science books and 9 math books. The books are selected at random.

**7.** If three dice are rolled, find the probability of getting a sum of 5.

**8.** All holly plants are dioecious—a male plant must be planted within 30 to 40 feet of the female plants in order to yield berries. A home improvement store has 12 unmarked holly plants for sale, 8 of which are female. If a homeowner buys 3 plants at random, what is the probability that berries will be produced?

GO ONLINE Go online for more practice problems.

# Chapter 4 Review

## Statistics Around Us: Analyze Results

**Probability of Selection** State Populations

1. Review your results from the *Use Statistics* exercises. Have you answered the questions you posed at the start of the chapter?
2. Based on what you have learned in this chapter, are there new questions or values to consider?
3. What have you learned about this topic?

## What did you learn?

After completing this chapter, you should be able to:

- Determine sample spaces and find the probability of an event, using classical probability or empirical probability. Classical probability uses sample spaces and assumes that all outcomes in the sample space are equally likely. Empirical probability uses frequency distributions and is based on observation. (Lesson 4–1)
- Find the probability of compound events, using the addition rules. To find the probability of two mutually exclusive events occurring, add the probability of each event. To find the probability of two events when they are not mutually exclusive, add the possibilities of the individual events and then subtract the probability that both events occur at the same time. (Lesson 4–2)
- Find the probability of compound events, using the multiplication rules. To find the probability of two independent events occurring, multiply the probabilities of each event. To find the probability that two dependent events occur, multiply the probability that the first event occurs by the probability that the second event occurs, given that the first event has already occurred. (Lesson 4–3)
- Find the conditional probability of an event. Conditional probability is the probability that a second event occurs, given that the first event has already occurred. (Lesson 4–3)
- Find the total number of outcomes in a sequence of events, using the fundamental counting rule. The fundamental counting rule states that in a sequence of $n$ events in which the first one has $k_1$ possibilities and the second event has $k_2$ and the third has $k_3$ and so forth, the total number of possibilities of the sequence will be $k_1 \cdot k_2 \cdot k_3 \cdots k_1$. (Lesson 4–4)

- Find the number of ways that $r$ objects can be selected from $n$ objects, using the permutation rule. When calculating permutations where one or more items are identical, division removes the duplicate counting of identical arrangements. (Lesson 4–4)
- Find the number of ways that $r$ objects can be selected from $n$ objects without regard to order, using the combination rule, ${}_nC_r = \frac{n!}{(n-r)!r!}$. (Lesson 4–4)
- Find the probability of an event, using the counting rules. (Lesson 4–5)

Go online for a detailed summary.

## Essential Question Revisited

How do we determine the probability of events?

## Chapter 4 Vocabulary

classical probability 207

combination 263

complement of an event 212

compound event 207

conditional probability 239

dependent events 239

disjoint events 223

empirical probability 214

equally likely events 207

event 207

fundamental counting rule 255

independent events 235

law of large numbers 218

mutually exclusive events 223

outcome 202

permutation 261

probability 202

probability experiment 202

sample space 202

simple event 207

subjective probability 218

tree diagram 204

Venn diagrams 213

# Chapter 4 Review

## Chapter 4 Formulas

TABLE 4–14 Chapter 4 Formulas

| Desired Value(s) | Formula or Rule |
|---|---|
| **Classical probability** | $P(E) = \dfrac{\text{total number of outcomes in E}}{\text{total number of outcomes in sample space}} = \dfrac{n(E)}{n(S)}$ |
| **Empirical probability** | $P(E) = \dfrac{\text{frequency for class}}{\text{total frequencies in distribution}} = \dfrac{f}{n}$ |
| **Addition rule 1: *P*(*A* or *B*)**<br><br>A, B mutually exclusive events | $P(A \text{ or } B) = P(A) + P(B)$ |
| **Addition rule 2: *P*(*A* or *B*)**<br><br>A, B not mutually exclusive events | $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$ |
| **Multiplication rule 1: *P*(*A* and *B*)**<br><br>A, B independent events | $P(A \text{ and } B) = P(A) \cdot P(B)$ |
| **Multiplication rule 2: *P*(*A* and *B*)**<br><br>A, B dependent events | $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$ |
| **Conditional probability** | $P(B \mid A) = \dfrac{P(A \text{ and } B)}{P(A)}$ |
| **Probability of complementary events, *E* and $\bar{E}$** | $P(\bar{E}) = 1 - P(E)$<br><br>$P(E) = 1 - P(\bar{E})$<br><br>$P(E) + P(\bar{E}) = 1$ |
| **Fundamental counting rule**<br><br>The total number of possibilities of a sequence of *n* events | In a sequence of *n* events in which the first one has $k_1$ possibilities, the second event has $k_2$ possibilities, the third has $k_3$ possibilities, etc., the total number of possibilities of the sequence will be:<br><br>$k_1 \cdot k_2 \cdot k_3 \cdots k_1$ |
| **Permutation rule 1**<br><br>The number of permutations of *n* objects taking *r* objects at a time when order is important | ${}_nP_r = \dfrac{n!}{(n-r)!}$ |

| Desired Value(s) | Formula or Rule |
|---|---|
| **Permutation rule 2**<br><br>The number of permutations of $n$ objects when $r_1$ objects are identical, $r_2$ objects are identical, . . . , $r_p$ objects are identical | $\dfrac{n!}{r_1!r_2! \cdots r_p!}$ |
| **Combination rule**<br><br>The number of combinations of $r$ objects selected from $n$ objects when order is not important | ${}_nC_r = \dfrac{n!}{(n-r)!r!}$ |

# Chapter 4 Review

## Chapter 4 Review Exercises

**For #1–14, find the requested probabilities.**

### Lesson 4–1

**1.** A twelve-sided die is rolled (containing the numbers 1, 2, 3, . . . , 10, 11, 12). Find the probability of each.

**a.** Rolling an even number

**b.** Rolling a multiple of 5

**c.** Rolling a number less than 7

**2.** When a card is selected from a standard deck of 52 cards, find the probability of each.

**a.** A face card

**b.** A black card

**c.** A 4 and a diamond

**d.** A spade

**3.** Five popular burger chains had the following cheeseburger sales for one day.

| | |
|---|---|
| Burgers Deluxe | 1,455 |
| Midwest Burgers | 1,568 |
| Hometown Burgers | 2,177 |
| Beefy Burgers | 756 |
| The Burger Hut | 436 |

Find each of the following for a single cheeseburger purchase.

**a.** The probability that the cheeseburger was from Hometown Burgers

**b.** The probability that the cheeseburger was from either Beefy Burgers or The Burger Hut

**c.** The probability that the cheeseburger was not from Midwest Burgers

**4.** When three coins are flipped, find each of the following:

**a.** The probability of flipping at least 1 tail

**b.** The probability of flipping exactly 2 tails

**c.** The probability of flipping no more than 2 heads



### Lesson 4–2

**5.** In a recent school survey, 17 students preferred having salad at lunch, 32 student preferred sandwiches, and 26 preferred the hot lunch special. Calculate the probability that a randomly selected student, from the survey group, prefers the hot lunch special.

6. A nearby carnival game has large stuffed animals as prizes. There are 15 giraffes, 21 orangutans, 12 lions and 6 polar bears to choose from. Calculate each of the following:

   a. The probability that a prize awarded is a polar bear

   b. The probability that a prize awarded is not a giraffe

   c. The probability that a prize awarded is either a giraffe, orangutan, or polar bear

7. A local junior high gathered data about what students' activities are after school. The results showed that the probability that a student skateboards after school is 0.32. The probability that a student plays video games is 0.73 and the probability that they skateboard and play video games is 0.21. Calculate the probability that a randomly chosen student either skateboards or plays video games after school.

8. A high school counselor examined the senior enrollment in courses and found that the probability that a senior was in choir was 0.16. The probability that a senior was taking physics was 0.44. The probability that a senior was enrolled in both classes was 0.09. Calculate the probability that a senior was not enrolled in either class.

## Lesson 4–3

9. A popular food chain is giving out superhero/villain trading cards with every purchase. 70% of the cards have a superhero on them, while 30% have a villain. Twenty percent of the superhero cards have a female character on them, while 5% of the villain cards have a female character on them. If cards are randomly distributed, what is the probability of getting a superhero card given that it is a female character?

10. Approximately 5% of the world's population are healthy carriers of a gene for sickle-cell disease. Choose 6 individuals at random. What is the probability that at least one of them carries the sickle-cell gene?
    *Source*: www.afro.who.int/health-topics/sickle-cell-disease

11. A small liberal arts college wanted to see if their exercise program was having an effect on students over their 4 years of attendance. The table below compares the exercise data collected for freshmen and seniors enrolled in the college.

| | Exercise 0-2 days/week | Exercise 3-5 days/week | Exercise 6-7 days/week |
|---|---|---|---|
| **Freshmen** | 335 | 215 | 50 |
| **Seniors** | 140 | 240 | 95 |

If a student is selected at random, calculate the following:

   a. The probability that the student exercises 3–5 days/week given that they are a senior

   b. The probability that the student exercises less than 3 days/week

   c. Given that the student is a freshman, what is the probability that they exercise 0–2 days/week?

d. Given that the student exercises 6–7 days/week, what is the probability that they are a freshman?

e. The probability that the student is a freshman who exercises at least 3 days/week

**12.** A city planning director found that 46% of local high school seniors drive and hold down a part-time job. Eighty-three percent of seniors in the high school drive. Find the probability that a senior has a job, given that they also drive.

**13.** Two cards are drawn from a standard deck of 52 cards *without* replacement. Find the probability of getting:

a. Two diamonds

b. A pair of jacks

c. An ace followed by a face card

**14.** At the Tokyo Olympics, the 6 countries with the most gold medals won are shown below.

| Country | Number of gold medals won |
|---|---|
| Australia | 17 |
| Great Britain | 22 |
| Japan | 27 |
| People's Republic of China | 38 |
| ROC | 20 |
| U.S.A. | 39 |

*Source:* https://www.teamusa.org/tokyo-2020-olympic-games/medal-tracker

A medal is chosen at random. Find the probability that:

a. The medal was won by a U.S. athlete.

b. The medal was won by either an Australian or British athlete.

c. The medal was won by a Japanese athlete, given that the athlete was from the continent of Asia.

**For #15–19, use counting techniques to determine the number of possibilities.**

## Lesson 4–4

**15.** A high school volleyball roster contains 5 seniors, 7 juniors, 3 sophomores, and 1 freshman. Only 6 players are allowed on the court at one time. How many different ways can the coach play 3 seniors, 1 junior, and 2 sophomores?

**16.** An ice cream shop offers the following: 20 flavors of ice cream and either a waffle cone or wafer cone. A patron can also choose to have the cone dipped in a chocolate or butterscotch hard-shell coating. What are the total number of different ice cream cones that can be made in the shop?

**17.** A special lock contains the letters from the word *ARRANGEMENT*. How many different permutations could you make to open the lock?

18. United States phone numbers consist of a 3-digit area code followed by a 7-digit number. The 7-digit number cannot begin with either a 0 or 1. How many different phone numbers can have the area code 909?

19. A silk screen company makes customizable T-shirts. Buyers can choose from the 30 major-league baseball team logos. They can order a shirt in either small, medium, or large sizes and either crew neck or V-neck styles. The colors available are black, white, navy, or grey. How many different T-shirts must the company be prepared to make?

**For #20–22, find probabilities using counting techniques.**

## Lesson 4-5

20. A statistics quiz has one multiple-choice question, with options (A), (B), (C), and (D), followed by 2 true/false questions. Draw a tree diagram for all the different ways a student could answer the three questions.

   a. What are the total number of different ways a student could answer?

   b. What is the probability that the correct answer to the multiple-choice question is (B)?

21. At a local Chinese food restaurant, they offer a 4-item plate special. All plates come with a choice of fried rice, brown rice, or chow mein, which is counted as one item. The entrée choices consist of 17 vegetable and 12 meat items.

   a. If a customer is allowed to select only 1 meat entrée item, how many different plate specials can they make?

   b. If a random selection of plates was chosen, what would be the probability of selecting either all vegetable or all meat entrées?

22. The Dewey Decimal System is a classification system used by libraries to organize books by subject. Each number given to a book begins with a 3-digit number followed by a decimal, 3 more digits, and then 3 letters. If a book belongs to the topic of mathematics, it must start with 51. Books that belong to the topic of probability are categorized with 519. A book is chosen randomly from the math section. What is the probability that the book is about probability and that its Dewey Decimal number contains the letters *AUL* at the end?

*Sources:* https://www.oclc.org/content/dam/oclc/webdewey/help/500.pdf; https://www.oclc.org/content/dam/oclc/dewey/resources/summaries/deweysummaries.pdf