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# *Reveal Math* K–5 Research Foundations:

# A Research Summary of Program Focused Outcomes

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## Table of Contents

# Introduction

As a learning science company, McGraw Hill designs all our products and services to unlock the potential of each learner. Not only do we strive to create products that improve and accelerate the learning process using rigorous and important content, but we also look to reflect a wide and diverse range of perspectives and approaches that cater to the whole child and each student's individual learning journey. Moreover, we support teachers as they work to create inclusive classrooms that embrace the needs of all learners.

The *Reveal Math* program is an exceptional example of how McGraw Hill puts that vision into practice. *Reveal Math* is built on principles that honor the full potential in each student mathematician by setting high expectations for all and providing support, extensions, and delivery options to incorporate the core values of curiosity, connections, communication, collaboration, and confidence. The overall instructional goal is to reveal the full potential in every student by empowering every teacher to orchestrate rich mathematics learning leading to enhanced student performance. To do this, we grounded the development of *Reveal Math* in salient research and evidence-based best practices.

At the core of *Reveal Math* are specific areas of focus that have emerged from numerous learning science domains essential to strengthening the teaching and learning of elementary mathematics (NCTM, 2017). The foci chosen for *Reveal Math* offer a balanced approach to mathematics instruction that encompasses both student-centered and teacher-facilitated instructional activities. Additionally, the focus areas are supported by research and inform how the program was crafted, starting from the development of the overarching program goals to the construction of the *Reveal Math* learning interactions and instructional model. A program logic model (see Figure 1), which delineates the path through which the program can meet the anticipated goals, was developed to build the program with the end in mind and provides a big picture overview of the main features of the *Reveal Math* program. The logic model is also an important component of the program research plan, as it guides the development of the program research foundation, program research questions, and effectiveness and efficacy studies.

**Inputs** · · · **Outputs** · · · **Outcomes** · · · **Long Term Goals**

### Inputs

**Launch—Prepare to Learn**
- Math Is...
- Ignite!
- *Be Curious*
- Focus, Coherence & Rigor Supports

**Explore & Develop–Teach and Learn together**
- Activity-Based Exploration
- Guided Exploration
- Instructional Routines
- Effective Teaching Practices

**Practice & Reflect—Practice, Apply, and Extend**
- Math Replay
- STEM Adventures
- Interactive Digital Practices
- Student Practice Book

**Assess–Evaluate and Apply Evidence**
- Exit Tickets
- Math Probes
- Assessments: Course, Unit, and Lesson

**Differentiate—Learning supports**
- Workstations: Games, Application, and Small Group
- Take Another Look
- eToolkit

### Outputs

**Students:**
- Think about and share what they know about math.
- Talk about and help create classroom norms, interactions, and expectations for learning math.
- Engage in self-reflection and discussion with others about classroom norms and expectations.
- Ask questions and talk with others about math problems, representations, strategies, or ideas.
- Learn about specific job & real-world related math skills.
- Apply math practices to solve problems.
- Regularly practice math skills and concepts.
- Self-reflect on how they learn math tasks/concepts and understand their areas of strength and growth.
- Engage in independent practice in areas of growth to enhance their own math learning.

**Teachers:**
- Model and foster meaningful math discussion practices.
- Use instructional routines to model specific mathematical practices.
- Monitor student progress and use data to make instructional decisions for student growth.
- Use standards, effective teaching practices, student data and interests to select appropriate lesson models and tasks.

### Outcomes

**Students:**
- **Problem-solving**–Make sense of problem situations and use sense-making to develop problem-solving skills.
- **Discourse**–Engage in class discourse, use appropriate math vocabulary, and learn to critique the math thinking of others.
- **SEL**–Use social and emotional learning competencies to become academically and socially engaged classroom members.
- **Fluency**–Develop fluency by using flexible strategies to practice math content.
- **Agency**–Draw on their agency through the expectation of ownership and accountability in their learning. Three areas support student agency:
  - *Growth Mindset*—Resilience in problem-solving and the learning process.
  - *Productive Struggle*—Deep engagement with mathematical ideas and relationships.
  - *Metacognition*—Promotion of student reflection on their learning through contemplative, communicative practice.
- **Reasoning**–Construct objective, logical arguments and share discipline-specific thought processes.
- **Sense-making**–Engage in the dynamic process of building or revising an explanation in order to "figure something out."
- **Perseverance**–Improve their ability to continue working on learning tasks even when difficult and/or tedious.
- **Mastery**–Successfully demonstrate learning (measured against a "mastery" benchmark) of a given skill, concept, or disciplinary disposition, typically achieved through individually paced learning experiences.

**Teachers:**
- Use instructional routines to provide structure and set expectations that create productive classroom interactions with students.
- Employ a variety of learner-focused practices to develop an equitable classroom designed for all students.

### Long Term Goals

**Students:**
- Consider multiple strategies, play with math, and practice perseverance.
- Students understand that math is not just a series of operations, but a rich language that calls fors specific ways of thinking and habits of mind.
- Engage in mathematical discourse as they listen actively, formulate thoughtful responses, and translate big ideas through their discussions.
- Respect and reflect different points of view, and support and inspire each other.
- Think about their own learning processes and develop agency as active and supported learners.

**Teachers:**
- Create a mathematically sound and equitable classroom through understanding students academically, socially, and personally.

Figure 1: *Reveal Math* logic model

Reaching the stated program goals requires substantial research and evidence-based support. Using the logic model outcomes as a guide, this paper provides theoretical and empirical foundational analyses that support the instructional underpinnings of *Reveal Math*:

- **Equitable Classrooms:** Teachers employ a variety of learner focused practices to develop an equitable classroom designed for all students.
- **Student Agency:** Students draw on their agency through the expectation of ownership and accountability in their learning. These areas support student agency:
  - *Metacognition:* Students use metacognition to reflect on their learning throughout the lesson.

    - *Productive Struggle:* Students engage in productive struggle as they grapple with mathematical ideas and relationships.
    - *Social and Emotional Learning:* Students use social and emotional learning competencies to become academically and socially engaged classroom members.
- **Classroom Discourse and Language:** Students engage in class discourse, use appropriate math vocabulary, and learn to critique the math thinking of others.
- **Sense-Making:** Students make sense of problem situations and use sense-making to develop problem-solving skills.
- **Fluency:** Students develop fluency by using flexible strategies to practice math content.
- **Instructional Routines:** Teachers use instructional routines to provide structure and set expectations that create productive classroom interactions with students.

Each section describes the foundational research for each outcome, the connections of the research to mathematics learning, and how the research is translated into practice through the *Reveal Math* instructional model.

## Equitable Classrooms

Educational equity has been, and continues to be, an essential foundation in our nation's schools and classrooms. Equity can be considered "the driving force behind ensuring that all students, everywhere, receive rigorous, rich educational experiences that are designed to meet their specific learning needs" (Snyder, Trowery, & McGrath, 2019, p. 3). The National School Board Association definition of educational equity highlights school district practices and resources—"the intentional allocation of resources, instruction, and opportunities according to need, requiring that discriminatory practices, prejudices, and beliefs be identified and eradicated" (NSBA, 2020). Geneva Gay (1988), in her work on designing relevant curricula for diverse learners, posits that a focus on the equitable outputs should lead the development and selection of the inputs, or materials and practices, used in classrooms: "…the real focus of equity is not sameness of content for all students, but equivalency of effect potential, quality status, and significance of learning opportunities" (p. 329). From a school mathematics perspective, the National Council of Teachers of Mathematics states, "Acknowledging and addressing factors that contribute to differential outcomes among groups of students is critical to ensuring that all students routinely have opportunities to experience high-quality mathematics instruction, learn challenging mathematics content, and receive the support necessary to be successful" (NCTM, 2020). Thus, an equitable classroom is one where all students are supported to learn rigorous academics and where teachers leverage the materials and practices needed to support positive academic outcomes for all students.

## Equity and Culturally Relevant, Responsive, and Sustaining Pedagogy

Research on culturally relevant, responsive, and sustaining pedagogies aligns with how equitable teaching and learning experiences are defined. Geneva Gay (1988) introduced **culturally responsive teaching**, which focuses on teacher practice and ways to make learning more relevant and effective for all students. She put forth a set of dimensions that guide teaching: being socially and academically empowering; setting high expectations for all students; engaging in multidimensional knowledge building, contributions, and perspectives; validating all students' cultures through diverse instructional strategies and materials; being socially, emotionally, and politically comprehensive in educating the whole child; using students' strengths to drive instruction; and being thoughtful and critical about how educational practices and ideals may form barriers to student success (Gay, 2010). Culturally relevant pedagogy, originally described by Gloria Ladson Billings, steps back from teaching practices and focuses more broadly on three key concepts or tenets of a classroom: **academic achievement**—in which teachers expect, develop, and reinforce students' academic excellence; **cultural competence**—in which students maintain their cultural integrity alongside academic excellence; and **critical consciousness**—in which students are expected to critically engage with the world around them (Ladson-Billings, 1995). Paris (2012) argues that culturally responsive or relevant pedagogies may not go far enough in the efforts to push schools to create spaces that are affirming and supportive of all students. Paris maintains that culturally sustaining pedagogies address the need in our pluralistic society to think about not only how to make instruction relevant and responsive, but also how to preserve, celebrate, share, and sustain the diverse cultures that our students bring to the learning experience.

Connecting research on culturally responsive, relevant, and sustaining pedagogies with research on educational equity provides a framework for building equitable classrooms. Research has helped uncover several factors that support classroom equity and echo the tenets of culturally responsive and sustaining practices: supporting high academic expectations for all students; a socially and emotionally positive and safe school and classroom climate; authentic and rigorous tasks; inclusive, relevant, and meaningful content; open and accepting communication; drawing from students' strengths, knowledge, culture, and competence; critically and socially aware inquiry practices; and strong teaching and teacher professional support for equity and inclusion (Aronson & Laughter, 2016; Gay, 2010; Krasnoff, 2016; Ladson-Billings, 2006; Morrison, Robbins, & Rose, 2008; NYSED, 2019; Saphier, 2017; Snyder, Trowery & McGrath, 2019; Waddell, 2014).

## Effective Practices in Equitable Math Classrooms

In the studies that focused on equitable teaching in mathematics classrooms, the findings are consistent with the work on educational equity as a whole (Brenner, 1998; Bonner, 2009; Gutstein, Lipman, Hernandez, & de los Reyes, 1997; Matthews, 2003; Nasir, 2002; Osisioma, Kiluva-ndunda, & Van Sickle, 2008; Tate, 1995). Boaler and Staples (2008) conducted a longitudinal study comparing how equitable teaching impacted students' math achievement in three high schools. In the school where the teachers taught mixed-level math classes, students were provided additional time to work together and grapple with more conceptually focused problems. Despite having begun the study with pre-test scores far below the comparison schools, students in this school outperformed the others in years 2 and 3 on post-test measures of math achievement. The researchers contend that because the focus school held high expectations for students; presented all students with a common, rigorous curriculum to support their learning; offered learning supports to struggling students; and enacted a high level of challenge in classroom tasks, inequalities in teaching practices were reduced, thus increasing students' math achievement levels (p. 635).

Other findings in the research on equitable and culturally relevant mathematics teaching demonstrate how teachers make effective connections to students' lives and communities with real-world applications of mathematics (Ensign, 2003; Enyedy & Mukhopadhyay, 2007; Gutstein, Lipman, Hernandez, de los Reyes, 1997; Rosa & Orey, 2010; Tate, 1995). Gutierrez (2009) posits that to move toward equitable mathematics teaching, teachers must know their students through a variety of lenses—academically, socially, personally—without essentializing who they are. Matthews (2003), in his work with four elementary mathematics teachers enacting culturally relevant teaching, suggests teachers should work to form an open relationship with their students so that informal/cultural knowledge and critical thinking in the classroom community can be used to build bridges to mathematics knowledge and the culture of school. In a study examining the effects of specific culturally relevant teaching practices on high school students' mathematics achievement, Langlie (2008) found that time teachers spent with students getting to know them outside of formal teaching and teachers who employed practices that encouraged students to see and use math as part of daily life were both factors that had statistically significant, positive effects on students' mathematics achievement. In creating equitable classrooms where all students have opportunities to learn math at high levels, the research demonstrates the influence classroom culture has on the math knowledge being shared with students as well as the impact classroom culture has on how math knowledge is learned by students (Waddell, 2014).

# Equity and *Reveal Math*

*Reveal Math* supports the development of equitable math classrooms through the variety of resources and practices embedded into the program. *Reveal Math* places an emphasis on creating a positive and productive classroom culture where all students have common access to rigorous instruction while supporting the student's development of growth mindset and a positive math identity. The Math Is... Unit, which is the first unit in each grade level, focuses on helping students see themselves as "doers of mathematics." Students develop thinking habits that are integral to mathematical problem-solving and co-create classroom norms that lead to a mutually supportive and productive learning environment.

**Social and Emotional Learning (SEL)** draws on equity and cultural relevance in the stated core competencies and supports teachers and students in self-awareness, social awareness, relationship skills, self-management, and responsible decision-making (CASEL, 2017). In *Reveal Math*, SEL objectives are integrated within every lesson, allowing teachers to support the whole child within the math classroom and understand who their students are and what skills and habits they bring to the classroom.

The lesson structure of *Reveal Math* is designed to assure all students will have access to rigorous instruction. Every lesson in *Reveal Math* highlights the Focus, Coherence, and Rigor of the content and sets the stage for establishing high academic expectations, a main tenet in culturally relevant pedagogy (Ladson-Billings, 1995). The Be Curious activity found in each lesson uses sense-making routines to engage students in a low floor, high ceiling discussion, creating an equitable classroom culture where all ideas are welcome and respected. Be Curious encourages classroom discourse and provides space for students to share their ideas and current understandings, listen to and learn about others, as well as make sense of mathematics for themselves, another important practice in equity teaching (Gay 2010; Paris, 2012).

For the lesson's main instruction, the teacher can choose between two equivalent methods of instruction: an Activity-Based Exploration or a Guided Exploration. These two options provide access to the same rigorous content while allowing for a variety of modalities to experience the math. Both methods offer students the opportunity to develop deep understanding of the material through meaningful discourse. Embedded teacher support helps to ensure that students have the appropriate scaffolds to understand and access the lesson content. These supports include English Learner scaffolds, math language routines, and questioning grounded in effective teaching practices.

For each lesson, *Reveal Math* provides rich differentiation resources, including intervention resources, to support all students in the learning process. Daily Exit Tickets can be a source of data, allowing teachers to make sound instructional decisions around which differentiation resource to assign to students: one that reinforces understanding, one that builds proficiency, or one that extends thinking. Each option includes multiple modalities, offering rich differentiation that not only supports students' understanding but also challenges them through STEM simulations and application cards. Targeted intervention aligns to item analysis of assessments to help support and target specific misunderstandings and gaps in learning. These features provide the foundation for all students to receive high-quality, rigorous mathematics instruction and reach their academic goals.

As discussion is infused throughout every lesson, teachers are encouraged to get to know their students beyond their math knowledge. Students start every school year taking a Math Attitude survey to communicate where they are on their journey to be a doer of mathematics and to reflect on how math is already part of their life. Students will interview their teacher about their own math story to better understand how everyone is a doer of mathematics. Daily reflections in the Student Edition and Exit Tickets allow teachers to understand student comfort and confidence level with the content to better inform instruction. These moments of reflection additionally help create a collaborative culture where all students can feel supported and safe within the math classroom as they begin to recognize the math around them.

## Student Agency

Agency is an important aspect of supporting all students in their efforts to learn. Researchers have many different descriptions of student agency, but most definitions converge on similar ideals. Agency is the capacity for individuals to make choices; those choices are thought to be bounded by a persons' habits and beliefs as well as external structures and events (Adie, Willis, & Van der Kleij, 2018; Giddens, 1984 in Deed et al., 2014; Klemencic, 2015; Vaughn, 2019). Agency is additionally time-bound: individuals draw on their patterns, habits, and identity to set goals or outcomes, create plans or actions toward reaching those goals and evaluate how well the plan and actions are helping to meet the goals in the current context (Adie, Willis, & Van der Kleij, 2018; Poon, 2018; Klemencic, 2015).

Student agency, or agency that is directed toward learning outcome goals and academic success in school, requires an acknowledgment of the structures and practices of the classroom. However, tension can arise when a student's desire to leverage agency in their own ways diverges from classroom and school-sanctioned ways of leveraging agency (Adie, Willis, & Van der Kleij, 2018). Some research focuses on specific teaching practices that look to support the development of these sanctioned ways of enacting agency in students (Ferguson, et al., 2015; Zieser, Scholz, & Cirks, 2018) without problematizing the ways in which the sanctioned enactment of agency may cause students agentic actions to be considered negative or unhealthy. While the role of the teacher is key, as research has shown, it is important that students be "positioned as knowledgeable leaders in the classroom and teachers work alongside their students to engage in flexible and adaptive teaching. Such contexts can provide rich learning spaces for students and teachers" (Vaughn, 2019, p.13). Having open discourse around how classroom practices support academic goal setting, teachers and students become active partners in developing the learning community and sets the foundation for students to exercise their agency toward their learning goals.

## Student Agency in *Reveal Math*

In *Reveal Math*®, student agency is surfaced in these areas: metacognition, productive struggle, and social and emotional learning. Briefly, growth mindset is the belief that abilities can be improved with effort. Research has shown that students with growth mindsets outperform those with fixed mindsets. In a study that followed 373 students transitioning to seventh grade, the research team monitored their math grades over the following two years. Their analysis showed significant improvement for students with growth mindset (Blackwell, Trzesniewski & Dweck, 2007).

Practices around metacognition, productive struggle, and social and emotional learning objectives are integrated into lessons and provide teacher guidance and support in building these competencies in students. It is critical to understand teacher implementation of these competencies and the impact on student behavior and learning outcomes. The following sections describe the research on metacognition, productive struggle, and social and emotional learning in schools and math classrooms.

## Metacognition

Metacognition refers to individuals' knowledge concerning cognitive processes and regulation of these processes in relation to cognitive objectives (Desoete & De Craene, 2019; Flavell, 1976; Jin & Kim, 2018). In other words, metacognition is the process of thinking about thinking. John Dewey defined reflection or reflective thought, which is key to metacognition, as the "active, persistent and careful consideration of any belief or supposed form of knowledge in the light of the grounds that support it and the further conclusions to which it tends" (1910, p.6). He proposed that reflective thinking comes about in situations of confusion and of wonder and curiosity. Such an awareness of one's own thinking can be key to improving learning.

Indeed, strategic metacognitive engagement has been shown to aid in performance in the classroom and overall academic achievement. For example, in one study, students' problem-solving processes were qualitatively shown to be supported by engaging in metacognitive regulation—the active monitoring and controlling of cognitive processes (Jin & Kim, 2018). Students were able to help monitor and adjust each other's thinking through their conversations. As students said things like, "This makes no sense" or "I don't understand this," other students would respond with, "Let's try to think of this another way." Desoete and De Craene (2019) noted that metacognitive skills were associated with mathematical accuracy. Reflection is also linked to social-emotional learning, as students can benefit from reflecting on the thoughts, feelings, and emotional aspects of what they have learned.

Metacognition strategies can be integrated into regular classroom instruction through collaborative activities, such as students working in groups while discussing solutions to a given problem (Jin & Kim, 2018), spending quality time on and having students engage in a thorough exploration of a topic (Denton, 2011), and via formative assessments (Denton, 2011), such as verbal discussions or written evaluations in which students complete a chart to explain how they feel about their learning.

## Metacognition in Mathematics

When studying mathematics, metacognitive practices can play an important role in knowledge acquisition, retention, and application. At the conceptual development stage, when students are first encountering new ideas and skills, thinking about the relationships between their prior knowledge and new knowledge tends to help students have better conceptual understanding (Mevarech & Kramarski, 2003). According to Gray (1991), "Metacognition as a component of mathematics instruction involves active learning to help students become aware of, reflect upon, and consciously direct their thinking and problem-solving efforts" (p. 24). It is important to note here that metacognitive skill development is critical for all learners, including those with learning disabilities. For example, Desoete and De Craene (2019) found that metacognitive activities can help students with learning disabilities build computational accuracy and mathematical reasoning.

There are several practical methods that students can use to reflect on their learning and engage in overall metacognition. Verbalizing and writing the steps to solve a problem is one method that helps students reflect on, monitor, and evaluate their problem-solving abilities and strategies. This has been shown to increase conceptual understanding and provide students the opportunity to evaluate their learning (Gray, 1991; Martin et al., 2017). Another method involves writing about their thinking, which contributes to their mathematical learning (Martin, Polly, & Kissel, 2017). For example, students may write math journal entries to think about what they learned and what they might not yet understand.

## Metacognition in *Reveal Math*

The first unit in every grade is the Math Is… Unit, which helps students and teachers understand math as a set of problem-solving strategies instead of an end result. It also defines classroom norms by incorporating meaningful supports for students to take ownership of their own learning journey, such as time for daily reflection, classroom discourse, collaboration opportunities, and daily Math Is… prompts. Teachers can best support students by utilizing prompts that encourage reflection and ask students to justify their reasoning and choice of strategy or elaborate on their high-level thought processes (Booth et al., 2017; Hattie, 2017, p. 152). *Reveal Math* provides additional metacognitive prompts in both student- and teacher-facing materials. For example, Math is… Mindset activities, found at critical points throughout each lesson, encourage students to think about their own thinking in relation to the information presented or a mathematical problem. These prompts support students as they plan and set goals before solving a problem, a process that helps build key metacognitive skills. The activities also support student reflection, prompting students to bridge their prior knowledge with new information presented in the lesson. Another opportunity for daily reflection is found at the end of each lesson with Reflect prompts. Students are asked to reflect on their learning, strategies they used, and ways they may approach problems. For example, one Reflect prompt encourages students to compare approaches by asking them to describe two strategies they might use and explain which is more efficient. Additionally, a Metacognitive Check is presented at the end of each lesson as part of the formative assessment Exit Ticket, which assesses students' understanding of the lesson concepts. Here, students are prompted to "Reflect On Your Learning," which allows them to consider how well they understand the lesson content and engage in thinking about their own thinking and how they feel about their learning.

This approach allows learners greater metacognitive insight into their own thinking—connecting intuition, modeling, and conceptual representation—and is at the very heart of the mathematical practices that foster deeper mathematical learning (Hattie, 2017, p. 136). Metacognition also empowers students to drive their own learning, building from the support of a teacher's modeling and moving toward independent practice of skills and concepts. An added benefit to this approach is that when teachers use strong focusing questions, they are also modeling how to ask clarifying questions in a way that will serve students better in later phases of learning, when they ask themselves those clarifying questions.

## Productive Struggle

Research identifies productive struggle as an essential component of effective mathematics classrooms (Boaler & Dweck, 2016; Preiss & Sternberg, 2010; Warshauer, 2014) which arises when teachers include opportunities for students to attempt solving problems that target concepts that are new to them, rather than limiting those opportunities to tasks with familiar/known skills (Hattie, 2017, p. 117). Drawing on the idea that students need to engage in thinking that has some perplexity, confusion, or doubt (Dewey, 1933), Hiebert and Grouws (2007) describe productive struggle as “the intellectual effort students expend to make sense of mathematical concepts, to figure out something that is not immediately apparent” (p. 387). In a study of ninth-grade students, Kapur (2014) also defined the notion of productive failure, which, similarly to productive struggle, notes the importance of students’ failure to generate correct solutions. However, this failure, if generated through the activation of students’ prior knowledge, “can be productive in preparing them to learn better from the subsequent instruction that follows” (p. 1009).

When defining productive struggle, it is important to note what it does *not* mean. Productive struggle should not result in unnecessary frustration derived from overly difficult tasks or challenges that are not mathematically appropriate or useful (Hiebert and Grouws, 2007; Warshauer, 2014). The goal is to allow students to engage in math thinking that causes some cognitive dissonance or disequilibrium but is within the students’ current ability to reason. However, research on productive struggle points to a certain tension between what we know is best for learners and what is a natural inclination to reduce discomfort and difficulty for them (Seeley, 2016, p. 22). Teachers can perceive student struggle or delay in arriving at correct answers as a reason to “rescue” students and show them an effective procedure too soon, ultimately to the detriment of students’ learning and conceptual understanding of new mathematical ideas (NCTM 2014, p. 48). Research has shown that “students’ struggles with learning mathematics are often viewed as a problem and cast in a negative light in mathematics classrooms” (Hiebert & Wearne, 1993; Borasi, 1996), which harms both engagement and learning outcomes. Allowing productive struggle to become an integral part of learning environments encourages struggle and occasional mistakes to become a natural and positive part of progressing toward understanding.

## Importance of Tasks in Productive Struggle

One way to ensure students are engaged in an appropriate level of struggle while learning math is by selecting or creating the correct tasks. Selected or created tasks should be within students' zone of proximal development, "the distance between the actual developmental level as determined by independent problem-solving and the level of potential development as determined through problem-solving under adult guidance, or in collaboration with more capable peers" (Vygotsky, 1978, p. 86). Supporting students to think and learn at levels right above their abilities facilitates increased learning and understanding. Thus, choosing the correct tasks becomes crucial for using productive struggle as a tool to increase student learning. Cognitive demand, a framework that describes the kind of thinking processes needed to solve a task, supports teachers' efforts to choose the appropriate type of task for students (Henningsen and Stein, 1997). While engaging in tasks of all cognitive demand levels—memorization, procedures without connections, procedures with connections, and doing math—is important in math learning, opportunities to develop productive struggle are most apparent in the levels of procedures with connections and doing math. Tasks at these higher levels of cognitive demand also have other features that support appropriate struggle: the tasks afford multiple solutions, strategies, and representations and activate students' prior knowledge (Kapur, 2014). Well-designed and cognitively demanding tasks can provide students opportunities to enhance their conceptual learning and offer up a wider view of what it means to do mathematics (Henningsen & Stein, 1997; Hiebert & Grouws, 2007).

Productive struggle relies not only on the cognitive level tasks can support but also on the ways in which students are expected to participate and with whom they collaborate in the learning process. By focusing on developing rich tasks with multiple avenues for engagement (and using productive struggle as part of that), teachers provide students opportunities to exercise their agency toward learning mathematics. Gresalfi et al. (2009) describe agency not as something a student "has" or "lacks," but "the ways in which he or she acts, or refrains from acting, and the ways in which her or his actions contribute to the joint action in the group in which he or she is participating" (p. 53). In a math classroom, how students choose to participate and the learning opportunities they are afforded directly impact how students will exercise their agency toward engaging with and doing math (Sengupta-Irving, 2015). Through productive struggle, students learn to reason about math, fail and make mistakes, and debate ideas and solutions, all of which can allow the expression of student agency toward the goal of productively understanding and doing mathematics (Sengupta-Irving, 2015).

## Impact of Productive Struggle in Classrooms

While research on productive struggle is relatively new in the field, some studies show the positive impact struggle affords students in learning mathematics. Kapur (2014) conducted two randomized-control studies of high school students in India to understand the impact of allowing students to engage in problem-solving on a new topic before instruction, which the author called productive failure. While both groups—the direct instruction students and the productive failure students —gained procedural knowledge, the productive failure group gained "significantly greater conceptual understanding and ability to transfer to novel problems than those who were taught first" (Kapur, 2014, p. 1008). One explanation for why the productive failure group may have outperformed the direct instruction group in conceptual understanding is that the productive failure group had to attend to "prior knowledge activation and differentiation during the problem-solving phase, which may have helped them better notice and attend to the critical features of the concept" (Kapur, 2014, p. 1018).

Warshauer (2014) analyzed 186 episodes of middle school students' productive struggles in proportional reasoning tasks to better understand the ways students can struggle and how teachers respond and support productive struggle in students. From this work, Warshauer developed a Productive Struggle framework that maps the progress of struggle from initiation, through student and teacher interactions, to resolution. Using the framework provides a way to categorize teachers' responses to students' productive struggles by mapping how teachers "maintain the task's level of cognitive demand, address the student struggle, and build on student thinking" (Warshauer, 2014, p. 387). Warshauer found that students' struggles with tasks emerged in different ways, including difficulties getting started on the problem, not knowing what process to carry out, being uncertain about how to explain their work and math thinking, or trying to solve problems based on a misconception. How teachers responded to these struggles determined the level of rigor the task ultimately entailed. Warshauer's study showed that teachers who responded to student struggles by telling students what steps to take or by directing students through a narrowed solution procedure lowered or minimally maintained the cognitive demand of the task. However, if the teacher supported student struggle by probing their thinking and asking questions with limited intervention, the cognitive demand of the task was maintained, or in some cases, raised. This study points to the need for teachers to be thoughtful in how they encourage and respond to productive struggle in students, so the struggle can be productive and not frustrating.

### Productive Struggle in *Reveal Math*

Among the hallmarks of productive struggle in practice is the notion that student-centered activities appear in the curriculum in a way that allows them to take center stage, rather than appearing as optional or time-permitting only (Cowen, 2016). The instructional model of *Reveal Math* incorporates this idea with the Explore and Develop activity-based exploration in each lesson, essentially posing a problem to students before teaching particular methods (Boaler & Dweck, 2016, p. 81). During these activities, students are introduced to a new concept by starting with a rich task that has multiple points of entry (Hattie, 2017)—or, put differently, a task that has a "low floor and high ceiling" (Boaler & Dweck, 2016, pp. 84–85). Individual learners are encouraged to suggest strategies for working through the problem, before procedures, formulas, and new concepts are formally introduced. Critically, during the activity-based explorations, students must engage in productive struggle while working toward solutions—drawing on their intuitions and existing knowledge and taking opportunities to engage in reasoning about the nature of the problem. Seely (2016) posits, "When students have some time to explore and even struggle with a problem, our role as teacher becomes one of facilitating and stimulating conversation among students to ensure that they uncover and discuss the important mathematical ideas that lie within the problem" (p.33). This instructional method is designed to maximize engagement and set the stage for new concepts, vocabulary, and procedures later on in the lesson. The *Reveal Math* Teacher Edition also provides teachers with scaffolded questions to guide students who might feel discouraged. These purposeful questions are located at point-of-use and help teachers find ways to alleviate frustration while still allowing students to explore and find their own paths through the problem.

## Social and Emotional Learning

When children learn and teachers teach, there is more that happens than just the transfer of content knowledge and information. Schools are dynamic and social environments in which both learners and teachers continuously interact, make decisions, and adapt to new circumstances. Developing the skills to successfully navigate the school (and later, the work and community) environments is a continuous and complex process that requires careful instruction and ongoing support for positive emotional, social, and behavioral skill development. A commonly used term for the development of these specific sets of skills is **Social and Emotional Learning**, or **SEL**.

The Collaborative for Academic, Social, and Emotional Learning (CASEL), a leading organization in the field, defines SEL as "the process through which children and adults understand and manage emotions, set and achieve positive goals, feel and show empathy for others, establish and maintain positive relationships, and make responsible decisions" (CASEL, 2017). Decades of research across a wide spectrum of educational settings have demonstrated that when educators support SEL, both in and out of the classroom, the positive benefits influence not only the trajectory of a student's success during the school years but also later on in post-secondary education, the workforce, and beyond.

Additional research on the economic benefits of SEL integration into education has demonstrated that the benefits of such integration outweigh the initial investment costs, with a reported 11:1 return for every dollar spent on SEL instruction (Dusenbury & Weissberg, 2017). Meta-analyses (analyses of multiple research studies) have shown that this high return on investment is due to the significant improvements in outcomes across several factors, ranging from academic achievement to reductions in bullying to improved workforce readiness (Durlak, et al., 2011; Taylor, Oberle, Durlak, & Weissberg, 2017).

## Elements of SEL Instruction

To support SEL in the classroom, it is important to first establish what this umbrella term encompasses. A commonly referenced framework developed by CASEL (2017) categorizes SEL skills across five major competencies:

- **Self-Awareness:** the identification of one's own emotions, thoughts, strengths, and weaknesses, as well as the development of a sense of self-confidence.
- **Self-Management:** the regulation of one's own behaviors, emotions, thoughts, and motivations; this competency can additionally encompass goal-setting.
- **Social Awareness:** the ability to understand and empathize with the perspectives and norms of others, including those from backgrounds different from one's own.
- **Relationship Skills:** the skills involved in communicating clearly, listening well, cooperating with others, resisting inappropriate social pressure, negotiating conflict constructively, and seeking and offering help when needed.
- **Responsible Decision-Making:** the practice of making constructive choices about personal behavior and social interactions based on ethical standards, safety concerns, and social norms.

Teachers can support the development of these competencies through stand-alone programs as well as through SEL integration into content-area instruction. Typically, standalone programs include curricula that provide a blend of instruction and activities, such as roleplaying, strategy instruction, discussions, projects, and writing prompts (C.F. Jones, et al. 2017; Panaviotou, Humphrey, & Wigelsworth, 2019). Other approaches are designed to pair content instruction with SEL, using elements of each domain to support learning in the other, such that SEL is supported at point-of-use and provides social and skill-building opportunities alongside content learning (Jones, et al., 2017).

In both approaches, it is critical that teachers are supported by their districts as they not only learn best practices for supporting SEL in their students but also explore and develop their own social, emotional, and behavioral capacities.

## SEL in Mathematics

As children move from the elementary grades into the middle and high school years, it becomes increasingly important to ensure that their interest, attitudes, and confidence in mathematics remain high (Wigfield, et al., 2006). Attending to students' social-emotional learning, specifically in mathematics learning contexts, has been shown to help students improve their math self-efficacy and attitudes toward math (Jones, Jones, & Vermette, 2009). Jones et al. (2009) conveyed that when teachers create a socially and emotionally supportive learning environment, there is a positive impact on student attitudes, behaviors, and academic performance. An SEL-conducive climate makes space for students to work with a diverse group of individuals (DeLay et al., 2016; Jones et al., 2009) and allows time for reflection (also noted in the Metacognition section).

SEL interventions have also been shown to improve performance and achievement in mathematics (DeLay et al., 2016; Jones et al., 2009). DeLay and associates, for example, conducted a relationship-building intervention in which students participated in 21 total activities across topics such as diversity and inclusion, problem-solving, critical thinking, and peer relationships. This intervention supports social and emotional learning by strengthening more positive peer influences in the classroom, reinforcing prosocial problem-solving strategies, and encouraging the resolution of divergent views. Through this intervention, students were found to be a more socially skilled group, allowing for peers to have greater influence over their fellow students' mathematical performance improvement. Using teacher-rated report grades from the first quarter (the pre-test measure) and the fourth quarter (the post-test measure), the authors found that as students developed skills by working together with others of varying ability levels, academic advances occurred.

SEL can be incorporated into each content area, including mathematics, from an early age to help foster social and emotional skills as well as positive attitudes and self-efficacy. By implementing effective SEL interventions in daily activities, teachers allow children the opportunity to experience SEL competencies in different contexts (McClelland, Tominey, Schmitt, & Duncan, 2017). For example, mindfulness is one technique used in SEL training in which students practice focusing on their breathing and attentive listening skills to improve their executive functions (cognitive control abilities that organize, sequence, and regulate behaviors). Student participants in a mindfulness intervention—previously shown in adults to increase self-awareness, empathy, regulation of emotions, and other social and emotional skills—showed improvement in self-reported measures of well-being as well as in math performance (Schonert-Reichl et al., 2015).

The teacher should also be ready with questions to prompt student thought and discussion of both the mathematics and the social and emotional aspects of learning (Charles A. Dana Center at The University of Texas at Austin & The Collaborative for Academic, Social, and Emotional Learning, 2016). Jones et al. (2017) provide examples of ways in which social and emotional competency instruction can be incorporated into the classroom, such as discussions in pairs or small groups, writing, and drawing, as well as increasing problem-solving skills by introducing problems related to everyday life.

Many state and national standards, such as the Common Core Mathematical Practices, make connections, both directly and indirectly, between mathematical practices and social and emotional learning competencies. As an example, the mathematical practice of making sense of problems and persevering in solving them is complemented by several of the SEL competencies. By asking students to be aware of their strengths and what they know (**self-awareness**), to resist impulses and regulate their thoughts and behaviors (**self-management**), and to manage their time and energy toward a goal while appraising their work (**self-management**), students can improve their problem-solving strategies and be more successful when grappling with challenging problems.

## SEL in *Reveal Math*

Social and Emotional learning objectives are integrated into every *Reveal Math* lesson, embedding the following strategies and techniques to help teachers and students build their social and emotional competencies. For example, Math Is… Mindset prompts appear in the student and teacher materials, keeping social and emotional learning at the top of students' minds as they interact and discuss throughout the lesson.

**Integrate**—Social and emotional learning competencies are called out to support students as they build an understanding of concepts and proficiency with skills. For example, teachers are provided with questions that help guide discussions and support students' ability to justify their thinking. As another example, students are presented with exercises in which they are shown several ways fictional students solved a problem and asked to take on others' perspectives, explaining how problems can be solved in different ways.

**Instruct**—The teacher edition of *Reveal Math* presents opportunities for teachers to provide explicit guidance and instruction in SEL competencies. *Reveal Math* provides teachers with support to encourage students to understand their strengths, stay motivated, be persistent, and develop organizational thoughts and strategies.

**Reflect**—Students reflect on their learning and think metacognitively at key points in lessons. For example, students might be instructed to write about if they were feeling frustrated and how they overcame that frustration. Teachers can encourage students to think back on their learning and ask questions pertaining to how they feel about the topic and the knowledge they've obtained as part of the self-management SEL competency.

# Classroom Discourse and Language

Classroom discourse is an important avenue for learning and an underused learning strategy in mathematics classrooms. Discourse encompasses interactions between members of the community and their attempts to develop shared meanings using a variety of tools, language, and norms (Bennett, 2014; Hicks, 1995; Lampert, Rittenhouse, & Crumbaugh, 1996; Moschkovich, 2012; Sherin, 2002; Yackel & Cobb, 1996). Sherin (2002) points out that "discourse is the process of how individuals communicate" (p.206), which is an important aspect of learning. In mathematics classrooms, "discourse requires students to evaluate and interpret the perspectives, ideas, and mathematical arguments of others as well as construct valid arguments of their own" (Bennett, 2014, p.20). This focus on building shared knowledge about mathematical ideas, language, representations, and symbols so all students can participate and learn mathematics, provides strong reason for developing discursive practices in mathematics classrooms. As described by Steele and Raith (2017), "Mathematical discourse should build on and honor student thinking, provide students with opportunities to share ideas, clarify understandings, develop convincing arguments, and advance the mathematical learning of the entire class" (p.123). Building a community of learners through a deliberate focus on classroom culture and norms allows the classroom community to socialize into new ways of interacting through discourse (Bennett, 2014; Lampert & Cobb, 2003).

## Discourse Practices

What are the interactions that define classroom discourse? O'Connor, Michaels, and Chapin (2015) describe the results of a four-year study of about 500 predominantly Latinx students in grades four through six who took part in Project Challenge, a program designed to identify fourth-grade students who have the potential to excel in math and provide them with a challenging curriculum. While the study was not originally focused on classroom discussions, the researchers and teachers in the study noted that the use of talk moves appeared to play a role in the positive academic gains for the students in the study. Talk moves—teacher revoicing, student restating, agree/disagree, adding on, wait time—are defined as ways teachers use questioning and probing when supporting student discussion and discourse. The goal of the talk moves is to "get students to make their own contributions, listen to other students, keep the focus on reasoning, and work respectfully and productively with the ideas of others" (p. 112). The notion of specific practices designed to support how teachers can guide the development of classroom discourse is supported in other research. Hufferd-Ackles, Fuson, and Sherin (2004) detail the development of a math-talk learning community—"a community in which individuals assist one another's learning of mathematics by engaging in meaningful mathematical discourse" (p. 81). Extending prior research on mathematical discourse and supporting teacher change, the authors conducted a year-long study on four teachers with predominantly Latinx classrooms and the teachers' work to develop whole-class discourse practices. Through coding and classifying the data collected from the classroom observations, teacher meetings, and telephone interviews, the researchers found that the development of the math community was linked to four categories of teacher actions—questioning, explaining math thinking, sources of mathematical ideas, and responsibility for learning (p. 87). These components were developed into a **Levels of Math-Talk Learning Community Framework**, which allowed teachers to see where they were in the journey of developing math talk in their classrooms and ways that they could improve their discourse practices. Both teachers and students showed significant growth in their ability to engage in higher levels of math talk by the end of the study. Continued work with the framework in professional learning sessions has shown the framework's usefulness in scaffolding teacher change in discourse practices.

## Equity and Classroom Discourse

Classroom discourse has also been shown to support equitable classroom environments. Engaging students in math discourse is important as a lever in advancing the mathematical learning of all members of the classroom community (NCTM, 2014). Moschkovich (2012) posits, "Classroom practices that support mathematical reasoning and broaden participation provide opportunities for students to use multiple semiotic resources to participate in, combine, and value multiple mathematical discourse practices. Equitable classroom practices also honor student resources, in particular the 'repertoires of practice' among students from nondominant communities" (p. 16). Using classroom discourse to share ideas, clarify understandings, construct arguments, develop language, and learn to see the perspective of others allows all students to participate, feel safe, and be empowered to take control of their learning (NCTM, 2014, p. 29). In two separate longitudinal studies, Boaler (2002) analyzed classrooms using a curriculum designed to support equitable teaching of mathematics. These classrooms were linguistically, ethnically, and racially diverse; the study matched each classroom with a comparable classroom that used a more procedural, teacher-focused curriculum and teaching methods. The findings showed that the schools using the open-ended approach achieved significant gains in achievement and performed better than the comparison groups. Additionally, the results showed that the gains were spread across different student groups and not contained to just higher achievers or more economically advantaged students (p. 247). The studies also highlighted the importance of enacting specific teaching practices in the effort to create equitable classrooms and improve student outcomes. In these studies, three specific practices were noted as crucial to the positive outcomes found: introducing activities through discussion, teaching students to explain and justify, and making real-world context accessible (Boaler, 2002). Multiple other studies conducted in classrooms with diverse learners—defined by ethnicity, achievement levels, and English language fluency—show academic improvement as well as increased participation in classroom activities (Bennett, 2014; Clark et al., 2015; Fenema et al., 1996; O'Connor, Michaels, & Chapin, 2015; Rubenstein-Avila et al., 2015). These studies show that teachers and schools must pay attention to the ways in which students are expected and supported to participate in learning mathematics, challenge deficit views around the discursive practice students bring with them to school, and build mathematics discourse practices starting where the students are (Aquino-Sterling, Rodriguez-Valls, & Zahner, 2016; Moschkovich, 2012).

While discourse involves more than language, Wagner, Herbel-Eisenmann, and Choppin (2012) point out that "language exemplifies and creates culture, and consequently, the language of instruction privileges culture associated with that language" (p. 2). By paying close attention to discourse practices in the classroom, teachers can surface the cultural knowledge and skills that inform the ways students use language in academic talk. In mathematics, students need to contend not only with new concepts and procedures that make up the school mathematics landscape, but also the mathematics vocabulary that accompanies such learning. As students engage in the discourse of the mathematics classroom, they begin to add the formal math language into their personal and informal language. Thus, teachers must leverage students' informal language as a bridge to the more formal aspects of mathematics terminology and ideas (Lampert & Cobb, 2003). Pierce & Fontaine (2009) suggest using research-based instructional practices from reading for vocabulary development in math classrooms to support the transition from informal language to academic language. Their work describes the practice of using rich and lively tasks to encourage deep processing of words and drawing on student-friendly definitions to develop meanings of academic vocabulary with students (p.239). For English language learners, the focus on language practices and discourse is a crucial step in encouraging the academic self-efficacy of those students. Teachers can support linguistic diversity in their classroom through pedagogical language knowledge, defined as "knowledge of language directly related to disciplinary teaching and learning and situated in the particular (and multiple) contexts in which teaching and learning take place" (Bunch, 2013 in Aquino-Sterling, Rodriguez-Valls, & Zahner, 2016). Enacting pedagogical language knowledge includes developing a culture of discourse, steeped in inclusion and democracy. This culture of discourse "acknowledges and validates diverse uses of language in the classroom, while at the same time creating opportunities for students to engage in linguistic exchanges proper of formal academic and disciplinary contexts" (Aquino-Sterling, Rodriguez-Valls, & Zahner, 2016, p. 94). By making explicit the connections between how students think about formal word meanings and how they relate (or not) to students' informal definitions, teachers privilege the knowledge students bring with them to school and demonstrate how they can use that knowledge to learn new ideas and concepts. Particularly for ELL students, "a culturally and linguistically responsive approach to mathematics education within linguistically diverse classroom contexts seamlessly builds bridges between the way emergent bilingual students talk and the discipline-specific ways with words students must learn in order to engage the discipline in deeper, expert ways" (Aquino-Sterling, Rodriguez-Valls, & Zahner, 2016, p. 100).

## Classroom Discourse in Mathematics Classrooms

The research on classroom discourse in mathematics points to its importance in improving student achievement. A study that analyzed effective strategies for at-risk kindergarten and first-grade students found that a focus on developing math verbalizations—defined as the opportunities students have to express their mathematical thinking—had a large effect on student achievement (Gersten et al., 2009 in Clarke et al., 2014). Similarly, a longitudinal study of 21 first-, second-, and third-grade teachers who participated in a four-year teacher development program designed to enhance teachers' skills in understanding their student's thinking showed academic gains in student concept and problem-solving (Fenema et al., 1996). While the study could not show a direct causal relationship, specific changes in the teachers' instruction over time could account for the increase in students' achievement: creating more time and space for students to grapple with concepts and ideas and providing opportunities for students to share their thinking regularly (p. 430). Focusing on teacher practices, such as using talk moves, and the impact on student learning, O'Connor, Michaels, and Chapin (2015) found that students in classrooms that effectively used talk moves improved in a pre-posttest design using state-wide assessments more than students in a non-math talk classroom. Beyond the improved content learning, observers noted that students in the math talk classrooms became much better at listening to the teacher and their peers, building on one another's ideas, explaining their own thinking, using appropriate mathematical vocabulary, engaging in mathematical argument, and sustaining discussions of complex ideas (p. 8).

## Classroom Discourse in *Reveal Math*

Math classrooms with rich discourse offer "students a way to express their ideas, reasoning, and thinking, and [they] also support the deepening of that thinking as students must make conjectures and back up their ideas with evidence" (Seely, 2017, p. 2). The instructional design of *Reveal Math* keeps the teacher as the facilitator and encourages rich discussion, participation, and reasoning from the students from the very beginning of the lesson. Every lesson includes a Number Routine as a warm-up activity. Designed to build students' proficiency with number and number sense, Number Routines provide teachers with prompts that encourage students to talk about their reasoning for solving unknown problems. Additionally, every lesson launches with a Be Curious activity that consists of a sense-making routine. Designed to develop students' ability to make sense of a situation, the activity is presented as a classroom discussion where students engage in collaborative conversations to connect and apply mathematics.

Furthermore, teachers lead whole-group discussions to connect concepts to strategies and procedures through examples and discussion during the **Explore and Develop** instructional moment. Regardless of the **Explore and Develop** pathway chosen—Guided or **Activity-Based Exploration**—students discuss and relate multiple representations, strategies, and procedures when solving problems. To support this discourse, ample resources are provided and meaningfully integrated into the lesson through Think About It and Math Is… Mindset prompts, which support student reflection and discussion. These are additional opportunities for students to engage in meaningful discourse as they learn new procedures and vocabulary while connecting and applying skills and strategies.

To promote effective self-questioning or collaboration, teachers can best support students by utilizing prompts that encourage reflection and ask students to justify their reasoning or choice of strategy—or at a minimum, to elaborate on their thought process (Booth et al., 2017; Hattie, 2017, p. 152). *Reveal Math* provides Math Is… prompts in both student- and teacher-facing materials. This approach allows learners greater metacognitive insight into their own thinking—connecting intuition, model, and conceptual representation—and is at the very heart of the mathematical practices that foster deep learning (Hattie, 2017, p. 136). Further, it aligns with a movement away from the teacher as the sole authority in the classroom, and towards a more effective and engaging mode of student-driven learning. Additionally, the instructional design of *Reveal Math* integrates all eight of NCTM's Effective Mathematics Teaching Practices throughout the lesson, providing teachers with open-ended questions to support meaningful classroom discussions.

*Reveal Math* was developed around the belief that mathematics is not just a series of operations, but a way of communicating and a way of thinking. Teachers will find language supports embedded at the unit and lesson levels to help all students build a shared language and communicate effectively about math. For example, the Language of Math prompts promote the development of key vocabulary terms that support how we talk about and think about math in the context of the lesson content. For English language learners, supporting their engagement in classroom discourse practices is crucial to their success in mathematics. McGraw Hill is committed to providing English learners appropriate support as they simultaneously learn content and language. In *Reveal Math*, English learners are supported through **English Learner Scaffolds**, which provide teachers with point-of-use practices that offer specialized instruction for EL students. These scaffolds help EL students cultivate meaning of math vocabulary as well as ideas and concepts in context while creating space for students to interact with math language through speaking, listening, reading, and writing. Additionally, English Learner Scaffolds offer three levels of scaffolding practices—Entering/Emerging, Developing/Expanding, and Bridging/Reaching—so teachers can provide the right level of support for each student.

# Sense-Making

The work of a mathematician is to solve problems. Key steps in solving problems include understanding the problem context, making sense of the issue at hand, and then finding or creating the tools needed to solve the problem. Developing these problem-solving skills is an important part of school mathematics. In particular, the idea of sense-making in mathematics problem-solving presents an interesting challenge to what we know about school mathematics. School mathematics is full of standards, skills, and facts to know; while these are fundamental to learning mathematics, being able to reason with these standards, skills, and facts moves them to being useful and usable and is at the heart of learning mathematics (Ball & Bass, 2003).

What does it mean to make sense of mathematics? Schoenfeld (1992) describes what it means to think mathematically as "…developing a mathematical point of view… and developing competence with tools of the trade, and using those tools in the service of the goal of understanding structure" (p.1). Put another way, "Understanding is the key to becoming a mathematician. Understanding what a problem is asking, understanding how to come up with a strategy to solve the problem, understanding enough to write or draw in detail how to solve the problem… is crucial to becoming a competent and confident mathematician" (Ostrow, 1999, p. 4). It is important for students to make sense of mathematical ideas themselves as they work toward mathematical proficiency. However, Ball and Bass (2003) point out that "making sense refers to making mathematical ideas sensible, or perceptible, and allows for understanding based only on personal conviction" (p. 29). Individual sense-making is just a first step in developing mathematical understanding and reasoning; opening up individual thinking to discussion and critique in a community of learners allows for the development of a collective set of practices and norms that is the backbone of mathematical reasoning (Yackel & Hanna, 2003). The mathematics classroom, then, becomes the community of practice within which students "develop the appropriate mathematical habits and dispositions of interpretation and sense-making" (Schoenfeld, 1992, p.13).

## Developing Sense-Making in the Classroom

How can teachers create a community of practice steeped in the idea that sense-making is crucial to learning mathematics with understanding? Research has shown practices teachers can employ that support the building of sense-making and reasoning in a classroom community. In a 2011 study, Mueller, Yankelewitz, and Maher examined two subsets of students from two larger studies—one study of sixth-grade students in an informal after-school math program and the second study from a longitudinal study of fourth- and fifth-grade students investigating counting and combinatorics—to find common features that led to improved student disposition toward mathematics.

The findings suggest that developing a classroom environment where students share their ideas, listen and respond to each other, and justify their thinking and solutions led to more positive dispositions around mathematics, which in turn led to stronger mathematical reasoning and understanding. These practices, or social norms, describe specific participation interactions needed to create a mathematics community in the classrooms that can support mathematical sense-making and reasoning. However, socio-mathematical norms are even more important as they focus more specifically on the development of students' mathematical ideas and mathematical thinking (Kazemi & Stipek, 2001; Yackel & Cobb, 1996).

In a study of fourth- and fifth-grade teachers and students, Kazemi and Stipek (2001) found that while all teachers enacted social norms that created a safe classroom environment for mathematical thinking and learning, only two of the teachers employed socio-mathematical norms in ways that supported deeper sense-making and reasoning by students in their efforts to solve problems. In those classrooms, students were required to use mathematical thinking, not just procedural steps, to justify answers, demonstrate an understanding of the relationship between strategies, use errors to reconceptualize a problem, and work collaboratively to understand problems through arguments and mathematical justifications (Kazemi & Stipek, 2001, p. 78). By focusing on more specific socio-mathematical practices and norms, teachers can create a community of practice that immerses students in math classrooms designed to make sense-making and reasoning a consistent and regular part of learning mathematics. In a fraction sense intervention study focused on understanding and reasoning about fraction meanings and relationships, students in the intervention outperformed control students on measures of fraction magnitude, fraction concepts, and fraction arithmetic, demonstrating the improved academic achievement of students engaged in sense-making as a regular part of the mathematics learning (Dyson et al., 2018).

A longitudinal study by Francisco and Maher (2005) highlighted the importance of enacting certain conditions, which coincide with the idea of socio-mathematical norms, that promote mathematical reasoning. The study followed 80 students, ranging from first-graders to postgraduates, over the course of three to 18 years using individual problem-solving sessions, interviews, and questionnaires to analyze how to promote sense-making and reasoning in problem-solving. An important finding in the study was the idea of student ownership of their mathematical activity. Privileging student ideas, thinking, and representations during learning tasks led to increased success in problem-solving over the long term. This focus on students building knowledge about mathematical ideas, rather than relying on the teacher to provide those ideas, helped learners develop a sense of responsibility to understand problems for themselves.

Also, providing students with complex tasks and problems that stimulate their sense-making and reasoning is shown to help build "durable mathematics knowledge" (Francisco & Maher, 2005, p. 371). Another important condition for increasing student sense-making and reasoning is developing the skill of justification. Encouraging students to provide mathematical support for their thinking, or to explain why their methods work, should be an integral part of problem-solving as it "promotes the building of personally meaningful arguments and ways of articulating them" (Francisco & Maher, 2005, p.368).

## Sense-Making in *Reveal Math*

*Reveal Math* supports teachers in their efforts to develop a classroom designed to allow students to make sense of problems and develop problem-solving skills. Drawing on the work from *Reveal Math* author Annie Fetter, *Reveal Math* incorporates sense-making routines into every lesson launch. Be Curious launches every lesson and is designed to encourage students' curiosity and ideas while they observe a situation, problem, or phenomenon. Students apply previously learned problem-solving strategies or knowledge to make sense of the problem at hand or to wonder about how they may approach the situation. Built to respect and welcome all ideas, the exercise permits students to discuss what they notice in the problem and what they don't know or understand. The focus is to engage the classroom community in making sense of the problem and context and encourage curiosity about the mathematics. In order to be curious, notice, and wonder about math problems, Be Curious moments in *Reveal Math* employ four practices, suggested by Fetter, to encourage sense-making: Get rid of the question; get rid of the question and the numbers; give students the answer; ask about ideas, not answers. These practices allow students to find entry into problems, connecting the knowledge they currently hold to the problem-solving discussion. Students have the opportunity to understand what they know and what they still need to know. Engaging in Be Curious moments as a consistent part of math teaching and learning goes a long way toward developing a mathematics community that supports thinking, reasoning, and communicating and sets the stage for using sense-making throughout all components of the *Reveal Math* program.

# Fluency

There is a good deal of research on mathematical fluency explaining what it is and why it is important for students' mathematical learning. While there is some theoretical debate in defining what fluency means and how it can be measured, Biancarosa and Shanley (2016) state that it should be treated as "a holistic description of a skilled performance" (p. 14). In other words, it is not one specific skill, nor is it simply about speed. Baroody (2011, as cited in Clarke, Nelson, & Shanley, 2016) defined fluency as the quick, accurate recall of facts and procedures, and the ability to use them efficiently (p. 71). Carr et al. (2011) similarly describe it as both the retrieval of math facts as well as the ability to quickly compute answers to more complex problems.

The repeated themes in fluency research seem to relate to accuracy and speed (Rhymer, Dittmer, Skinner, & Jackson, 2000) as well as efficiency. Efficiency and speed are somewhat related in that, as students develop and use more efficient strategies to solve problems, they are likely to increase their speed. Thus, for the purposes of this paper, fluency refers to the accuracy and speed at which a student computes a mathematical computation.

Fluency, not to be confused with automaticity, involves the application of automatic computation. For example, multi-digit addition or long division requires the application of memorized computations while fluently carrying out the procedure (Hasselbring & Bausch, 2017). This means that although automatic recall of math facts is important, students must also be able to quickly and accurately conduct procedures to be fluent in more complex mathematical computations. In this way, fluency also takes into account the relationships among conceptual understanding, procedural knowledge, and basic fact recall (Clarke, Nelson, & Shanley, 2016).

## Importance of Fluency

Empirical literature on fluency in the elementary grades has shown many benefits to students increasing their computation fluency. One such benefit is that fluency supports mathematical proficiency (Clarke, Nelson, & Shanley, 2016). "When students are fluent in computation, they are more likely to develop the number sense that underlies more complex mathematics problem-solving" (Gersten & Chard, 1999, as cited in Carr, Taasoobshirazi, Stroud, & Royer, 2011). Fluency also frees up working memory that can be used for higher-order activities (Carr et al., 2011; Clarke, Nelson, & Shanley, 2016; Hasselbring & Bausch, 2017; Ramos-Christian, Schleser, & Varn, 2008). When the load on working memory is reduced, students have a greater capacity to think about and solve more complex problems and tasks.

By supporting proficiency and creating the means for students to conduct higher-order tasks, fluency has been shown to predict greater mathematics achievement (Carr et al., 2011). According to Greene, Tiernan, and Holloway (2018), students who have greater fluency are better able to retain and maintain what they have learned, are more able to focus on a task even when distractions arise, and are more able to apply their learning to new contexts. Inversely, students who lack fluency have been shown to have persisting mathematical difficulties.

How can teachers support students' fluency? Practice, practice, practice. Practice is key. High levels of performance can be attributed to deliberate practice, which differs from drill and practice exercises. Deliberate practice involves activities that are designed to improve performance, have set objectives just above one's current level of competence, provide feedback, and involve repetition. A teacher or coach should guide students in correctly learning and practicing a skill (Hasselbring & Bausch, 2017, p. 58). Effective practice strategies include incremental rehearsal, such as flashcards-based activities; practice sessions with modeling; and self-management strategies, such as cover, copy, and compare. These approaches have been shown to be productive in supporting students' mathematical fluency (Clarke, Nelson, & Shanley, 2016).

Feedback is also an important aspect of fluency practice and support. Providing immediate feedback (rather than after students have completed an entire practice session) prevents students from practicing incorrect responses (Berrett & Carter, 2018; Rhymer et al., 2000). Upon receiving feedback, students are able to take corrective measures and then continue practicing with this improved understanding. Peer-tutoring has also been shown to improve fluency by individualizing support based on students' specific needs while also providing feedback and maintaining engagement in the task (Greene et al., 2018). Students who engage in peer mentoring benefit by being both the mentee and the mentor. Peer mentors can also help students find efficient strategies for more quickly solving problems. Teachers should then help students to connect their own strategies and methods to more efficient procedures (NCTM, 2014).

Computer-based interventions have been shown to improve fluency (Carr et al., 2011). Research demonstrates that computer-assisted instruction (CAI), when used appropriately, can function as an effective supplementary tool "by providing opportunities for added practice and by differentiating the educational experience of each child. Other advantages of incorporating CAI in the classroom include immediate feedback, automated progress monitoring and adaptive instruction, increased engagement, and high accessibility" (Berrett & Carter, 2018, p. 226).

### Fluency in *Reveal Math*

Each lesson in *Reveal Math* begins with a Number Routine. These short activities are designed to help students activate their prior knowledge and to practice skills that will be needed for the new mathematical content. Often, these activities include problems that aid in increasing students' computational accuracy and speed.

At the end of each lesson, students are provided practice problems. By completing these problems and receiving immediate feedback, students can work on their speed and accuracy with computations and become more efficient with corresponding mathematical tasks. Additionally, engaging Digital Games and Spiral Review practice offer further opportunities to build fluency while providing immediate feedback.

At the end of each unit, students are provided "Fluency Practice." These pages contain the following sections:

**Fluency Strategy:** Students are presented a strategy to help them recall their prior learning. They are asked questions related to the strategy, which help with conceptual understanding of the strategy and computational skill.

**Fluency Flash:** Students are provided one or two quick problems that may involve mathematical models and asked to write or solve a problem. These types of problems also aid in developing conceptual understanding.

**Fluency Check:** Students complete practice problems designed to increase speed and accuracy with specific computational skills.

**Fluency Talk:** Students are given a prompt and space to write about their strategies and explain their thinking.

## Instructional Routines

Most teachers establish classroom routines during the first few days of school. These routines can help students understand expected behaviors and reduce the cognitive demands of learning new concepts (Leinhardt, Weidman, & Hammond, 1987). Well-practiced and understood classroom routines allow students to maintain focus on their learning without diverting attention to the more general rules and activities (Lampert, Beasley, Ghousseini, Kazemi, & Franke, 2010; Leinhardt et al., 1987). As defined by Leinhardt and colleagues (1987), "Routines... are fluid, paired, scripted segments of behavior that help movement toward a shared goal. Routines can have explicit descriptors, can be modeled or, more commonly, can simply evolve through shared exchange of cues" (p. 136). When implementing routines, it is critical that students are aware of and involved in the learning process, with clear roles and expectations (Bulgren & Scanlon, 1998).

These can be placed into categories of routines, including management, instructional support, and teacher-student exchange (Leinhardt & Steele, 2005).

Of importance to curriculum programs are routines that support instruction. According to Yinger (1979), “Instructional routines are methods and procedures established by the teacher to carry out specific instructional moves” (p. 166). Instructional moves are steps a teacher takes when conducting and carrying out activities. Yinger gives examples such as, “giving instructions, questioning, presenting information, monitoring, evaluating student performance, and offering feedback” (p. 165). Critical aspects of instructional routines include frequency of use, closeness to classroom practice, positive impacts to the learning of all students, and the ability to teach the routines in multiple settings (Hiebert & Morris, 2012). For example, several different routines can be used to support instructional dialogue, a practice in which “an explanation is co-constructed by the teacher and students in the class during an instructional conversation” (Lampert et al. 2010, p. 131). In their work with instructional dialogue, Leinhardt and Steele (2005) used what they call “exchange” routines to help when explanations are necessary. These include the call-on routine, the revise routine, and the clarification routine. The call-on routine involves an “open invitation to discussion” (2005, p. 143), often followed by prompts to further explain, clarify, or give other comments. The revise routine can be coupled with the call-on routine, wherein students can revise or expand on previous statements. The clarification routine could be used when there is confusion around a topic or idea. Each of these routines has its own established rules so that every student can feel protected and willing to share and contribute to the dialogue. The ability to facilitate meaningful discussions using routines takes training and understanding by the teachers.

## Pedagogical Content Knowledge and Instructional Routines

As teachers employ routines, they will need to draw on a deep understanding of content knowledge and pedagogical strategies of their subject matter to choose effective instructional moves. Combining these two bodies of knowledge, Shulman (1986) characterized the idea of pedagogical content knowledge (PCK) as the “blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented and adapted to the diverse interests and abilities of learners, and presented for instruction” (p. 8). Ball and Forzani (2010), in their work of defining what makes a skillful teacher, also describe a related concept called specialized expertise, the ability to unpack a method or idea in a way that makes it accessible to the students, as an important aspect of teaching all students effectively. Studies have shown that students in classrooms with teachers having higher proficiency in PCK performed significantly better on large-scale assessments than students whose teacher was not as strong in PCK (Hill, Rowan, & Ball, 2005; Knauss, Baumert & Blum, 2008).

Knauss, Baumert, and Blum (2008) also found that PCK had stronger effects on student learning and outcomes than simply a strong understanding of the content alone.

Pedagogical content knowledge allows teachers to become fluent in the instructional moves that make up instructional routines, thus providing the structure and organization that support student learning. For instructional routines, teachers choose and use effective instructional moves that make those routines effective, such as uncovering content misconceptions in student ideas; connecting ideas between content concepts; breaking apart difficult topics or ideas and scaffolding them for students; and drawing on student ideas, partial understandings, and conjectures to further learning (Ball and Forzani, 2010; Frey & Fisher, 2010; Hiebert & Morris, 2012; Hill, Rowan & Ball, 2005).

## Instructional Routines in Mathematics

While some routines maintain classroom behaviors, others may be more subject- or content-specific. Here, PCK plays an important role, as Berry (2018) notes when he describes routines known to support the development of mathematical proficiency, which include conceptual understanding, strategic competence, adaptive reasoning, productive disposition, and procedural fluency. He states,

> If the goal in mathematics teaching and learning is to support student success with mathematical proficiency, then we must be explicit about using instructional routines that focus on student engagement in activities that support reasoning and sensemaking, communication with and about mathematical ideas, making meaningful connections, building procedural fluency from conceptual understanding, and productive struggle.

## Instructional Routines in *Reveal Math*

*Reveal Math* provides three types of instructional routines throughout the program: Sense-Making Routines, Number Routines, and Math Language Routines. By establishing these routines early on and adapting them as students progress to the next grade level, teachers can help reduce cognitive load, and students can focus on their mathematical thinking and learning. The following are some of the instructional routines used throughout the program:

**Sense-Making Routines:** As previously shown in the Sense-Making section of this paper, in order to become problem solvers in mathematics, students must understand the problem context, make sense of the issue at hand, and then find the tools needed to solve the problem. As such, *Reveal Math* includes the following sense-making routines. While each routine has essential components and purposes, the descriptions below focus more on the teacher's role in carrying out these routines.

- **Notice and Wonder™**—Developed at the Math Forum, the Notice and Wonder routine has teachers present students with an image or problem scenario without any questions, data, or answers to the problem. Students write or draw the things they notice and the things they wonder. The teacher can then engage students in a class discussion about the things they notice and wonder, and record their comments on the board. The teacher should allow for students' comments to be non-mathematical in nature but should eventually want to steer the conversation to focus on mathematical wonderings.
- **Numberless Word Problems**—This routine requires students to look for relationships among the objects given in the problem and to talk with others about the things they notice. It begins by presenting a problem or image without any numbers. Teachers serve an important role in guiding students in this routine as they may be confused by the lack of numbers. The teacher should help widen student thinking about problems, beyond solving and numbers, and help students make connections to other students' thinking and strategies.
- **Which Doesn't Belong?**—In this routine, students look for similarities and differences among numbers, images, or terms, and determine which one doesn't belong with the others in the group. The teacher begins by presenting three to no more than six numbers or images with attributes, such as color or size, then gives students time to think about the similarities and differences and determine which one doesn't belong. The teacher should encourage students to find more than one solution.
- **Is It Always True?**—In this routine, students are presented with one or more images or situations and think about the relationships among the objects in the image. Students consider whether the relationships always hold true or whether they are unique to this particular image or situation.

**Number Routines:** The purpose of number routines is to develop a better sense of numbers and how they function. Without a solid foundation in number sense, it is difficult to learn and understand geometry and statistics, for example. Using number routines, students are able to make more sense of math rather than following rigid sets of rules (Shumway, 2011). *Reveal Math* includes the following number routines:

- **About How Much?**—Students build estimating skills by explaining their strategies and then comparing and analyzing their estimates to the actual value.
- **Break Apart/Decompose It**—Students build flexibility with numbers by decomposing them, sharing their thinking, and discussing patterns.
- **Can You Make the Number?**—Students build flexibility and efficiency with operations by building expressions with a value for the given target number.

- **Find the Pattern, Make a Pattern**—Students build efficiency by determining the rules for a given pattern, then continue the pattern or create a new pattern.
- **Find the Missing Values**—Students build their identification of patterns and efficiency with solving equations by analyzing a series of equations, looking for patterns, and finding missing values.
- **Greater Than or Less Than**—Students build place value sense, estimations skills, and comparing skills by estimating or evaluating the value of an expression and comparing it to a target benchmark number.
- **Let's Count**—Students build proficiency with skip counting by counting forward or backward using a given counting interval.
- **Mystery Number**—Students build mathematical reasoning and thinking by looking at clues one at a time, proposing possible solutions, and eliminating solutions that are no longer viable.
- **What Did You See?**—Students build visual discrimination, quantitative reasoning, and mathematical discourse by viewing images and then describing and discussing what they saw.
- **What's Another Way to Write It?**—Students build number sense by writing alternative expressions to a given expression and looking at relationships among the different expressions.
- **Where Does It Go?**—Students build estimating skills by placing a target number on a number line and justify their reasoning.
- **Which Benchmark Is It Closest To?**—Students enhance rounding and reasoning skills by determining which benchmark a given number is closest to and explaining their reasoning.
- **Would You Rather?**—Students build number sense and enhance decision-making by choosing between two options, both of which require mental math, and giving the rationale for their choice.

Kindergarten also includes the following number routines: Counting Things, Start and Stop, The Counting Path, and The Match.

**Math Language Routines**—Mathematical language routines (MLRs) are structured but adaptable formats for amplifying, assessing, and developing students' mathematical language. These routines were developed by the Stanford University UL/SCALE team:

- **Stronger and Clearer Each Time**—Students revise and refine their ideas and their verbal and written output.
- **Collect and Display**—Students' oral words and phrases are collected into a reference for them to refer to later.
- **Critique, Correct, and Clarify**—Students analyze, reflect on, and develop a piece of writing that is not their own.
- **Information Gap**—Students communicate with partners or team members to convey missing pieces of necessary information.
- **Co-Craft Questions and Problems**—Students use conversation skills to generate, choose, and improve questions and problems before producing answers.
- **Three Reads**—Students reflect on the ways mathematical questions are presented.
- **Compare and Connect**—Students identify, compare, and contrast different mathematical approaches, representations, concepts, examples, and language.
- **Discussion Supports**—Students have supported discussions about mathematical ideas, representations, contexts, and strategies.

# Conclusion

The foundation on which *Reveal Math* has been built reflects decades of work by educators, mathematicians, and researchers. Based on that work, the entire *Reveal Math* program is designed to spark curiosity, make connections among math concepts, build and support mathematical communication, encourage collaboration, and instill confidence in students.

A strong foundation begins with creating and supporting an equitable classroom. This involves classroom experiences designed to meet the specific needs of each and every learner. Furthermore, the integration of social and emotional learning into the mathematics curriculum can also lead to greater learning outcomes. Part of social and emotional learning involves the act of reflecting on one's own learning and thoughts. Opportunities for students to engage in metacognition are woven throughout the program at key moments as well as at the beginning and end of each lesson. Supports for students to engage in classroom discourse are included at key points in the curriculum as well.

A strong mathematical foundation begins with research in the field of mathematics education. This is also imperative for student success. Mathematical sense-making, using problem-solving skills to make sense of problems and situations, is a focus of the *Reveal Math* program. Students also have opportunities to engage in productive struggle as they explore and develop new mathematical concepts. Furthermore, to support new learning and concept development, students need to have a solid footing in accurately completing mathematical operations and procedures. The ability to quickly and efficiently execute these procedures provides students greater cognitive capacity to focus on new ideas. As such, each unit in each grade includes fluency practice.

Lastly, teachers are given supports throughout the *Reveal Math* Teacher Edition with included instructional routines. These are explained and described in the Math Is… Unit at the beginning of the course and then utilized throughout each course. This includes routines related to pedagogical content knowledge. Predictability in classroom and instructional routines allows students to focus on new learning, which can lead to greater success.

The *Reveal Math* program is designed to encourage and support teachers and students in their daily mathematical routines. Through the use of this math curriculum, students can succeed in developing life skills that will translate not only in their educational careers but throughout their lives.

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