NUMBER WORLDS®

Research-Proven Math Intervention for Grades PreK–8

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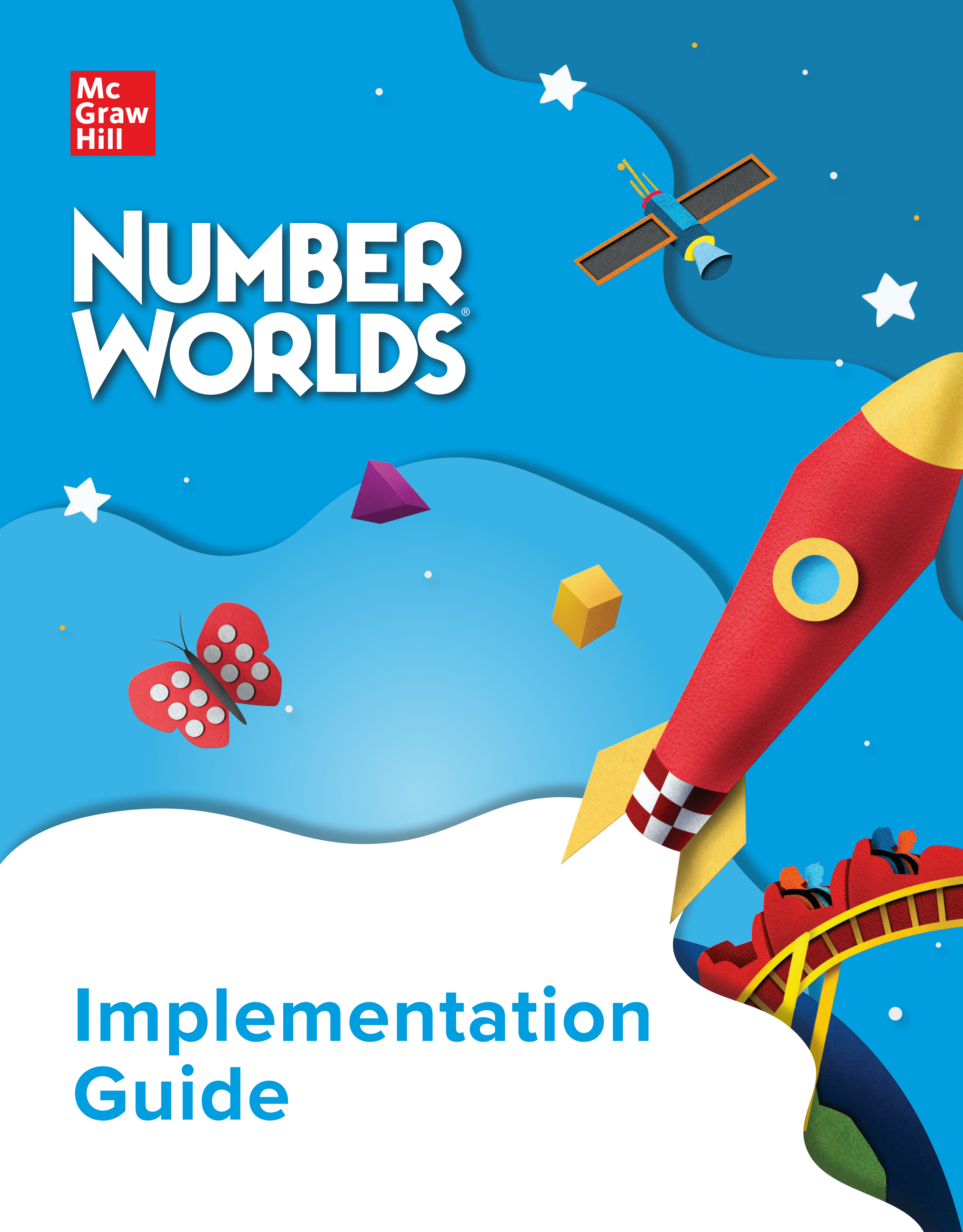
NUMBER WORLDS®



Implementation Guide

Table of Contents

Program Preview

Number Worlds® is a comprehensive math prevention/intervention program, proven to bring struggling students up to grade-level performance and beyond. *Number Worlds*©2024 retains the strengths of earlier editions, while enhancing the program's ability to help all students master mathematical standards. *Number Worlds* accelerates mathematical understanding through five researched-based instructional principles: (1) build upon children's current knowledge; (2) follow natural developmental progressions when selecting new knowledge to teach; (3) teach computational fluency and conceptual understanding; (4) provide plenty of opportunity for hands-on exploration, problem-solving, and communication; and (5) expose students to major ways we represent and talk about numbers. Based on these principles, the curriculum can effectively close achievement gaps for students in grades PreK to 8.

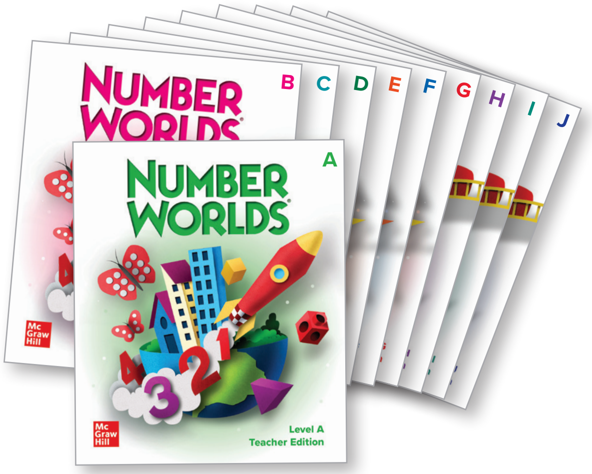
In *Number Worlds,* learners are actively engaged in developing computational fluency and conceptual understanding by building skills and mastering strategies through a variety of interactive learning experiences designed to bring fun into the math classroom.

- Hands-on learning and digital games help students apply concepts in a variety of contexts.
- Purposeful activities bring real-world scenarios into every lesson, making math more tangible for students.
- Meaningful discussions promote a deeper understanding of math concepts.
- Project-Based Learning materials challenge students to extend their learning beyond simple practice and application, encouraging critical thinking and preparing students to take more responsibility for their own learning.
- Differentiated instruction and alternative grouping options provide the tools needed to tailor instruction to the needs of each classroom and student.

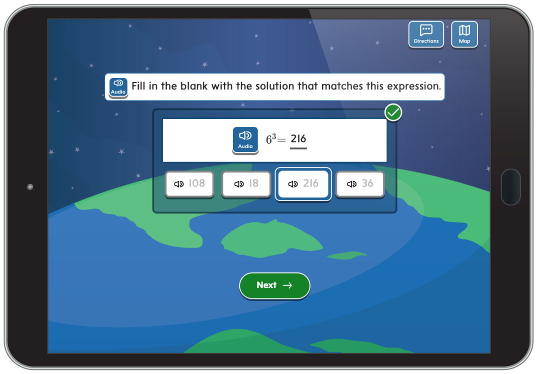
Digital resources, including virtual manipulatives, digital math tools, and *Building Blocks* Adaptive make the research-based instruction even more powerful. Online assessments, progress monitoring, and reporting all make tracking student progress and accelerating math success even easier.

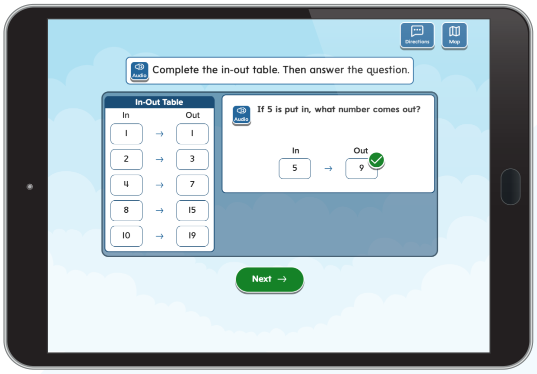
The easy-to-use placement and assessment resources help pinpoint where students should begin the curriculum, and allow teachers to monitor at-risk students. Through identification and development of key math concepts, *Number Worlds* accelerates learning and student achievement. Weekly and unit tests, along with informal assessments, ensure accurate progress monitoring, and help teachers move students into the regular math curriculum.

Prevention Levels (A–C) focus on foundations in number sense to support students who start school with a weak conceptual background in mathematics. Intervention Levels (D–J) help intervention students unlock mathematical concepts by focusing on key standards at each grade level (2–8) and the supporting foundational skills and concepts needed to access standards.



Building Blocks Adaptive is a digital component that uses a variety of activities to develop early mathematical knowledge in children. Children will use *Building Blocks* as an independent practice opportunity to help make connections between informal mathematical knowledge and formal mathematical concepts.





A Program Designed for Students to Succeed

Number Worlds was designed with prevention levels (A–C) and intervention levels (D–J) to support Grades PreK–8.

Prevention: Levels A–C

Prepare PreK–1 students with foundational skills and concepts necessary to be successful with more complex mathematics in the future. Each Prevention Level consists of 32 weeks of daily instruction including lessons on time and money.

Level A Building Foundations for Grade PreK	Level B Grade K Key Standards	Level C Grade 1 Key Standards
Students acquire well-developed counting and quantity schemas.	Students develop a well-consolidated central conceptual structure for single-digit numbers.	Students link their central conceptual structure of the number to the formal number system.

Intervention: Levels D–J

Help students in Grades 2 through 8 learn the foundational skills and concepts needed to master key mathematical standards. Designed for flexibility, units can be taught in any order or in isolation with placement tests to help identify student needs.

	Level D Grade 2	Level E Grade 3	Level F Grade 4	Level G Grade 5	Level H Grade 6	Level I Grade 7	Level J Grade 8
Unit 1	Number Sense within 100	Number Sense	Number Sense	Number Sense	Number Sense	Number Sense	Number Sense
Unit 2	Number Sense to 1,000	Addition	Addition & Subtraction	Multiplication & Division	Operations Sense	Operations Sense	Operations Sense
Unit 3	Addition	Subtraction	Multiplication	Operations with Decimals	Algebra	Algebra	Algebra
Unit 4	Subtraction	Multiplication & Division	Division	Operations with Fractions	Statistical Analysis	Statistical Analysis	Statistical Analysis
Unit 5	Geometry & Measurement	Geometry & Measurement	Geometry & Measurement	Geometry & Measurement	Geometry & Measurement	Geometry & Measurement	Geometry & Measurement

Prepare to Meet Mathematical Standards

Standards identify the knowledge students should attain and the concepts students should understand at each grade level. Domains are used to group related Math Standards. For Grades K–8, the standards are organized into 4 or 5 domains at each grade level.

Domains

Grades K–5	• Counting & Cardinality • Operations & Algebraic Thinking • Number & Operations in Base Ten	• Number & Operations—Fractions • Measurement & Data • Geometry
Grades 6–8	• Ratios & Proportional Relationships • The Number System • Expressions & Equations	• Functions • Geometry • Statistics & Probability

Engage Your Students

Throughout the *Number Worlds* program students are engaged in activities that employ mathematical thinking. These standards define the ways in which all students should engage in mathematics as they progress through a mathematics curriculum.

The activities in *Number Worlds* keep students engaged as they employ the Mathematical Practices. Assessment types range from paper-and-pencil tests to online assessments with auto grading where appropriate, games, and unit projects based on real-world problems.

Engage with Variety

A hallmark of *Number Worlds* is that students who participate in the program become more and more engaged in mathematics. When students are engaged in learning, they will be more successful.

Engage is the heart of the daily routines that comprise *Number Worlds* lesson instruction. Instead of a predictable pattern of activity in math class of studying a model-worked example and practicing that model, **Engage** offers a variety of instructional techniques that are appropriate for the concept being taught and appeal to a variety of learners.

Interactive activities:

- Hands-On Activities
- Games
- Digital Activities
- Math eTools
- Guided Discussions
- Free Response Activities
- Writing Prompts
- Paper-and-Pencil Activities

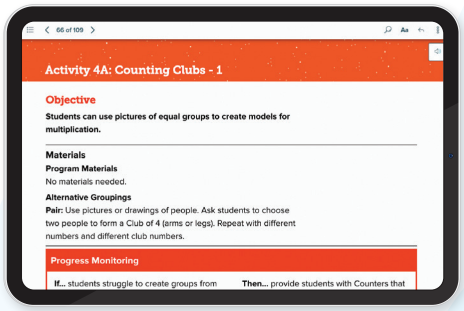
Engage through Collaboration and Higher-Order Thinking

A major focus for mathematical success is the ability to communicate mathematical ideas. Mathematically proficient students use language and symbols precisely to explain solutions, justify conclusions, and critique the reasoning of others. *Number Worlds* provides students with daily opportunities to communicate mathematical ideas and engage in higher-order thinking.

The Teacher Edition and instructional Activity Cards suggest alternative groupings that support oral communication, as well as discussion questions that promote higher-order thinking. The Reflect questions in the Student Workbook (Interactive and Print) provide students with an opportunity to communicate about mathematical ideas in writing, and challenge students to gain deeper understandings of lesson concepts.

Number Worlds weaves problem solving throughout each lesson week. Students are encouraged to find their own strategies to solve problems, and to work collaboratively to develop additional problem-solving strategies. Throughout *Number Worlds,* students develop, compare, contrast, and evaluate their own efforts to solve problems.

Through Project-Based Learning activities, students take this one step further, applying and translating mathematical skills and strategies in many different contexts. These activities encourage learners to take a deeper ownership of the concepts while, at the same time, require students to become thinkers and communicators of mathematics. This ongoing flow of ideas within the classroom is a powerful practice for ensuring student mastery of mathematics.



Engage with Digital Experiences

The *Number Worlds* program includes online resources for you and your students. Digital resources are designed to make classroom management effective and efficient, as well as make instruction fun and engaging.

The teacher experience for *Number Worlds* allows for instant access to instructional lessons, supporting resources (e.g., eTools), and assessments. Additionally, teachers can easily view reports for assessments and *Building Blocks* as well as student work in interactive workbooks.

Digital resources for students include digital versions of workbooks, game boards, playing cards, manipulatives, Math eTools, and *Building Blocks* Adaptive activities. The table below outlines the Math eTools that are available for you and your students to explore concepts and solve problems.

Math eTools

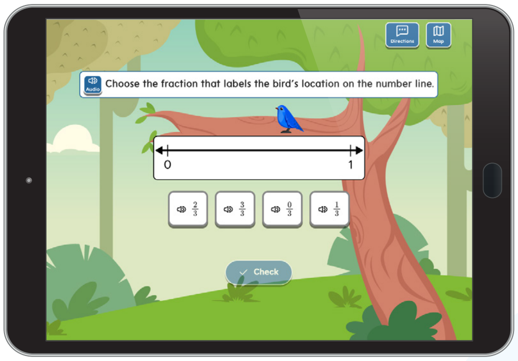
Calculation and Counting Tools	100 Table, Addition Tables, Arrays, Base-10 Blocks, Calculator, Coins and Money, Fractions, Function Machine, Multiplication Tables, Number Line, Number Stairs, Sets Former
Data Organization and Display Tools	Coordinate Grid, Graphing and Spreadsheet, Pictograph, Venn diagram
Geometric Exploration Tools	Geoboard, Geometry Sketch, Net Tool, Pythagorean Theorem, Shape Creator, Shape Tool, Tessellation
Measurement and Conversion Tools	Calendar, Double Pan Balance, Estimating Proportions, Metric and Customary Conversion, Pitcher, Stopwatch, Thermometer
Probability Tools	Coin Flip, Probability Tool

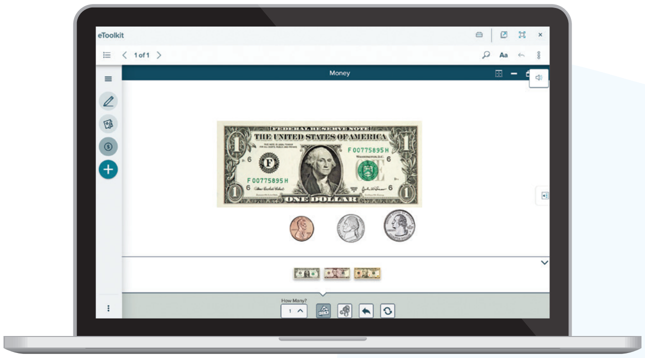
Engage with All-New Digital Content

When students get excited about math, nothing can hold them back. With *Number Worlds,* you can build that excitement no matter how much acceleration your students need.

This program is packed with an all-new Interactive Student Workbook, eTool Kit, *Building Blocks* Adaptive practice, and games to accommodate a range of learning styles.

> *Students are much more confident; they are excited about math; their participation has increased; and their behavior has improved—because they understand the material.*
>
> **–Math Interventionist, Ladue School District, Missouri**





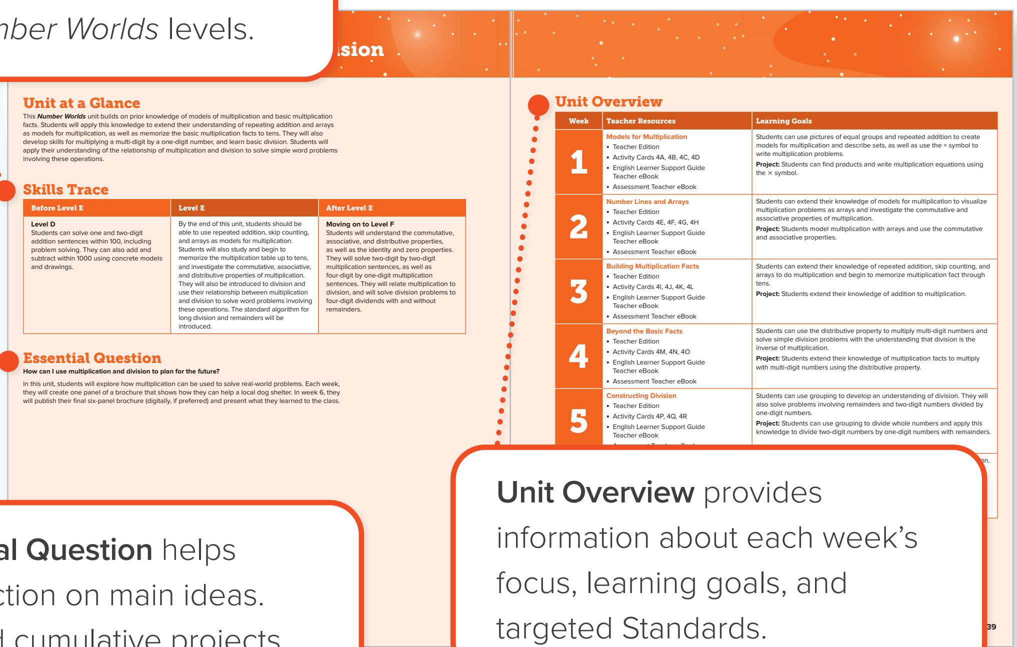
Engage with Effective, Efficient Planning

Lesson plans, digital or print, provide an overview of the focus of each lesson as well as the resources and materials needed for instruction. Lessons can be aligned to your teaching time. There are 30-minute, 45-minute, and 60-minute instructional options.

Unit Overviews and weekly planners can be used to plan lessons and prepare learning materials for students. Unit Overviews indicate the instructional focus of each week, and a summary of the learning goals. Unit Overviews also list Activity Cards needed for each week.

Each Unit Overview also contains a summary of key mathematical concepts and a Skills Trace. Each Skills Trace describes skills learned in the previous level and how the skills learned in the unit will be used in the next level. This feature ensures coherence of instruction as students progress through the math curriculum.

Skills Trace gives an overview of the skills and concepts students will explore in the unit, and describes how unit topics connect to similar topics in earlier and later *Number Worlds* levels.



The Essential Question helps focus instruction on main ideas. Lessons and cumulative projects are developed around each unit's Essential Question.

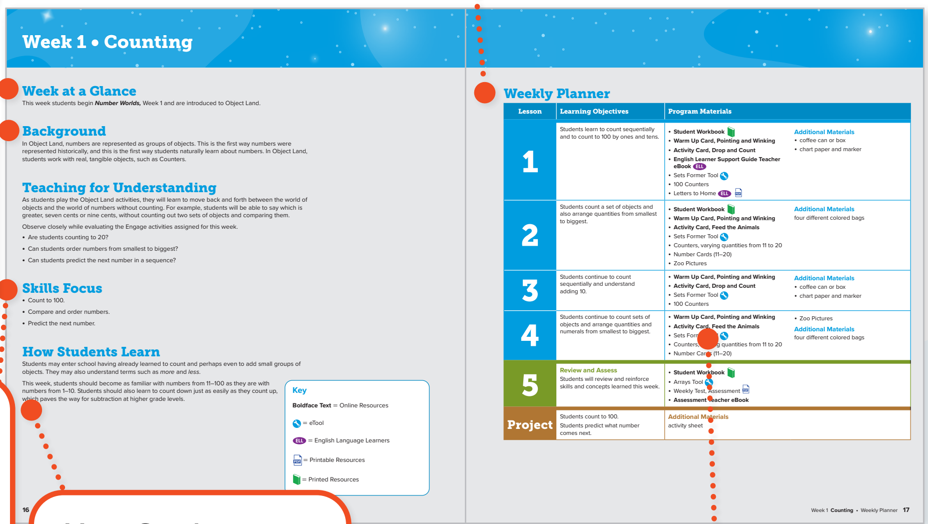
Unit Overview provides information about each week's focus, learning goals, and targeted Standards.

Weekly Planners provide background information for key mathematical concepts and offer lesson-by-lesson overviews of the learning goals that are the focus of each lesson. In addition to listing the Activity Cards, Student Workbook pages, Practice pages, and Assessment pages needed for instruction, the Weekly Planners indicate additional materials that will be used and technology that supports and enhances lessons. Each weekly planner also directs you to a Math at Home activity.

In the Teacher Editon, Lesson Plans indicate each lesson's objective, standard, and vocabulary. Each Lesson Plan also provides a 4-part lesson structure outlining that day's instruction and offers suggestions for creating context for English Learners.

Weekly Planner lists objectives that explain how the key concepts are developed lesson by lesson and which resources can be used in each lesson.

Week at a Glance gives an overview of the week's goals.



Background gives teachers math context for the lesson.

Skills Focus gives a snapshot of the weekly learning objectives.

How Students Learn provides a refresher of the mathematics principles relevant to the unit.

eTools are developmentally appropriate digital manipulatives used to demonstrate, explore, and build specific skills and concepts.

Engage by Using the 4-Part Lesson Structure

Every lesson in *Number Worlds* is organized in the same way. Lessons begin with an overview of the day's objective and with the materials needed to facilitate the day's activities.

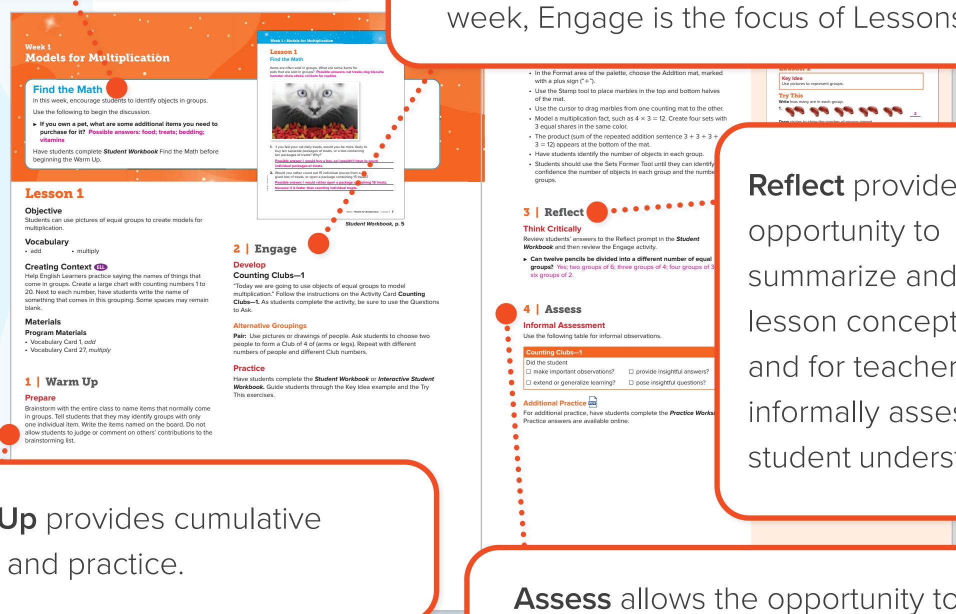
In Levels D–J, each lesson week begins with an opportunity for students to connect mathematics with real-world experiences. In Find the Math, students respond to questions based on a real-world context that relates to the week's mathematical focus.

At all levels, lesson instruction is divided into four distinct sections for simplified time management in the classroom: **Warm Up, Engage, Reflect,** and **Assess.**

Both the responsive digital Teacher's Edition and the print Teacher's Edition offer point-of-use suggestions to differentiate instruction, ask effective questions, and form collaborative groups. Strategies are provided to monitor student progress during instruction and assess end-of-the lesson understanding.

Find the Math helps students relate math concepts to real-world situations with a weekly discussion.

Engage is the heart of the lesson instruction and provides instructional models, hands-on activities, discussion, and strategy-building exercises. Each week, Engage is the focus of Lessons 1–4.



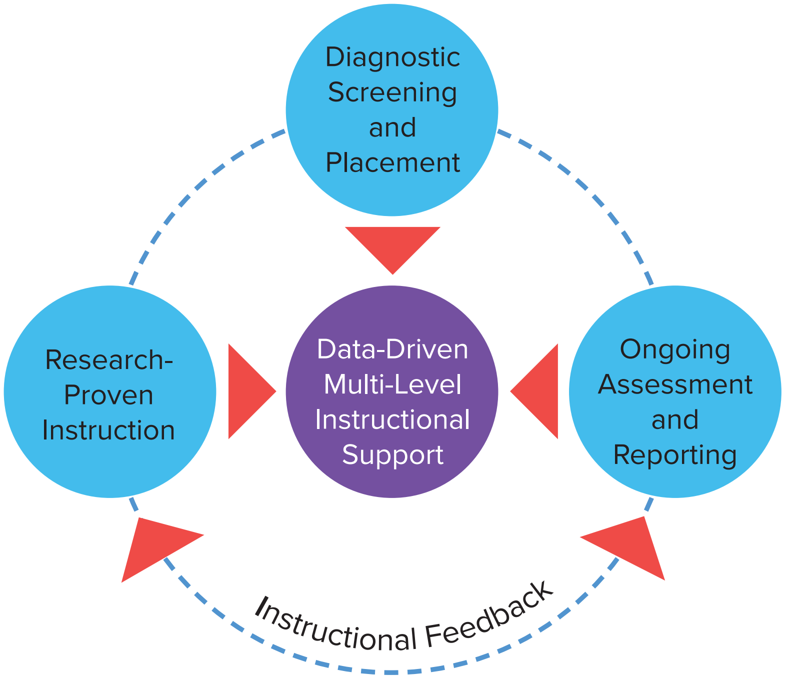
Reflect provides opportunity to summarize and apply lesson concepts and for teachers to informally assess student understanding.

Warm Up provides cumulative review and practice.

Assess allows the opportunity to formally assess student progress. Assess is the focus of Lesson 5.

Engage through Intervention

An intervention is any instructional or practice activity designed to help students who are not making adequate progress. *Number Worlds* is designed for both Tier 2 and Tier 3 students. Levels A–C are also appropriate for on-level Tier 1 students.



Layers of intervention are organized in tiers to enable teachers to respond to student needs. Tier 2 students may spend a brief time in the program learning a key concept and then be quickly reintegrated back into the core instructional program. Students in Tier 3 will most likely need to complete the entire *Number Worlds* curriculum at their learning levels. Tier 3 requires intensive intervention for students with low skills and a sustained lack of adequate progress within Tiers 1 and 2.

Number Worlds mathematics intervention offers flexible implementation options, instruction built on the math standards, engaging resources, and assessment and reporting for placed students that differentiate instruction and measure progress toward learning goals and standards.

Engage with Flexible Implementation Options

The 4-part structure of each lesson allows for flexible implementation to meet your students' needs. The table below gives suggested time allocations for 30-minute, 45-minute, and 60-minute instructional periods. A 60-minute instructional period is recommended for Tier 3 students.

Engage is the heart of Lessons 1–4, and **Assess** is the focus of Lesson 5. **Engage** and **Assess** should receive the greatest allocation of time on Days 1–4 and Day 5, respectively.

If You Have 30 Minutes			Days 1–4	Day 5 Assessment and Project-Based Learning
	Engage	Develop	15	
		Practice	15	
	Assess	Practice		15
		Conduct Formal Assessment		15

If You Have 45 Minutes			Days 1–4	Day 5 Assessment and Project-Based Learning
	Engage	Develop	15	
		Practice	15	
		Supported Differentiation	15	
	Assess	Practice		15
		Conduct Formal Assessment		15
	Project-Based Learning			15–20

If You Have 60+ Minutes			Days 1–4	Day 5 Assessment and Project-Based Learning
	Warm Up	Prepare	5	5
	Engage	Develop	15	
		Practice	15	
		Supported Differentiation	15	
	Reflect	Critical Thinking	5	5
		Real-World Application	5	5
	Assess	Practice		15
		Conduct Formal Assessment		15
	Project-Based Learning			15–20

Because of *Number Worlds'* hands-on, engaging activities, the program can be used effectively in a variety of instructional settings. The table below offers suggestions for program use in situations ranging from whole class instruction to tutoring and summer school settings.

Situation	Suggested Use
Whole Class	Levels A–C, in particular, are designed for whole-class settings, but all levels can be used for whole-class instruction.
Math Intervention	*Number Worlds* is effective in small group sessions with a teacher or paraprofessional.
Special Education	Students with special needs benefit from *Number Worlds'* intensive, concrete developmental progression of abstract concepts.
After School	Because the activities are engaging, *Number Worlds'* intensive math activities work well for both intervention and regular education students in after-school programs.
Summer School	*Number Worlds* is ideal for mathematics summer school.
Tutoring	*Number Worlds* is highly effective when used in an individualized tutoring session in which a teacher or a paraprofessional participates in activities.

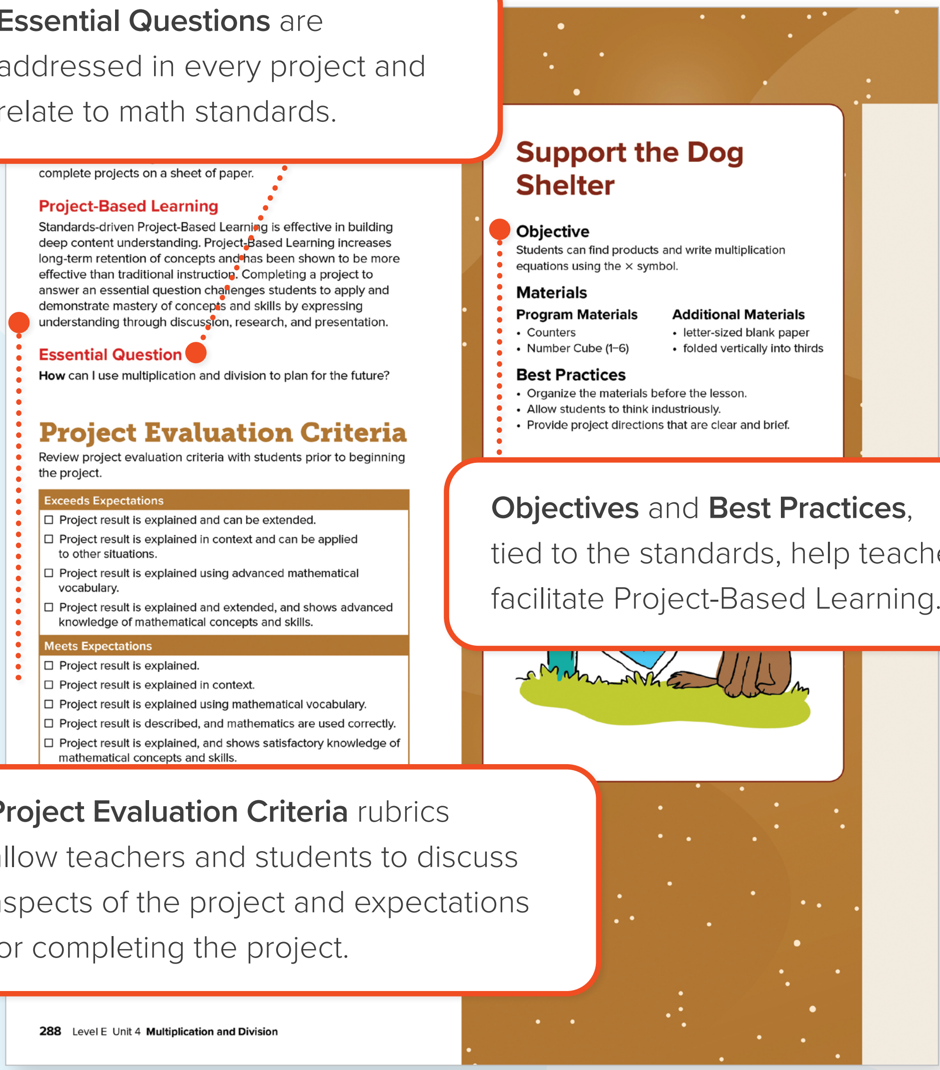
Assessment

Assess with Project-Based Learning

Project-Based Learning, or PBL, is an instructional approach in which students work collaboratively on a real-world based problem or challenge. As students explore real-world situations, group activities are facilitated by a teacher. At the conclusion of an investigation, students demonstrate mastery of concepts and skills in a variety of ways, such as creating displays, making presentations, and creating models.

Project-Based Learning has been shown to increase student motivation and enhance student engagement. In addition, because students are engaging in authentic learning, Project-Based Learning increases long-term retention of concepts and has been shown to be more effective than traditional instruction.

Essential Questions are addressed in every project and relate to math standards.



complete projects on a sheet of paper.

Project-Based Learning

Standards-driven Project-Based Learning is effective in building deep content understanding. Project-Based Learning increases long-term retention of concepts and has been shown to be more effective than traditional instruction. Completing a project to answer an essential question challenges students to apply and demonstrate mastery of concepts and skills by expressing understanding through discussion, research, and presentation.

Essential Question

How can I use multiplication and division to plan for the future?

Project Evaluation Criteria

Review project evaluation criteria with students prior to beginning the project.

Exceeds Expectations

- ☐ Project result is explained and can be extended.
- ☐ Project result is explained in context and can be applied to other situations.
- ☐ Project result is explained using advanced mathematical vocabulary.
- ☐ Project result is explained and extended, and shows advanced knowledge of mathematical concepts and skills.

Meets Expectations

- ☐ Project result is explained.
- ☐ Project result is explained in context.
- ☐ Project result is explained using mathematical vocabulary.
- ☐ Project result is described, and mathematics are used correctly.
- ☐ Project result is explained, and shows satisfactory knowledge of mathematical concepts and skills.

288 Level E Unit 4 **Multiplication and Division**

Support the Dog Shelter

Objective

Students can find products and write multiplication equations using the × symbol.

Materials

Program Materials	Additional Materials
• Counters	• letter-sized blank paper
• Number Cube (1–6)	• folded vertically into thirds

Best Practices

- Organize the materials before the lesson.
- Allow students to think industriously.
- Provide project directions that are clear and brief.

Objectives and **Best Practices**, tied to the standards, help teachers facilitate Project-Based Learning.

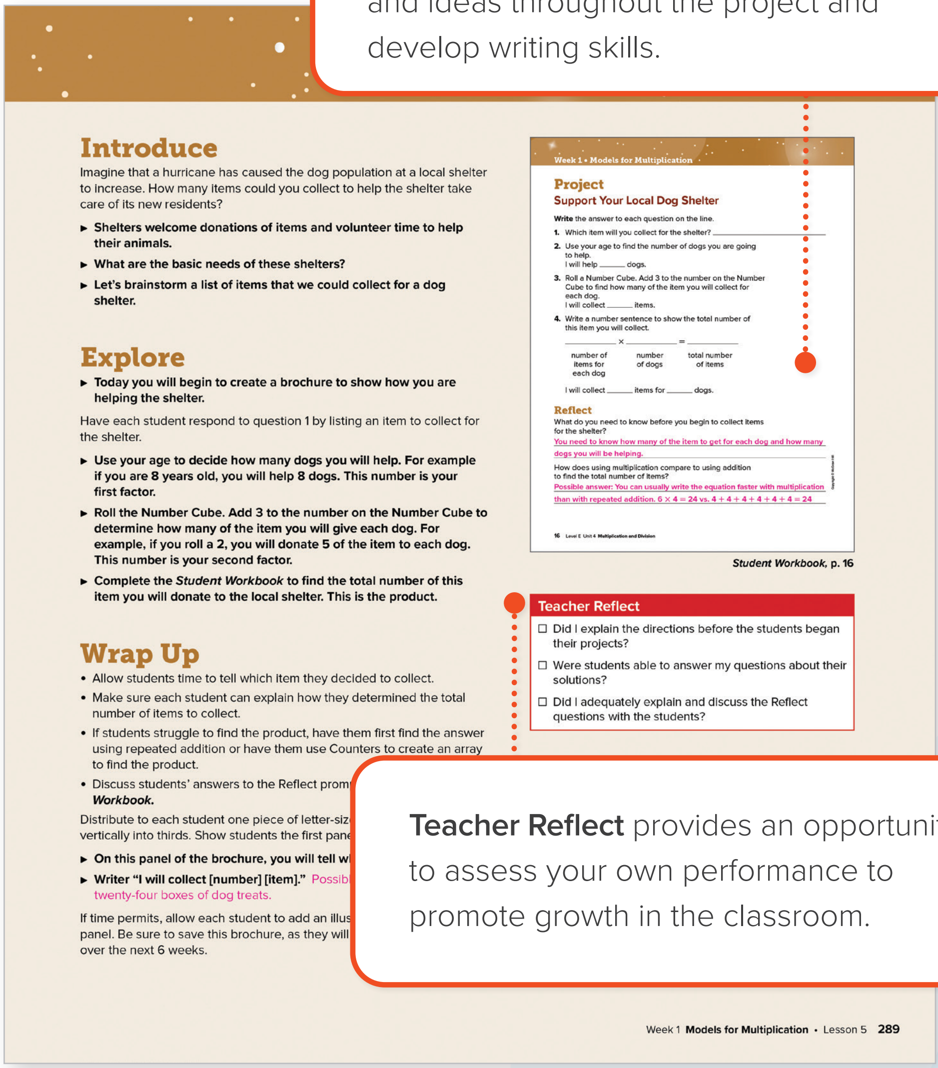
Project Evaluation Criteria rubrics allow teachers and students to discuss aspects of the project and expectations for completing the project.

Number Worlds' lessons are developed around Essential Questions, which can be found on the Project pages for Levels A–C and on the Unit Opener pages for Levels D–J. In addition, each Lesson 1 in Levels D–J begins with a Find the Math real-world connection. Essential Questions and Find the Math features are first links to *Number Worlds* projects.

In Levels A–C, students participate in activities for three weeks that build to a cumulative project at the end of every fourth week. In Levels D–J, students participate in weekly activities that culminate in Unit Projects.

For each project, the Teacher Edition provides guiding questions and directions you can use to facilitate student explorations. It also provides a rubric to evaluate student projects.

Student Workbooks provide a place for students to communicate thoughts and ideas throughout the project and develop writing skills.



Introduce

Imagine that a hurricane has caused the dog population at a local shelter to increase. How many items could you collect to help the shelter take care of its new residents?

- ▶ **Shelters welcome donations of items and volunteer time to help their animals.**
- ▶ **What are the basic needs of these shelters?**
- ▶ **Let's brainstorm a list of items that we could collect for a dog shelter.**

Explore

- ▶ **Today you will begin to create a brochure to show how you are helping the shelter.**

Have each student respond to question 1 by listing an item to collect for the shelter.

- ▶ **Use your age to decide how many dogs you will help. For example if you are 8 years old, you will help 8 dogs. This number is your first factor.**
- ▶ **Roll the Number Cube. Add 3 to the number on the Number Cube to determine how many of the item you will give each dog. For example, if you roll a 2, you will donate 5 of the item to each dog. This number is your second factor.**
- ▶ **Complete the *Student Workbook* to find the total number of this item you will donate to the local shelter. This is the product.**

Wrap Up

- Allow students time to tell which item they decided to collect.
- Make sure each student can explain how they determined the total number of items to collect.
- If students struggle to find the product, have them first find the answer using repeated addition or have them use Counters to create an array to find the product.
- Discuss students' answers to the Reflect prom[…] ***Workbook.***

Distribute to each student one piece of letter-siz[…] vertically into thirds. Show students the first pan[…]

- ▶ **On this panel of the brochure, you will tell w[…]**
- ▶ **Writer "I will collect [number] [item]."** Possib[…] twenty-four boxes of dog treats.

If time permits, allow each student to add an illus[…] panel. Be sure to save this brochure, as they will […] over the next 6 weeks.

Week 1 • Models for Multiplication

Project
Support Your Local Dog Shelter

Write the answer to each question on the line.

1. Which item will you collect for the shelter? ______
2. Use your age to find the number of dogs you are going to help.
   I will help ______ dogs.
3. Roll a Number Cube. Add 3 to the number on the Number Cube to find how many of the item you will collect for each dog.
   I will collect ______ items.
4. Write a number sentence to show the total number of this item you will collect.
   ______ × ______ = ______
   number of items for each dog / number of dogs / total number of items
   I will collect ______ items for ______ dogs.

Reflect

What do you need to know before you begin to collect items for the shelter?
You need to know how many of the item to get for each dog and how many dogs you will be helping.

How does using multiplication compare to using addition to find the total number of items?
Possible answer: You can usually write the equation faster with multiplication than with repeated addition. $6 \times 4 = 24$ vs. $4 + 4 + 4 + 4 + 4 + 4 = 24$

16 Level E Unit 4 Multiplication and Division

Student Workbook, p. 16

Teacher Reflect

- ☐ Did I explain the directions before the students began their projects?
- ☐ Were students able to answer my questions about their solutions?
- ☐ Did I adequately explain and discuss the Reflect questions with the students?

Week 1 **Models for Multiplication** • Lesson 5 **289**

Teacher Reflect provides an opportunity to assess your own performance to promote growth in the classroom.

Assess to Meet Students' Needs

There are two placement tests that can be used to place students into the appropriate *Number Worlds* program level: the Number Knowledge Test and the Placement Test. The Number Knowledge Test can be used as an initial screener in Grades PreK to 1 to place students in one of the three prevention levels. The Number Knowledge Test is a validated, research-based test that assesses students' basic understanding of number sense and operations. The test is separated into four sections, with a minimum score for each section that students need to reach or exceed in order to move on to the next section. It is an oral test which is given individually and takes about 5 to 10 minutes to complete. Once a student completes the test or fails to reach the maximum score for one of the sections, a raw score conversation chart can be used to place the student in Levels A–G.

The *Number Worlds* Placement Test can be used for student placement in any of the program levels. Much like the Number Knowledge Test, the Placement Test is separated into sections. Each section represents a program level, with each section containing a maximum of 10 questions. A student's raw score on a section will either place that student in that specific level or indicate that another section of the test should be completed to determine an appropriate placement. If a score is too low or too high, a student should take the section before or after the section just completed. Although this is a manually scored test, multiple students can be given the Placement Test at the same time, even if students begin the test at different sections.

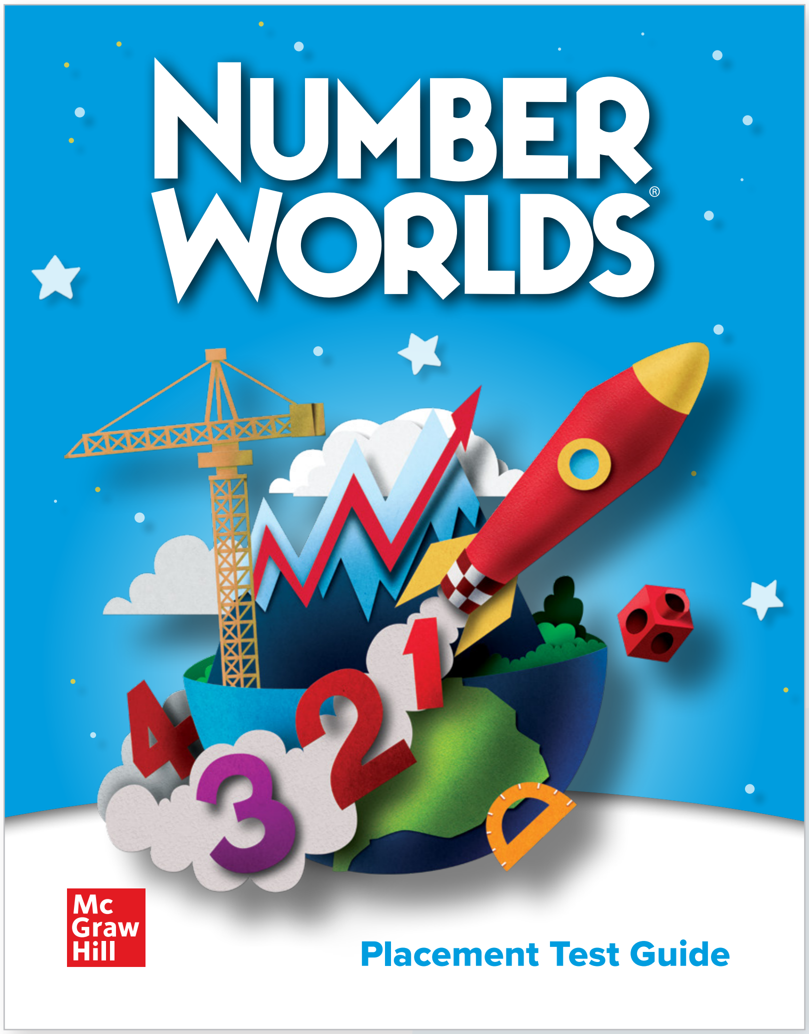
The Number Knowledge Test's Developmental Conversion Chart

Raw Test Score	Developmental Age Score (Chronological Age Equivalents)	Grade Level Equivalents	*Number Worlds* Level
1–6	3–4 years	Preschool	Level A
7–8	4–5 years	PreK–K	Level B
9–14	5–6 years	K–1	Level C
15–19	6–7 years	1–2	Level D
20–25	7–8 years	2–3	Level E
26–28	8–9 years	3–4	Level F
29–30	9–10 years	4–5	Level G

After using the Placement Test to identify students' levels within *Number Worlds,* teachers can assign a unit pretest to determine students' baseline measures for the selected unit. When students have completed the unit, teachers can evaluate students' mastery by assigning the posttest.

Although items in each unit's pretest and posttest differ, the tests are identical in structure and content. Available online, both assessments are multiple choice tests. Both are scored automatically, and teachers may view students' results in the Reporting application of the teacher experience.

Unlike the *Number Worlds* Placement Test, the pretests are not placement tests; instead, they focus on specific topics at particular levels. Finally, the posttests are valuable progress-monitoring tools, especially when teachers use them in conjunction with other data sources, such as **Student Workbook** and **Practice Book** pages, Weekly Tests, Projects, and in-class observations.



Assess to Monitor Progress

Each Prevention level contains weekly and cumulative assessments. For each level, there are 32 weekly assessments and eight cumulative assessments, available in digital and print versions. Assessments contain a wide variety of question types that reflect concepts learned in each week. If digital assessments are assigned, scores will be recorded in the Reports section of the Teacher Experience. If print assessments are used, scores can be manually entered and accessed in the Reports application.

Each Intervention level contains weekly assessments, unit pretests, and unit posttests. For each level, there are 30 weekly assessments, five unit pretests, and five unit posttests. All of these assessments are offered both digitally and in print. These assessments have the same assignment and scoring features as the Prevention level assessments. The unit pretests can also be effectively used to place students in Tier 2, module-based learning. Using the program in this way allows for concentrated, topic-based instruction.

Assess with Digital Resources

Number Worlds also has a report section that offers student, class, school- and district-reports. The class reports allow teachers and administrators to see how an entire class is performing on specific skills and topics. The student reports allow teachers to see which skills an individual student has mastered and where each student can improve. There are also administrator reports for users at the school and district levels.

Get Started with Program Features

Unit Overviews and Weekly Planners in the **Teacher Edition** provide general information about each unit's big ideas as well as the details needed for daily instruction. The **Teacher Edition** includes specific information about academic vocabulary terms, lesson resources, instructional activities, and strategies for differentiating instruction. In addition, the section called Creating Context offers strategies for supporting English Learners.

All lessons in Levels A–C, and Lessons 2–5 in Levels D–J, begin with **Warm Up,** activities that review and practice previously learned concepts and skills. In Levels D-J, Lesson 1 of each week begins with Find the Math and then the **Warm Up.** Find the Math presents a real-world connection for the lesson's content in context. After students reply to the guided discussion question, they answer several related questions in the **Student Workbook** (print or digital).

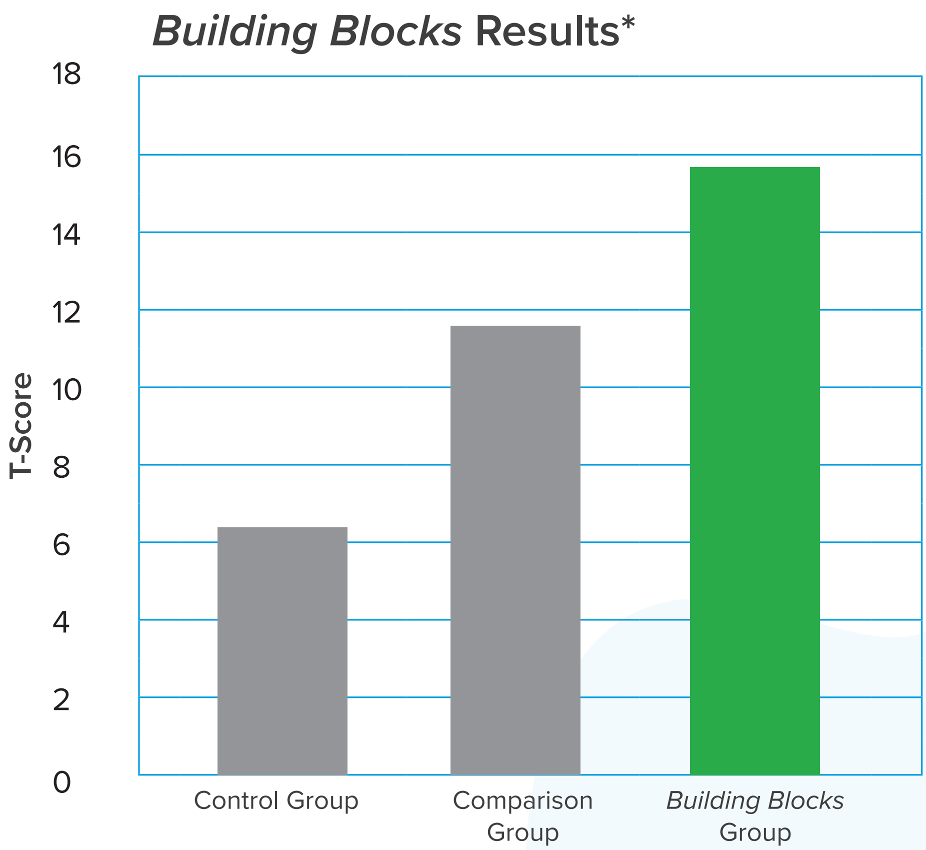
Engage is the heart of Lessons 1–4 of each week. During **Engage,** students participate in hands-on activities, discussions, and strategy-building exercises that provide a firm foundation on targeted skills. To help teachers introduce and facilitate meaningful activities and discussions, the program includes Activity Cards. These cards offer explicit instruction on how to introduce, demonstrate, practice, and assess activities. Progress monitoring appears on each card via an If-Then recommendation to provide immediate remediation when needed.

Following **Engage,** the **Reflect** section provides opportunities for students to summarize and apply lesson concepts and engage in critical thinking. Students explain their thinking in multiple ways including discussion, drawing, writing, and modeling with manipulatives. The **Reflect** section provides opportunities for students to summarize and apply lesson concepts. Students answer a critical thinking question in multiple ways, such as drawing, discussion, writing, or modeling with manipulatives. If students are in a level with a workbook, the **Reflect** section will be in the workbook pages for that day's lessons.

For Lessons 1–4 of each week, **Assess** provides several tools to evaluate student competency and progress, and to record informal observations. Students in need of additional practice can complete the lesson's **Practice** page. Each week, **Assess** is the core section of Lesson 5 and offers both digital and print formal assessment options. Weekly test results can be used to further differentiate instruction for students.

A National Science Foundation-funded study on randomly assigned classrooms tested *Building Blocks* against a comparable math program and a no-treatment control group. Students using *Building Blocks* significantly outperformed both the comparison group and the control group. In these research studies, *Building Blocks* was shown to increase young children's knowledge of multiple essential skills.

Another study tests *Building Blocks* against a comparable preschool math program and a no-treatment control group. All classrooms were randomly assigned; the gold standard of scientific evaluation. *Building Blocks* significantly outperformed both the comparison group and control group. Results indicate strong positive effects with achievement gains near or exceeding those recorded for individual tutoring.



*Source: Clements, Douglas H., Julie Sarama, and Ann-Marie DiBiase eds. *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education.* Mahwah, NJ: Lawrence Erlbaum Associates, 2004.

Meet the Authors

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During the past 25 years, Dr. Griffin has received several research awards to study learning and achievement among at-risk children, to teach number sense, and to enable mathematics teachers to enhance their students' learning and achievement.

Dr. Griffin is the author of *Number Worlds*, a PreK to grade 8 Prevention-Intervention mathematics curriculum, and the co-author of the book, *What Develops in Emotional Development?* She is also the author of numerous articles on the development of number sense, and in the fields of cognitive development, emotional development, and mathematics education.

Dr. Griffin has also served on several national and international advisory boards on projects designed to enhance the cognitive, mathematical, and language development of children, from birth through the elementary school years. As a member of the Mathematical Sciences Education Board at the National Academies of Science (NAS) as well as the Center of Education Research and Innovation at the Organization for Economic Collaboration and Development (OECD), she helped shape the direction of education research and policy for the United States, Canada, the United Kingdom, and several European countries.

Dr. Douglas Clements, Kennedy Endowed Chair in Early Childhood Learning and Professor at the University of Denver, is widely regarded as "the major scholar" in the field of early childhood mathematics education, with equal relevance to the academy, to the classroom, and to the educational policy arena. At the national level, his contributions have led to the development of new mathematics curricula, teaching approaches, teacher training initiatives, and models of "scaling up" interventions. His contributions have also had a tremendous impact on educational planning and policy, particularly in the area of mathematical literacy and access. Most recently, Dr. Clements was selected to sit on the National Research Council Committee on Science of Children Birth to Age 8: Deepening and Broadening the Foundation for Success for The National Academies of Sciences' Institute of Medicine.

Dr. Julie Sarama, Kennedy Endowed Chair in Innovative Learning Technologies and Professor at the University of Denver, conducts research on young children's development of mathematical concepts and competencies, the implementation and scale-up of educational reform, professional development models and their influence on student learning, and the implementation and effects of software environments in mathematics classrooms. She has been Principal or Co-Principal Investigator on seven projects funded by the National Science Foundation (NSF), and she is Principal Investigator on her latest NSF award, entitled, *"Early Childhood Education in the Context of Mathematics, Science, and Literacy."* Dr. Sarama is currently co-directing three large-scale studies funded by the U.S. Education Department's Institute of Educational Studies (IES).

A Mathematics Program for Young Children

What is number sense? We all know number sense when we see it but, if asked to define what it is and what it consists of, most of us, including the teachers among us, would have a much more difficult time. Yet this is precisely what we need to know to teach number sense effectively. Consider the answers three kindergarten children provide when asked the following question from the Number Knowledge Test (Griffin & Case, 1997): “Which is bigger: seven or nine?”

Brie responds quickly, saying “Nine.” When asked how she figured it out, she says, “Well, you go, ‘seven’ (pause) ‘eight’, ‘nine’ (putting up two fingers while saying the last two numbers). That means nine has two more than seven. So it’s bigger.”

Leah says, hesitantly, “Nine?” When asked how she figured it out, she says, “Because nine’s a big number.”

Caitlin looks genuinely perplexed, as if the question was not a sensible thing to ask, and says, “I don’t know.”

Kindergarten teachers will immediately recognize that Brie’s answer provides evidence of a well-developed number sense for this age level and Leah’s answer, a more fragile and less-developed number sense. The knowledge that lies behind this “sense” may be much less apparent. What knowledge does Brie have that enables her to come up with the answer in the first place and to demonstrate good number sense in the process?

1. Knowledge that underlies number sense

Research conducted with the Number Knowledge Test and several other cognitive developmental measures (see Griffin, 2002; Griffin & Case, 1997 for a summary of this research) suggests that the following understandings lie at the heart of the number sense that 5-year-olds like Brie are able to demonstrate on this problem. They know (a) that numbers indicate quantity and therefore, that numbers, themselves, have magnitude; (b) that the word “bigger” or “more” is sensible in this context; (c) that the numbers 7 and 9, like every other number from 1 to 10, occupy fixed positions in the counting sequence; (d) that 7 comes before 9 when you are counting up; (e) that numbers that come later in the sequence—that are higher up—indicate larger quantities and therefore, that 9 is bigger (or more) than 7.

Brie provided evidence of an additional component of number sense in the explanation she provided for her answer. By using the Count-On strategy to show that nine comes two numbers after seven and by suggesting that this means “it has two more than seven,” Brie demonstrated that she also knows (f) that each counting number up in the sequence corresponds precisely to an increase of one unit in the size of a set. This understanding, possibly more than any of the others listed above, enables children to use the counting numbers alone, without the need for real objects, to solve quantitative problems involving the joining of two sets. In so doing, it transforms mathematics from something that can only be done out there (e.g., by manipulating real objects) to something that can be done in their own heads, and under their own control.

This set of understandings, the core of number sense, forms a knowledge network that Case and Griffin (1990), see also Griffin and Case (1997), have called a ***central conceptual structure for number***. Research conducted by these investigators has shown that this structure is central in at least two ways (see Griffin, Case, & Siegler, 1994). First, it enables children to make sense of a broad range of quantitative problems across contexts and to answer questions, for example, about two times on a clock (Which is longer?), two positions on a path (Which is farther?), and two sets of coins (Which is worth more?). Second, it provides the foundation on which children's learning of more complex number concepts, such as those involving double-digit numbers, is built. For this reason, this network of knowledge is an important set of understandings that should be taught in the preschool years, to all children who do not spontaneously acquire them.

2. How can this knowledge be taught?

Number Worlds, a mathematics program for young children (formerly called *Rightstart*), was specifically developed to teach this knowledge and to provide a test for the cognitive developmental theory (i.e., Central Conceptual Structure theory; see Case & Griffin, 1990) on which the program was based. Originally developed for kindergarten, the program (see Griffin & Case, 1995) was expanded to teach a broader range of understandings when research findings provided strong evidence that (a) children who were exposed to the program acquired the knowledge it was designed to teach (i.e., the central conceptual structure for number), and (b) the theoretical postulates on which the program was based were valid (see Griffin & Case, 1996; Griffin, Case, & Capodilupo, 1995; Griffin et al., 1994). Programs for grades one and two were developed to teach the more complex central conceptual structures that underlie base-ten understandings (see Griffin, 1997, 1998) and a program for preschool was developed (see Griffin, 2000) to teach the "precursor" understandings that lay the foundation for the development of the central conceptual structure for number.

Because the four levels of the program are based on a well-developed theory of cognitive development, they provide a finely graded sequence of activities (and associated knowledge objectives) that recapitulate the natural developmental progression for the age range of 3–9 years, and allow each child to enter the program at a point that is appropriate for his or her own development. To progress through the program to teach 20 or more children at any one time, every effort has been made in the construction of the *Number Worlds* program to make it as easy as possible for teachers to accommodate the developmental needs of individual children (or groups of children) in their classroom. Five instructional principles that lie at the heart of the program are described below and are used to illustrate several features of the program that have already been mentioned and several that have not yet been introduced.

2.1. Principle 1: Build upon children's current knowledge

Each new idea that is presented to children must connect to their existing knowledge if it is going to make any sense at all. Children must also be allowed to use their existing knowledge to construct new knowledge that is within reach—that is one step beyond where they are now—and a set of bridging contexts and other instructional supports should be in place to enable them to do so.

In the examples of children's thinking presented earlier, three different levels of knowledge are apparent. Brie appears to have acquired the knowledge network that underlies number sense and to be ready, therefore, to move on to the next developmental level: to connect this set of understandings to the written numerals (i.e., the formal symbols) associated with each counting word. Leah appears to have some understanding of some of the components of this network (i.e., that numbers have magnitude) and to be ready to use this understanding as a base to acquire the remaining understandings (e.g., that a number's magnitude and its position in the counting sequence are directly related). Caitlin demonstrated little understanding of any element of this knowledge network, and she might benefit, therefore, from exposure to activities that will help her acquire the "precursor" knowledge needed to build this network, namely knowledge of counting (e.g., the one-to-one correspondence rule) and knowledge of quantity (e.g., an intuitive understanding of relative amounts). Although all three children are in kindergarten, each child appears to be at a different point in the developmental trajectory and to require a different set of learning opportunities—ones that will enable each child to use her existing knowledge to construct new knowledge at the next level up.

To meet these individual needs, teachers need (a) a way to assess children's current knowledge, (b) activities that are multi-leveled so children with different entering knowledge can all benefit from exposure to them, and (c) activities that are carefully sequenced and that span several developmental levels, so children with different entering knowledge can be exposed to activities that are appropriate for their level of understanding. These are all available in the *Number Worlds* program and are illustrated in various sections of this paper.

2.2. Principle 2: Follow the natural developmental progression when selecting new knowledge to be taught

Researchers who have investigated the manner in which children between the ages of 3 and 9 years construct number knowledge have identified a common progression that most, if not all, children follow (Griffin, 2002; Griffin and Case, 1997). As suggested earlier, by the age of 4 years, most children have constructed two "precursor" knowledge networks—knowledge of counting and knowledge of quantity—that are separate in this stage and that provide the base for the next developmental stage. Sometime in kindergarten, children become able to integrate these knowledge networks—to connect the world of counting numbers to the world of quantity—and to construct the central conceptual understandings that were described earlier. Around the age of 6 or 7 years, children connect this integrated knowledge network to the world of formal symbols and, by the age of 8 or 9 years, most children become capable of expanding this knowledge network to deal with double-digit numbers and the base-ten system. A mathematics program that provides opportunities for children to use their current knowledge to construct new knowledge that is a natural next step, and that fits their spontaneous development, will have the best chance of helping children make maximum progress in their mathematics learning and development.

Because there are limits in development on the complexity of information children can handle at any particular age/stage (see Case, 1992), it makes no sense to attempt to speed up the developmental process by accelerating children through the curriculum. However, for children who are at an age when they should have acquired the developmental milestones but for some reason haven't, exposure to a curriculum that will give them ample opportunities to do so makes tremendous sense. It will enable them to catch up to their peers and thus, to benefit from the formal mathematics instruction that is provided in school. Children who are developing normally also benefit from opportunities to broaden and deepen the knowledge networks they are constructing, to strengthen these understandings, and to use them in a variety of contexts.

2.3. Principle 3: Teach computational fluency as well as conceptual understanding

Because computational fluency and conceptual understanding have been found to go hand in hand in children's mathematical development (see Griffin, 2003; Griffin et al., 1994), opportunities to acquire computational fluency, as well as conceptual understanding, are built into every *Number Worlds* activity. This is nicely illustrated in the following activities, drawn from different levels of the program.

In *The Mouse and the Cookie Jar Game* (created for the preschool program and designed to give 3- to 4-year-olds an intuitive understanding of subtraction), children are given a certain number of counting chips (with each child receiving the same number but a different color) and told to pretend their chips are cookies. They are asked to count their cookies and, making sure they remember how many they have and what their color is, to deposit them in the cookie jar for safe keeping. While the children sleep, a little mouse comes along and takes one (or two) cookies from the jar. The problem that is then posed to the children is "How can we figure out whose cookie(s) the mouse took?"

Although children quickly learn that emptying the jar and counting the set of cookies that bears their own color is a useful strategy to use to solve this problem, it takes considerably longer for many children to realize that, if they now have four cookies (and originally had five), it means that they have one fewer and the mouse has probably taken one of their cookies. Children explore this problem by counting and recounting the remaining sets, comparing them to each other (e.g., by aligning them) to see who has the most or least, and ultimately coming up with a prediction. When a prediction is made, children search the mouse's hole to see whose cookie had been taken and to verify or revise their prediction. As well as providing opportunities to perfect their counting skills, this activity gives children concrete opportunities to experience simple quantity transformations and to discover how the counting numbers can be used to predict and explain differences in amount.

The *Dragon Quest Game* that was developed for the Grade 1 program teaches a much more sophisticated set of understandings. Children are introduced to Phase 1 activity by being told a story about a fire-breathing dragon that has been terrorizing the village where children live. The children playing the game are heroes who have been chosen to seek out the dragon and put out his fire. To extinguish this dragon's fire (as opposed to the other, more powerful dragons they will encounter in later phases) a hero will need at least 10 pails of water. If a hero enters into the dragon's area with less than 10 pails of water, he or she will become the dragon's prisoner and can only be rescued by one of the other players.

To play the game, children take turns rolling a die and moving their playing piece along the colored game board. If they land on a well pile (indicated by a star), they can pick a card from the face-down deck of cards, which illustrate, with images and symbols (e.g., +4) a certain number of pails of water. Children are encouraged to add up their pails of water as they receive them and they are allowed to use a variety of strategies to do so, ranging from mental math (which is encouraged) to the use of tokens to keep track of the quantity accumulated. The first child to reach the dragon's lair with at least 10 pails of water can put out the dragon's fire and free any teammates who have become prisoners.

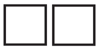
As children play this game and talk about their progress, they have ample opportunity to connect numbers to several different quantity representations (e.g., dot patterns on the die; distance of their pawn along the path; sets of buckets illustrated on the cards; written numerals also provided on the cards) and to acquire an appreciation of numerical magnitude across these contexts. With repeated play, they also become capable of performing a series of successive addition operations in their heads and of expanding the well pile. When they are required to submit formal proof to the mayor of the village that they have amassed sufficient pails of water to put out the dragon's fire before they are allowed to do so, they become capable of writing a series of formal expressions to record the number of pails received and spilled over the course of the game. In contexts such as these children receive ample opportunity to use the formal symbol system in increasingly efficient ways to make sense of quantitative problems they encounter in the course of their own activity.

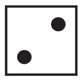
2.4. Principle 4: Provide plenty of opportunity for hands-on exploration, problem-solving, and communication

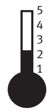
Like the *Dragon Quest Game* that was just described, many of the activities created for the *Number Worlds* program are set in a game format that provides plenty of opportunity for hands-on exploration of number concepts, for problem-solving and for communication. Communication is explicitly encouraged in a set of question prompts that are included with each small group game (e.g., How far are you now? How many more buckets do you need to put out the dragon's fire? How do you know?) as well as in a more general set of dialogue prompts that are included in the teacher's guide. Opportunities for children to discuss what they learned during game play each day, to share their knowledge with their peers, and to make their reasoning explicit are also provided in a Wrap-Up session that is included at the end of each math lesson.

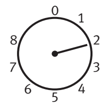
Finally, in the whole group games and activities that were developed for the Warm-Up portion of each math lesson, children are given ample opportunity to count (e.g., up from 1 and down from 10) and to solve mental math problems, in a variety of contexts. In addition to developing computational fluency, these activities expose children to the language of mathematics and give them practice using it. Although this is valuable for all children, it is especially useful for English learners, who may know how to count in their native language but not yet in English. Allowing children to take turns in these activities and to perform individually gives teachers opportunities to assess each child's current level of functioning, important for instructional planning, and gives children opportunities to learn from each other.

2.5. Principle 5: Expose children to the major way number is represented and talked about in developed societies









In ***Object Land,*** students explore the world of numbers by counting and comparing sets of objects or pictures of objects. In Object Land you might ask: ■ How many or few do you have? ■ Which is bigger or smaller?	In ***Picture Land,*** numbers are represented as sets of stylized, semi-abstract dot-set patterns, such as in a die, tally marks, or numerals. In Picture Land you might ask: ■ What did you roll/pick? ■ Which has more or less?	In ***Line Land,*** number is represented as a point on a path or line. The language used for numbers in Line Land refers to a particular place on a line and also to moves along a line. In Line Land you might ask: ■ How far did you go? ■ Do you go forward or backward?	In ***Sky Land,*** number is represented as a position on a vertical scale such as on a thermometer or in a bar graph. In Sky Land you might ask: ■ How high or low are you now? ■ Who is above or below?	In ***Circle Land,*** number is represented as a point on a dial, such as a clock face or a sundial. In Circle Land you might ask: ■ How many times did you go around? ■ What did you land on?

Number is represented in our culture in five major ways: as a group of objects, a dot-set pattern, a position on a line, a position on a scale (e.g., a thermometer), and a point on a dial. In each of these contexts, number is also talked about in different ways, with a larger number (and quantity) described as "more" in the world of dot-sets, as "further along" in the world of paths and lines, as "higher up" in the world of scale measures, and as "further around" in the world of dials. Children who are familiar with these forms of representation and the language used to talk about number in these contexts have a much easier time making sense of the number problems they encounter inside and outside of school.

In the *Number Worlds* program, children are systematically exposed to these forms of representations as they explore five different "lands." Learning activities developed for each land share a particular form of number representation while they simultaneously address specific knowledge goals for each grade level. Many of the games, like *Dragon Quest,* also expose children to multiple representations of number in one activity so children can gradually come to see the ways they are equivalent.

3. Discussion

Children who have been exposed to the *Number Worlds* program do very well on number questions like the one presented in the introduction and on the Number Knowledge Test (Griffin & Case, 1997) from which this question was drawn. In several evaluation studies conducted with children from low-income communities, children who received the *Number Worlds* program made significant gains in conceptual knowledge of number and in number sense, when compared to matched-control groups who received readiness training of a different sort. These gains enabled them to start their formal schooling in grade one on an equal footing with their more advantaged peers, to perform as well as groups of children from China and Japan on a computation test administered at the end of grade one, and to keep pace with their more advantaged peers (an even outperform them on some measures) as they progressed through the first few years of formal schooling (Griffin & Case, 1997).

Teachers also report positive gains from using the *Number Worlds* program and from exposure to the instructional principles on which it is based. Although all teachers acknowledge that implementing the program and putting the principles into action is not an easy task, many claim that their teaching of all subjects has been transformed in the process. They now facilitate discussion rather than dominating it; they pay much more attention to what children say and do; and they now allow children to take more responsibility for their own learning, with positive and surprising results. Above all, they now look forward to teaching math and they and their students are eager to do more of it.

Griffin, S. “Building Number Sense with Number Worlds: A mathematics program for young children.” In Early Childhood Research Quarterly 19 (2004) 173–180. Elsevier.

References

Case, R. (1992). *The mind's staircase: Exploring the conceptual underpinnings of children's thought and knowledge.* Hillsdale, NJ: Erlbaum.

Case, R., & Griffin, S. (1990). Child cognitive development: The role of central conceptual structures in the development of scientific and social thought. In E.A. Hauert (Ed.), *Developmental psychology: Cognitive, perceptuo-motor, and neurological perspectives* (pp. 193–230). North-Holland: Elsevier.

Griffin, S. (1997). *Number Worlds: Grade one level.* Durham, NH: Number Worlds Alliance Inc.

Griffin, S. (1998). *Number Worlds: Grade two level.* Durham, NH: Number Worlds Alliance Inc.

Griffin, S. (2000). *Number Worlds: Preschool level.* Durham, NH: Number Worlds Alliance Inc.

Griffin, S. (2002). The development of math competence in the preschool and early school years: Cognitive foundations and instructional strategies. In J. Royer (Ed.), *Mathematical cognition* (pp. 1–32). Greenwich, CT: Information Age Publishing.

Griffin, S. (2003). Laying the foundation for computational fluency in early childhood. *Teaching children mathematics,* 9, 306–309.

Griffin, S. (1997). *Number Worlds: Kindergarten level.* Durham, NH: Number Worlds Alliance Inc.

Griffin, S., & Case, R. (1996). Evaluating the breadth and depth of training effects when central conceptual structures are taught. *Society for research in child development monographs,* 59, 90–113.

Griffin, S., & Case, R. (1997). Re-thinking the primary school math curriculum: An approach based on cognitive science. *Issues in Education,* 3, 1–49.

Griffin, S., Case, R., & Siegler, R. (1994). Rightstart: Providing the central conceptual prerequisites for first formal learning of arithmetic to students at-risk for school failure. In K. McGilly (Ed.), *Classroom lessons: Integrating cognitive theory and classroom practice* (pp. 24-49). Cambridge, MA: Bradford Books MIT Press.

Griffin, S., Case, R., & Capodilupo, A. (1995). Teaching for understanding: The importance of central conceptual structures in the elementary mathematics curriculum. In A. McKeough, I. Lupert, & A. Marini (Eds.), *Teaching for transfer: Fostering generalization in learning* (pp. 121–151). Hillsdale, NJ: Erlbaum.

Leverage Learning Trajectories

Children follow natural developmental progressions in learning, developing mathematical ideas in their own way. Curriculum research has revealed sequences of activities that are effective in guiding children through these levels of thinking. These developmental paths are the basis for *Building Blocks* Learning Trajectories. Learning Trajectories have three parts: a mathematical goal, a developmental path through which children develop to reach that goal, and a set of activities matched to each of those levels that help children develop the next level. Thus, each learning trajectory has levels of understanding, each more sophisticated than the last, with tasks that promote growth from one level to the next. The *Building Blocks* Learning Trajectories give simple labels, descriptions, and examples of each level. Complete Learning Trajectories describe the goals of learning, the thinking and learning processes of children at various levels, and the learning activities in which they might engage. This document provides only the developmental levels.

Frequently Asked Questions

1. Why use Learning Trajectories?

Learning Trajectories allow teachers to build students' mathematical knowledge along natural developmental pathways, to help students move through stages of understanding. Since all activities in *Building Blocks* Adaptive are tied to a specific Learning Trajectory level, we know that students will be able to successfully work within their developmental capacities.

2. When are children "at" a level?

Children are at a certain level when most of their behaviors reflect the thinking—ideas and skills—of that level. As students learn new skills, they may continue to show behaviors from the previous trajectory level as well as some from the next level.

3. Can children work at more than one level at the same time?

Yes, although most children work mainly at one level or in transition between two levels (naturally, if they are tired or distracted, they may operate at a much lower level). Levels are not "absolute stages." They are "benchmarks" of complex growth that represent distinct ways of thinking.

4. How do these developmental levels support teaching and learning?

The levels help teachers, as well as curriculum developers, assess, teach, and sequence activities. Teachers who understand Learning Trajectories and the developmental levels that are at their foundation are more effective and efficient. Through planned teaching and encouraging informal, incidental mathematics, teachers help children learn *at* an appropriate and deep level.

5. Why do the Learning Trajectories keep changing?

Learning Trajectories are living things! That is, we are always learning more and adding new ideas as we continue our research and deepen our understanding of how children naturally develop math skills. Changes to the Learning Trajectories build on new information from our research and wisdom of expert practice.

Developmental Levels for Counting

To count a set of objects, children must learn to count verbally—saying the list of number names in order—well as learning the system that generates new number names. Counting includes the ability to say number words in correspondence with objects (enumerate objects), understanding that the last number word said when counting refers to how many items have been counted (cardinality), and using counting strategies to solve problems.

Age Range	Level Name	Level	Description
1	Number Word Sayer (Foundations)	1	At the earliest level a child shows no verbal counting. The child may name some number words with no sequence.
1–2	Chanter	2	At this level, a child may sing-song or chant indistinguishable number words.
2	Reciter	3	A child at this level verbally counts with distinct words, not necessarily in the correct order above "five". They may say "One, two, three, four, five, seven."
3	Reciter (10)	4	A child at this level may verbally count to ten with some correspondence with objects. He or she may either continue an overly rigid correspondence or exhibit performance errors (e.g., skipping, double-counting).
3	Corresponder	5	At this level, a child may keep one-to-one correspondence between counting words and objects-at least for small groups of objects laid in a line.
4	Counter (Small Numbers)	6	A child at this level accurately counts objects in a line to five and answers the "how many" question with the last number counted, understanding that this represents the total number of objects (the cardinal principle).
4	Producer (Small Numbers)	7	At this level, a child can count out objects to five. He or she recognizes that counting is relevant to situations in which a certain number must be placed. The child can produce a group of four objects.
4	Counter (10)	8	This child counts arrangements of objects to ten with understanding of the cardinal principle. He or she may be able to read and write numerals to represent one to ten. The child may also be able to tell the number just after or just before another number, but only by counting up starting at one. Verbal counting to twenty is developing.
5	Counter and Producer (10+)	9	At this level, a child counts and counts out objects accurately to ten, then beyond (to about thirty). Has explicit understanding of cardinality. Keeps track of objects that have and have not been counted, even in different arrangements. Writes or draws to represent one to ten (then, 20, then 30). Separates the decade and the ones part of a number word and begins to relate each part of a number word/numeral to the quantity to which it refers. Recognizes errors in others' counting and can eliminate most errors in own counting (point-object) if asked to try hard.
5	Counter Backward from ten	10	Another milestone at about age five is being able to count backward from ten to one, verbally, or when removing objects from a group.
6	Counter from N (N+1, N-1)	11	At this level a child may count verbally and with objects from numbers other than one. He or she can immediately determine numbers just after or just before.
6	Skip Counting by tens to 100	12	A child at this level may count by tens to 100 or beyond with understanding. For example, the child can "see" groups of ten within a quantity and counts those groups by ten (this relates to multiplication and algebraic thinking). "ten, 20, 30 . . . 100."
6	Counter to 100	13	A child at this level may count by ones to 100. He or she can make decade transitions (for example, from 29 to 30) starting at any number.
6	Counter on Using Patterns	14	At this level, a child may keep track of a few counting acts by using numerical patterns (spatial, auditory, or rhythmic).

6	Skip Counter	15	At this level, the child can count by fives and twos with understanding.
6	Counter On Keeping Track	16	A child at this level counts forward or back from a given number keeping track of counting acts numerically, first using objects, then by "counting counts."
6	Counter of Quantitative Units/Place Value	17	At this level, a child understands the base ten numeration system and place value concepts, including ideas of counting in units and multiples of hundreds, tens, and ones. When counting groups of ten, he or she can decompose into "ten ones" if that is useful. The child understands value of a digit according to the place of the digit within a number and counts unusual units, such as "wholes" when shown combinations of wholes and parts.
6	Counter Beyond 100	18	At this level, a child accurately counts beyond 100, recognizing the patterns of ones, tens, and hundreds.
7	Number Conserver	19	At this level a child conserves number (i.e., believes number has been unchanged) even if a group of objects has been rearranged. For example, if there is a row of ten buttons, the child understands there are still ten without recounting, even if they are rearranged in a long row or a circle.
8	Counter Forward and Back (10s and 1s)	20	A child at this level may count in either direction and recognize that sequence of decades mirrors single-digit sequence. He or she switches between sequence and composition views of multi-digit numbers easily.

Developmental Levels for Comparing Numbers

Children learn to compare perceptually and numerically, with increasing ability to estimate quantities of larger numbers with more precision. Comparing and ordering sets is a critical skill for children as they determine whether one set is larger than another in order to make sure sets are equal and "fair." The ability to compare and order sets with fluency develops over the course of several years.

Age Range	Level Name	Level	Description
0–1	Comparison Senser (Foundations)	1	From the first months of life, children are sensitive to a change in the number, either of a change of very small collections, such as one vs. two, or large changes in larger collection, such as double the number. Therefore, we know infants have an unconscious, innate sensitivity to such simple equivalence comparisons.
1–3	Early Comparison Corresponder	2	Many-to-One Corresponder: Consciously recognizes that two very small collections have the "same number" by intuitively making a correspondence between the items in each collection. At this level, in certain situations, children may also put objects, words, or actions in one-to-one or many-to-one correspondence or a mixture. One-to-One Object Corresponder: Puts objects into one-to-one correspondence when it is clear the materials are a physical "pair." In other situations, such as setting the table, may start to do one-to-one, but then may keep on passing out items until they are all dispersed, or may skip some (due to the lack of clear matching, such as cups "near" plates). The child is sensitive to the relationships of "more than" and "less than" when working with very small numbers (from one to two years of age). Uses words to include "more," "less," or "same." Object Corresponder: Puts objects into one-to-one correspondence, although they may not fully understand that this creates equal groups
2–3	Perceptual Comparer	3	At this level, a child can compare collections that are quite different in size (for example, one is at least twice the other) and know that one has more than the other. Compares collections more similar in number but only for very small numbers (one, two, and sometimes three). Compares collections using number words "one" and "two."
3	First-Second Ordinal Counter	4	At this level, the child can identify the "first" and often "second" object in a sequence.
3	Early Comparer of Similar Items	5	At this level, a child can compare collections of one to four items verbally or nonverbally ("just by looking"). The items must be the same. May recognize that different organizations of the same number are equal and different from other sets.
3–4	Early Comparer of Dissimilar Items	6	At this level, a child can match small, equal collections of dissimilar items, such as shells and dots, and show that they are the same number.
4	Matching Comparer	7	As children progress, they begin to compare groups of one to six by matching. For example, a child gives one toy bone to every dog and says there are the same number of dogs and bones.
4	Counting Comparer (Same Size)	8	At this level, children make accurate comparisons via counting, but only when objects are about the same size and groups are small (about one to five items). Children at this level also accurately count two equal collections, but, when asked, may say that the collection of larger items has more.
5	Spatial Extent Estimator (Small/Big)	9	At this level, children estimate which set is "more" or "less", if the differences are clear (e.g., one is double the other). They name a "small number" (e.g., from one to four) for sets that cover little space and a "big number" (ten to twenty or more) for sets that cover a lot of space. Children classify numbers as "little" or "big." This may change with the size of the to-be estimated objects.
5	Counting Comparer (5)	10	As children develop their ability to compare sets, they compare accurately by counting, even when a larger collection's objects are smaller. A child at this level can figure out how many "more" or "less" when comparing sets.

4–5	Mental Number Line to 5	11	A child at this level uses knowledge of counting number relationships to determine relative size and position of numbers up to about five, when given perceptual support.
4–5	Serial Orderer to 5	12	At this level, children order quantities (dots) or numerals up to five. Orders lengths marked into units.
5	Ordinal Counter	13	At this level, a child identifies and uses ordinal numbers from "first" to "tenth." For example, the child can identify who is "third in line."
5	Counting Comparer (10)	14	This level can be observed when the child compares sets by counting, even when a larger collection's objects are smaller, up to ten. A child at this level can accurately count two collections of nine items each, and says they have the same number, even if one collection has larger blocks.
5	Mental Number Line to ten	15	As children move into this level, they begin to use mental images and knowledge of number relationships to determine relative size and position. For example, a child at this level can answer which number is closer to six, four, or nine without counting physical objects.
6	Serial Orderer to 6+	16	At this level, the child orders quantities (dots) or numerals to six and beyond. Similarly, he or she orders lengths marked into units. A significance advance is the ability to correctly determine whether numerals that are not in count-list order are in order or not (e.g., some are skipped; five, seven, nine).
7	Place Value Comparer	17	Children at this level, compare numbers with place value understanding. For example, a child at this level can explain that "63 is more than 59 because six tens is more than five tens, even if there are more than three ones."
7	Mental Number Line to 100	18	Children demonstrate the next level when they can use mental images and knowledge of number relationships, including ones embedded in tens, to determine relative size and position. For example, when asked, "Which is closer to 45—30 or 50?", a child at this level may say "45 is right next to 50, but 30 isn't."
7	Scanning with Intuitive Quantification Estimator	19	At this level, a child can scan a group of objects and relate the results to a mental number line to perform a useful numerosity estimation.
7–8	Mental Number Line to 1,000s	20	A child at this level uses internal images and knowledge of number relationships, including place value, to determine relative size and position. For example, when asked, "Which is closer to 3,500—2,000 or 7,000?", a child at this level may say "70 is double 35, but 20 is only 15 from 35, so 20 hundreds, 2,000, is closer to 3,000."
8	Benchmarks Estimator	21	Children at this level can count a portion of the to-be-estimated collection and uses that as a benchmark from which an estimate is made, intuitively or using repeated addition or multiplication. Scanning can be linked to recalled benchmarks.
8	Composition Estimator	22	At this level a child decomposes or partitions to-be-estimated set into convenient subset sizes then recomposes the numerosity. Initially, this is done with regular arrangements using repeated addition or multiplication. Later, the process can be done with irregular arrangements and children more consistently use multiplication skills to recompose.

Developmental Levels for Subitizing

Subitizing is quickly recognizing and naming the number in a group without counting. "Subitize" comes from the Latin "to arrive suddenly." It begins in infancy with a sensitivity to number. Then children learn to recognize very small numbers. Later, they learn to do it quickly—perceptual subitizing. Another qualitative progression is their ability to see several groups and combine them quickly into one quantity—conceptual subitizing.

Age Range	Level Name	Level	Description
0–1	Number Senser (Foundations)	1	A child has inborn specific "sensers" for numbers from the first months of life without explicit knowledge of numbers. He or she intuitively distinguishes between groups of one and two (and possibly two and three). Children may also show sensitivity to ratios of quite large numbers. (Approximate Number System, or ANS). These are pre-mathematical, foundational abilities.
1–3	Very Small Number Recognizer	2	At this level, children begin connecting small quantities to number words to form an explicit idea of cardinality, or "how-many-ness." Only over time do they begin to understand that all groups labeled with the same number word have the same amount.
2–3	Small Collection Namer	3	The first sign occurs when the child can name groups of one to two, sometimes three. For example, when shown a pair of shoes, this young child says, "two shoes."
2–3	Maker of Small Collections	4	At this level, a child can nonverbally make a small collection, via mental model, (no more than four, usually one to three) with the same number as another collection. For example, when shown a collection of three, the child makes another collection of three. May not recognize spatial structures at first and may count this.
3–4	Perceptual Subitizer to 4	5	Progress is made when a child instantly recognizes collections up to four and verbally names the number of items. For example, when shown four objects briefly, the child says "four."
4	Perceptual Subitizer to 5	6	This level is the ability to instantly recognize collections up to five and verbally name the number of items. For example, when shown five objects briefly, the child says "five."
4	Conceptual Subitizer to 5	7	At this level, the child can verbally label all arrangements to about five, when shown only briefly. For example, a child at this level might say, "I saw two and two, and so I saw four."
4–6	Conceptual Subitizer to 7	8	At this level a child can verbally label all arrangements to six, then seven, when shown only briefly.
5–6	Conceptual Subitizer to 10	9	This step is when the child can verbally label most arrangements to six shown briefly, then up to ten, using groups. For example, a child at this level might say, "In my mind, I made two groups of three and one more, so seven."
6–7	Conceptual Subitizer to 20	10	Next, a child can verbally label structured arrangements up to 20 shown briefly, using groups. For example, the child may say, "I saw three fives, so 5, 10, 15."
7	Conceptual Subitizer with Place Value	11	At this level, a child can use groups, skip counting, and place value to verbally label structured arrangements shown briefly. For example, the child may say, "I saw groups of tens and twos, so ten, 20, 30, 40, 42, 44, 46—46!"
8+	Conceptual Subitizer with Place Value and Multiplicative Thinking	12	Verbally labels structured arrangements, shown only briefly, using groups, multiplicative thinking, and place value. This level builds on the previous level, such that children are able to use the base ten system to conceptually subitize larger numbers. Children can also verbalize the quantity of tens they see.

Developmental Levels for Composing Numbers

Composing and decomposing are combining and separating operations that allow children to build concepts of "parts" and "wholes"—recognizing six as three and three, five and one, or zero and six. Most children follow a natural developmental progression in learning to compose and decompose numbers with recognizable stages or levels.

Age Range	Level Name	Level	Description
0–1	Actor on Parts (Foundations)	1	A child at this level displays actions that show intuition about parts and wholes such as gathering objects together. The child can only nonverbally recognize parts and wholes. He or she may also recognize that sets can be combined in different orders but may not explicitly recognize that groups are additively composed of smaller groups.
1–3	Parts Combiner	2	At this level, a child recognizes that sets can be combined in different orders but may not explicitly recognize that groups are additively composed of smaller groups. The toddler also recognizes Part-Whole relations in nonverbal, intuitive, perceptual situations and can nonverbally represent parts that make a whole.
3–4	Inexact Part-Whole Recognizer	3	A sign of development is that the child knows a whole is bigger than parts but does not accurately quantify. For example, when shown four red blocks and two blue blocks and asked how many there are in all, the child may name a "large number" such as five or ten.
4–5	Composer to 4, then 5	4	At this level, a child knows number combinations. A child at this level quickly names parts of any whole, or the whole given the parts. For example, when shown four, then one is secretly hidden, and then shown the three remaining, the child may quickly say "one" is hidden.
5–6	Composer to 7	5	The next sign of development is when a child knows number combinations to totals of seven. A child at this level quickly names parts of any whole, or the whole when given parts, and can double numbers to ten. For example, when shown six, then four are secretly hidden, and then shown the two remaining, the child may quickly say "four" are hidden.
5–6	Composer to 10	6	This level is when a child knows number combinations to totals of ten. A child at this level may quickly name parts of any whole, or the whole when given parts, and can double numbers to twenty. For example, this child would be able to say "nine and nine is eighteen."
7	Composer with Tens and Ones	7	At this level, the child understands and two-digit numbers as tens and ones, can count with dimes and pennies, and can perform two-digit addition with regrouping.
6–7	Deriver +/−	8	At this level, students use flexible strategies such as Break Apart to Make Ten and derived combinations (e.g., "7 + 7 is 14, so 7 + 8 is 15") to solve all types of problems. He or she can simultaneously think of numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another. Students can also solve simple cases of multi-digit addition (and often subtraction) by incrementing tens and/or ones. The goal for this level goes beyond simply memorizing math facts. The goal is knowing the "breakaparts," which means the paired values that compose a target number.
7	Problem Solver +/−	9	Students at this level solve all types of problems, with flexible strategies and known combinations. Multi-digit problems may be solved by incrementing or combining tens and ones.
7–8	Multi-digit +/−	10	Children at this level use composition of tens and all previous strategies to solve multi-digit addition and subtraction problems.

Developmental Levels for Adding and Subtracting

Research has shown that addition and subtraction have their roots in counting, counting on, number sense, composing and decomposing numbers, and place value. Children learn that addition is the putting together of sets, here by counting (for example, the sum of 4 + 2 is the whole number that results from counting 2 numbers starting at four) and subtraction is the separating of sets (counting backward).

Age Range	Level Name	Level	Description
0–1	Arithmetic Senser (Foundations)	1	Very young children are sensitive to combining or separating perceptual groups. An infant may observe, point, or make sounds while someone else introduces a quantity of objects. They will notice the effects of increasing or decreasing small collections by one item and may also be sensitive to the results of combining larger groups.
2–3	Preverbal +/–	2	A child at this level adds and subtracts very small collections (totals up to three) often making a collection rather than answering verbally.
4	Small Number +/–	3	This level is when a child can find sums for joining problems up to 3 + 2 by counting with objects. For example, when asked, “You have two balls and get one more. How many in all?” the child may count out two, then count out one more, then count all three: “one, two, three — three!” A child at this level may also solve problems for result unknown and separate.
4–5	Find Result +/–	4	Addition: Evidence of this level in addition is when a child can find sums for joining and part-part-whole problems by direct modeling, counting all, with objects. For example, when asked, “You have two red balls and three blue balls. How many in all?”, the child may count out two red, then count out three blue, then count all five. Subtraction: In subtraction, a child can also solve take-away problems by separating with objects. For example, when asked, “You have five balls and give two to Tom. How many do you have left?”, the child may count out five balls, then take away two, and then count the remaining three.
4–5	Find Change +/–	5	Addition: At this level, a child can find the missing addend (5 + _ = 7) by adding on objects. For example, when asked, “You have five balls and then get some more. Now you have seven in all. How many balls did you get?” The child may count out five, then count those five again starting at one, then add more, counting “six, seven,” then count the balls added to find the answer, two. Subtraction: A child can compare by matching in simple situations. For example, when asked, “Here are six dogs and four balls. If we give a ball to each dog, how many dogs will not get a ball?”, a child at this level may count out six dogs, match four balls to four of them, then count the two dogs that have no ball.
4–5	Make It N	6	A significant advancement occurs when a child is able to count on. This child can add on objects to make one number into another without counting from one. For example, when told, “This puppet has four balls, but she should have six. Make it six.”, the child may put up four fingers on one hand, immediately count up from four while putting up two fingers on the other hand, saying, “five, six,” and then count or recognize the two fingers.
5–6	Counting Strategies +/–	7	This level occurs when a child can find sums for joining and part-part-whole problems with finger patterns or by adding on objects or counting on. For example, when asked “How much is four and three more?”, the child may answer “four—five, six, seven—seven!” Children at this level can also solve missing addend (3 + _ = 7) or compare problems by counting on. When asked, for example, “You have six balls. How many more would you need to have eight?”, the child may say, “six, seven [puts up first finger], eight [puts up second finger]. Two!”
6	Part Whole +/–	8	Further development has occurred when the child has part-whole understanding. This child can solve problems using flexible strategies and some derived facts (for example, “5 + 5 is 10, so 5 + 6 is 11”), can sometimes do start unknown problems (_ + 6 = 11), but only by trial and error. When asked, “You had some balls. Then you get six more. Now you have eleven balls. How many did you start with?”, this child may lay out six, then three, count, and get nine. The child may put one more, say ten, then put one more. The child may count up from six to eleven, then recount the group added, and say, “five!”

6–7	Numbers-in-Numbers +/−	9	Evidence of this level is when a child recognizes that a number is part of a whole and can solve problems when the start is unknown with counting strategies. For example, when asked, "You have some balls, then you get four more balls, now you have nine. How many did you have to start with?", this child may count, putting up fingers, "five, six, seven, eight, nine." The child may then look at his or her fingers and say, "five!"
6–7	Deriver +/−	10	At this level, a child can use flexible strategies such as Break Apart to Make Ten and derived combinations (e.g., "7 + 7 is 14, so 7 + 8 is 15") to solve all types of problems. He or she can simultaneously think of three numbers within a sum, and can move part of a number to another, aware of the increase in one and the decrease in another. The child may solve simple cases of multi-digit addition (sometimes subtraction) by counting by tens and/or ones.
7	Problem Solver +/−	11	As children develop their addition and subtraction abilities, they can solve by using flexible strategies and many known combinations. Multi-digit problems may be solved by incrementing tens and ones by counting.
8+	Multi-digit +/−	12	Further development is shown when children can use composition of tens and all previous strategies to solve multi-digit +/− problems. For example, when asked, "What's 37 − 18?" this child may say, "Take one ten away from the three tens. That's two tens. Take seven ones away from the seven. That's two tens and zero ones—twenty. I have one more to take away. That's nineteen." Or, when asked, "What's 28 + 35?" this child may think, 30 + 35 would be 65. But it's twenty-eight, so it's two less—sixty-three.

Developmental Levels for Multiplication and Division

Multiplication and division builds on addition and subtraction understanding and is dependent upon counting and place value concepts. Children learn that multiplying can be seen as the process of repeated addition and division as the splitting of a number into equal parts or groups. Finding and using patterns aids in learning multiplication and division facts with understanding.

Age Range	Level Name	Level	Description
0–2	Nonquantitative Sharer (Foundations)	1	Multiplication and division concepts begin very early with the problem of sharing. Early evidence of these concepts can be observed when a child gives some, but not necessarily an equal number to each person.
3	Beginning Grouper and Distributive Sharer	2	Progression to this level can be observed when a child is able to make small groups (fewer than five). This child can share by “dealing out,” but often only between two people, although he or she may not appreciate the numerical result. For example, to share four blocks, this child may give each person a block, check that each person has one, and repeat.
4–5	Grouper and Distributive Sharer	3	The next level occurs when a child makes small equal groups (fewer than six). This child can deal out equally between two or more recipients but may not understand that equal quantities are produced. For example, the child may share six blocks by dealing out blocks to herself and a friend one at a time.
5	Concrete Modeler ×/÷	4	As children develop, they are able to solve small-number multiplying problems by grouping-making each group and counting all. At this level, a child can solve division/sharing problems with informal strategies, using concrete objects-up to 20 objects and two to five people—although the child may not understand equivalence of groups. For example, the child may distribute 20 objects by dealing out two blocks to each of five people, then one to each, until the blocks are gone.
6	Parts and Wholes ×/÷	5	A new level is evidenced when the child understands the inverse relation between divisor and quotient. For example, this child may understand “If you share with more people, each person gets fewer.”
7	Skip Counter ×/÷	6	As children develop understanding in multiplication and division, they may use repeated adding, additive doubling, or skip counting to solve multiplication problems and for measurement division (finding out how many groups). A child at this level may also use trial and error for partitive division (finding out how many in each group). This child can also predict, demonstrate, and justify outcomes of equipartitioning collections.
8–9	Deriver ×/÷	7	At this level, children use strategies, patterns, composition, decomposition, and derived combinations. For example, by nine as $10 - 1$, or 7×8 from $7 \times 7 + 7$. The child may also solve multi-digit problems by operating on tens and ones separately and can quantitatively predict the effects of changes in the number of people sharing on the size of the shares.
8–9	Array Quantifier / Problem Solver ×/÷	8	Children at this level can solve many types of multiplicative problems, with flexible strategies and known combinations. Students may solve multi-digit problems using combinations separately on ones and tens.
8–9	Partitive Divisor	9	This level can be observed when a child is able to figure out how many are in each group. The child may first repeatedly add a divisor until the dividend is reached.

Developmental Levels for Fractions

The development of conceptual and procedural knowledge of fractions is critical for children. An understanding of fractions will include clear concepts of whole numbers versus fractions, including the idea that a single unit fraction of ½ is larger than ¼, despite the denominator in one-fourth being larger. Children also must learn how the arithmetic of numerators and denominators operate, with help to notice that adding numerators and denominators (such as ¾ + ⅓ yielding 4⁄6) leads to a sum that is smaller than the parts with which it began (not noticing that 4⁄6 is less than ¾). Learning opportunities are key, and they differ considerably for different children.

Age Range	Level Name	Level	Description
0–2	Early Proportional Thinker (Foundations)	1	At this level, a child has an intuition about proportions. (Ex. A child play with fruit sets, pulling them apart to make two pieces and putting them back together to make one piece.)
3–4	Shape Equipartitioner	2	Children at this stage can equipartition a whole shape, such as a circle or rectangle. (Ex. Children can draw lines on paper "cakes" to share equally with a friend and can split food equally with a friend.)
4–5	Half Recognizer	3	At this level, children recognize "halves" at least in continuous representations, especially in the context of fair shares. They also recognize the need for ½ when sharing an odd number of objects. Children can intuitively and visually combine regions that are a part of a whole, showing initial foundations for addition. (Ex. Recognizes what is and is not "half" when sharing a sandwich.)
4–6	Unit Fraction Recognizer	4	A child at this level, recognizes unit fractions in simple discrete (countable) and maybe continuous representations for ½, ⅓, and ¼. The child understands intuitively that fractions are formed by dividing a whole into equal parts and can name these shares.
7	Fraction Recognizer	5	At this level a child recognizes simple (small number denominators) fractions in familiar continuous and discrete contexts.
7	Fraction Maker from Units	6	Children at this level can create a fraction representation with equal parts and the correct number of repetitions of a unit fraction. They are also able to label that fraction with written fraction notation and compare fraction representations and states which is the larger number.
8	Fraction Maker	7	At this level, children can create a fraction representation with equal parts and the correct number of repetitions of a unit or non-unit fraction (as long as they are not greater than a whole–fraction may only be a "part-of-a-whole"). They may also compare simple common fractions using physical models.
8	Fraction Repeater	8	A child at this level can create a fraction representation with repetitions of unit and non-unit fractions, including results greater than a whole. This student moves beyond fraction as a "part-of-a-whole" to fraction as a number with a relation to the reference whole and understands that fractions with the same denominator can be added or subtracted using the units of the unit fraction. The child may also compare simple common fractions using models such as the number line—understanding that two fractions are equal when they represent the same portion of a whole or have the same length on the number line.
8	Fraction Arithmetic +/-	9	Children at this level can add and subtract simple common fractions using physical models. These children may also have enough understanding of the fractions as numbers that they can estimate adding the fractions without using the procedures of finding a common denominator.
8	Fraction Arithmetic ×/÷	10	At this level, a child is able to multiply simple common fractions using a rectangle array model.
8	Fraction and Integer Sequencer	11	A child at this level can represent simple ratios as percentages, fractions, and decimals. He or she also orders integers, positive fractions, and decimals.

Developmental Levels for Patterns, Structure, and Algebraic Thinking

Children learn to find and use mathematical regularities and structures. Children learn to identify, duplicate, and extend sequential patterns such as ABCABCABC, but also to find regularities and structures in number and geometry. Repeating patterning skills predicts later math knowledge even after controlling for prior math knowledge. It is a critical math topic!

Age Range	Level Name	Level	Description
0–2	Intuitive Patterner (Foundations)	1	A child at this level detects and uses patterning implicitly and intuitively, such as in movement activities or common nursery rhymes that repeat words and action. He or she may be attentive to repeating patterns without recognizing them explicitly or accurately, often attending to individual attributes such as color.
2–3	Pattern Recognizer	2	At this level, the child can recognize a simple sequential pattern (usually ABABAB) as a pattern, even if he or she doesn't yet name or describe it. For example, a child at this level may say, "I'm wearing a pattern" about a shirt with black, white, black, white stripes.
3–4	Patterner AB	3	At this level the child recognizes, describes, and builds repeating ABAB patterns. He or she is able to fix and duplicate and extend AB patterns.
4	Patterner	4	A child at this level recognizes, describes, and builds repeating patterns, including AB but also patterns with core units such as AAB, ABC, and AABC.
4–5	Pattern Translator and Unit Recognizer	5	Children at this level translate patterns into new media or using new materials; that is, abstract and generalize the pattern. They are also able to identify the smallest core unit of a repeating pattern.
5–7	Numeric Patterner	6	At this level, a child describes a pattern numerically, can translate between geometric and numeric representation of a series. For example, given objects in a geometric pattern, child describes the numeric progression.
5–7	Beginning Arithmetic Patterner	7	A child at this level recognizes and uses relatively transparent arithmetic patterns with perceptual or pedagogical support, first that involve properties of zero. Accepts number sentences not in the form of $a + b = c$ (e.g., $c = a + b$, or $a + b = c + d$); this represents a move from Pre-Equivalence, "equals-as-an-answer" to Equal Numbers Relater. For example, child recognizes and uses patterns (e.g., that can be symbolized as $a + b - b = a$).
6–7	Relational Thinker +/-	8	At this level, a child recognizes and uses patterns that involve addition and subtraction and can compare two sides of a number sentence with reasoning without actually carrying out the computations. For example, the child recognizes $3 + 6 - 3 = 6$ as a true statement without preforming computations. In functional thinking, the child creates functional relationships between two data sets but only for specific cases. He or she may use letters to represent numbers, but only as representing objects or fixed values.
6–7	Relational Thinker – Symbolic +/–	9	A child at this level recognizes and uses patterns that involve addition and subtraction and an understanding of equality. Can compare two sides of a number sentence with reasoning, even when the quantities are represented by variables, such as $a + b = b + a$. In functional thinking, generalizes functional relationships between two data sets, at first just noticing, and later a quantitative relationship. Uses letters for unknown numbers, an initial algebraic notion.
6–8	Relational Thinker with Multiplication	10	Children at this level recognize and use patterns that involve multiplication as repeated addition and use of the distributive property to partition number facts. In functional thinking, they generalize functional relationships between two data sets, using letters as variables to represent this relationship.
6–8	Functional Algebraic Thinker	11	In functional thinking, the child generalizes functional relationships between two data sets, understanding the boundaries of generalizability and thus understanding the function as a math object. For example, given a situation involving two data sets, the child relates the columns in a t-chart, describing the specific math transformation in a formula and understands that changing any conditions would change the relationship. He or she uses variables to show the change, such as $C = S + S$ changed to $C = S + S + 1$ if the engine is counted.

Developmental Levels for 2-D Shapes

Children learn to match, name, describe, represent, construct, and classify two-dimensional ("flat") geometric shapes, increasingly using mathematical properties of the shapes.

Age Range	Level Name	Level	Description
0–2	"Same Thing" Comparer (Foundations)	1	Perceives the shape and size of objects. Forms intuitive mental prototypes of shapes, especially those that are symmetric and closed. Uses shape to form intuitive categories of objects in the world. Toddlers use the shape of objects to learn new vocabulary words. Comparing: Compares real-world objects. Judges two shapes the same if they are visually similar in any way.
0–2	Shape Matcher—Identical, Orientations & Sizes	2	Identical: Matches familiar shapes (circle, square, typical triangle) that have the same size and orientation. In specific congruence tasks, may think that two shapes are the same if they are more visually similar than different. Orientations: Matches familiar shapes with different orientations. Sizes: Matches familiar shapes with different sizes. When working on specific congruence tasks, children at this level may think that two shapes are the same if they are more visually similar than different. They may also think that two shapes are the same after matching one side on each.
3	Shape Recognizer—Typical	3	Recognizes and names typical circle, square, and sometimes a typical triangle. May physically rotate shapes presented in atypical orientations to visually match them to a prototype. Sometimes names different sizes, shapes, and orientations of rectangles, but also calls some shapes "rectangles" that look rectangular but are not rectangles.
3–4	Shape Matcher—More Shapes; Sizes & Orientations; Combinations	4	Shape Matcher—More Shapes: Compares and matches a wider variety of shapes with the same size and orientation. Shape Matcher—Sizes and Orientations: Compares and matches a wider variety of shapes with different sizes and orientations. Shape Matcher—Combinations: Compares and matches combinations of shapes to each other. When working on specific congruence tasks, children at this level may look for similarities in shapes' attributes, examining full shapes, but not considering all their attributes. For example, they may say, "These are the same" as they compare two rectangles with the same height but slightly different width.
4	Shape Recognizer—Circles, Squares, and Triangles	5	Recognizes some less typical squares and triangles and may recognize some rectangles, but usually not rhombuses (diamonds). May not distinguish between sides and corners. Differentiates between 2-D and 3-D shapes and recognizes faces of 3-D shapes as 2D shapes.
4	Constructor of Shapes from Parts—Looks Like	6	Uses manipulatives representing parts of shapes, such as sides, to make a shape that "looks like" a goal shape. May think of angles as a corner (which is "pointy").
4–5	Shape Recognizer—All Rectangles	7	As children develop understanding of shape, they recognize more sizes, shapes, and orientations of rectangles.
4–5	Side Recognizer	8	Identifies sides as distinct geometric objects. May say two shapes are the same by comparing many of their attributes but not all.
4–5	Most Attributes Comparer	9	Looks for differences in attributes, examining full shapes, but may ignore some spatial relationships.
4–5	Corner (Vertex, Angle) Recognizer	10	Recognizes angles as separate geometric objects, at least in the limited context of "corners."

5	Shape Recognizer—More Shapes	11	Recognizes most familiar shapes and typical examples of other shapes, such as hexagon, rhombus (diamond), and trapezoid.
6	Shape Identifier	12	Names most common shapes, including rhombuses, without making mistakes such as calling ovals "circles." Recognizes (at least) right angles, so distinguishes between a rectangle and a parallelogram without right angles.
7	Angle Recognizer—More Contexts	13	Can recognize and describe contexts in which angle knowledge is relevant, including corners (can discuss "sharper" angles), crossings (e.g., a pair of scissors), and, later, bent objects and bends (sometimes bends in paths and slopes). Only later can a child explicitly understand how angle concepts relate to these contexts (e.g., initially may not think of bends in roads as angles; may not be able to add horizontal or vertical lines to complete the angle in slope contexts; may even see corners as more or less "sharp" without representing the lines that constitute them). Often does not relate these contexts and may represent only some features of angles in each (e.g., oblique line for a ramp in a slope context).
7	Parts of Shapes Identifier	14	At this level, the child can identify shapes in terms of their parts. For example, the child may say, "No matter how what size it is, a triangle has three sides."
7	Congruence Superposer	15	Moves and places objects on top of each other to determine congruence with knowledge that this has matched all the shapes' attributes. May also determine congruence by comparing all attributes and all spatial relationships, such as comparing every one of their sides and angles.
7	Constructor of Shapes from Parts-Exact	16	Uses manipulatives representing parts of shapes, such as sides and angle "connectors," to make a shape that is completely correct, based on knowledge of components and relationships. Construction is consistent with side lengths, right angles, symmetry, and so forth.
8	Shape Class Identifier	17	As children develop, they begin to use class membership not explicitly based on properties. For example, a child at this level may say, "I put the triangles over here, and the quadrilaterals, (including squares, rectangles, rhombuses, and trapezoids) over there."
8	Shape Property Identifier	18	At this level, a child can use properties explicitly. For example, the child may say, "I put the shapes with opposite sides parallel over here and the shapes with four sides but not opposite parallel sides over there." He or she can also see the invariants in the changes of state or shape but maintaining the shape properties.
8	Angle Representer	19	Represents various angle contexts as two lines, explicitly including the reference line (horizontal or vertical for slope; a "line of sight" for turn contexts) and, at least implicitly, the size of the angle as the rotation between these lines (may still maintain misconceptions about angle measure, such as relating angle size to the length of side's distance between endpoints and may not apply these understandings to multiple contexts).
8	Congruence Representer	20	Continued development is evidenced as children refer to geometric properties and explain with transformations. For example, a child at this level may say, "These must be congruent because they have equal sides, all square corners, and I can move them on top of each other exactly."
8	Property Class Identifier	21	Uses class membership for shapes (e.g., to sort or consider shapes "similar") explicitly based on properties, including angle measure. Is aware of restrictions of transformations and also of the definitions and can integrate the two. Sorts hierarchically, based on properties.
8	Angle Synthesizer	22	As children develop understanding of shape, they can combine various meanings of angle (turn, corner, slant) including angle measure. For example, a child at this level could explain, "This ramp is at a 45° angle to the ground."

Developmental Levels for Composing 2-D Shapes

Children learn to compose (put together) two-dimensional shapes to make other shapes, solving puzzles or making pictures or designs, or decompose ("take apart") shapes into parts (simpler shapes).

Age Range	Level Name	Level	Description
0–3	Separate Shapes Actor (Foundations)	1	Infants and toddlers manipulate shapes as individuals but are unable to combine them to compose a larger shape. Toddlers may decompose only by trial and error.
4	Piece Assembler	2	Children at this level make pictures in which each shape represents a unique role (e.g., one shape for each body part) and shapes touch. They may also fill simple puzzles in which all shapes are outlined, often using trial and error.
4–5	Picture Maker	3	At this level, a child puts several shapes together to make one part of a picture (e.g., two shapes for one arm). He or she uses trial and error and does not anticipate the creation of the new geometric shape. The child can choose shapes using "general shape" or side length. He or she may also fill easy Pattern Block Puzzles that suggest the placement of each shape and some internal lines are still given.
4–5	Simple Decomposer	4	A significant step occurs when a child is able to decompose ("take apart" into smaller shapes) simple shapes that have obvious clues as to their decomposition.
4–5	Shape Composer	5	A sign of development is when a child composes shapes with anticipation ("I know what will fit!"). A child at this level chooses shapes using angles as well as side lengths. Rotation and flipping are used intentionally to select and place shapes.
5–6	Substitution Composer	6	As sign of development is when a child is able to make new shapes out of smaller shapes and use trial and error to substitute groups of shapes for other shapes to create new shapes in different ways. At this level, children solve Pattern Block Puzzles in which they must substitute shapes to fill an outline in different ways.
6	Shape Decomposer (with Help)	7	As children develop, they can decompose shapes using imagery that is suggested and supported by the task or environment. The child is "helped" by the suggested decompositions. For example, with a hexagon, decomposition to triangles may be suggested while decomposition to squares is not.
6–7	Shape Composite Repeater	8	A child this level can construct and duplicate units of units (shapes made from other shapes) intentionally. He or she understands each as being both multiple small shapes and one larger shape. For example, the child may may continue a pattern of shapes that leads to tiling.
7	Shape Decomposer with Imagery	9	A significant sign of development is when a child is able to decomposes shapes flexibly using independently generated imagery. That is, decomposition is intentionally specified by the child.
8	Shape Composer—Units of Units	10	Children demonstrate further understanding when they are able to build and apply units of units (shapes made from other shapes). For example, in constructing spatial patterns, children can extend patterning activity to create a tiling with a new unit shape (a unit of unit shape) that they recognize and consciously construct.
8	Shape Decomposer with Units of Units	11	As children develop understanding of shape, they can decompose shapes flexibly using independently generated imagery and planned decompositions of shapes that themselves are decompositions.

Developmental Levels for Disembedding 2-D Shapes

Children learn to compose (put together) two-dimensional shapes to make other shapes, solving puzzles or making pictures or designs, or decompose ("take apart") shapes into parts (simpler shapes).

Age Range	Level Name	Level	Description
3	Intuitive Disembedder (Foundations)	1	A child at this level can remember and reproduce only one, or a small collection of non-overlapping (isolated) shapes. For example, an infant is able to find their parent's face in a picture.
4	Simple Disembedder	2	Children at this level can identify the frame of a complex figure. For example, they may trace the outer frame of a picture that contains two figures. At this level, children are also able to find some shapes in arrangements in which figures overlap, but not in which figures are embedded in others. For example, when given a picture of a smaller circle inside a larger circle, children trace the outline of the larger circle.
5–6	Shapes-in-Shapes Disembedder	3	As a children develop, they can identify shapes embedded within other shapes, such as concentric circles and/or a circle in a square. The children are able to immediately find primary structures inside complex figures. For example, when shown a square inside a circle, a child names "a square" as the smallest shape in the picture
7	Secondary Structure Disembedder	4	Children demonstrate further understanding when they can identify embedded figures even when they do not coincide with any primary structures of the complex figure. For example, they can find and trace figures, lines, or structures that are embedded and do not coincide with segments of the primary structure.
8	Complete Disembedder	5	At this level, a child can successfully identify all varieties of complex arrangements. For example, a child can find and trace all segments, figures, and shapes for both primary and secondary structures.

Developmental Levels for 3-D Shapes

Children learn about 3-D shapes—starting with common object names such as "ball" and later understanding mathematical terms such as "sphere." Three-dimensional shape understanding also includes distinguishing between the faces of an object (rectangle) from the object as a whole (rectangular prism).

Age Range	Level Name	Level	Description
0–2	3-D Perceiver: Foundations	1	Children can perceive 3-D shapes accurately from infancy; however, this competency is limited to continuously moving objects, rather than single or even multiple static views of the same object.
3–4	3-D Prototype Recognizer	2	Children at this level can recognize some prototypical 3-D shapes, such as the sphere and cube, using formal or informal names. However, they may use 2-D vocabulary to name some 3-D shapes and describe solids using a variety of informal characteristics, such as "pointiness" or "slenderness."
5–6	3-D Shape Recognizer	3	At this level a child recognizes more 3-D shapes (solids), using informal and some formal names. The child can also recognize faces as 2-D shapes.
7	3-D Face Counter	4	As sign of development is when a child can recognize all faces of a solid (3-D shape) as 2-D shapes and count the faces accurately.
8	3-D Shape Identifier	5	A child at this level identifies most solids (3-D shapes) and can name several of their attributes. He or she can also identify the common solid created by a particular net.
8	3-D Constructor of Shapes from Parts	6	Continued development is evident when a children use manipulatives to represent parts of three-dimensional shapes, such as faces ("sides" that are 2-D shapes) and angle "connectors," to make a 3-D shape (a "solid") that is completely correct, based on knowledge of components and relationships (such as right angles.) For example, with some manipulatives, such as toothpicks or similar materials (e.g., D-Stix), children may first build one or more 2-D shapes and then connect them. With those that are the 2-D shapes (e.g., connect magnetically) they can build 3-D shapes directly.
8	3-D Shape Class Identifier	7	Children demonstrate further understanding when they can identify most solids (3-D shapes), based on their properties.

Developmental Levels for Composing 3-D Shapes

Children learn to compose (put together) three-dimensional shapes to make other 3-D shapes.

Age Range	Level Name	Level	Description
0–1	Separate Blocks Actor (Foundations)	1	At this level, a child either places blocks randomly, or manipulates shapes as individuals, but does not combine them to compose a larger shape. He or she may pound, clap together, or slide blocks or use single blocks to represent an object, such as a house or truck.
1	Stacker	2	A child at this level shows use of the spatial relationship of "on" to stack blocks but chooses blocks unsystematically.
1	Line Maker	3	A child at this level understands and applies the concept of "next to" to make a line of blocks.
2	Same Shape Stacker	4	Children at this level show use of relationship of "on" to stack congruent blocks, or those that show a similarly helpful relationship to make stacks or lines.
2–3	Piece Assembler (3-D)	5	At this level, a child builds vertical and horizontal components within a building, but within a limited range, such as building a "floor" or simple "wall." These then are two-dimensional structures.
3–4	Picture Maker (3-D)	6	Children at this level use multiple spatial relations, extending in multiple directions and with multiple points of contact among components, showing flexibility in integrating parts of the structure. Starting at 30 months, children produce arches/bridges, enclosures, corners, and crosses, although may use unsystematic trial and error and simple addition of pieces.
4–5	Shape Composer (3-D)	7	At this level, children compose shapes with anticipation, understanding what 3-D shape will be produced with a composition of two or more other (simple, familiar) 3-D shapes. Can produce arches (with vertical interior space), enclosures (with internal horizontal space), corners, and crosses systematically. They build enclosures and arches with several blocks. Later in this level, children add depth to make 3-D structures, and they add roofs across structures multiple blocks high (but they may have no internal spaces).
5–6	Substitution Composer and Shape Composite Repeater (3-D)	8	A child at this level substitutes a composite for a congruent whole. He or she builds complex bridges with multiple arches, with ramps and stairs at the ends. Structures are 3-D, often including roofs and multiple internal spaces.
6–8	Shape Composer—Units of Units (3-D)	9	Children at this level make complex towers or other structures, involving multiple levels with ceilings (fitting the ceilings), and adult-like structures with blocks, including arches and other substructures.

Developmental Levels for Spatial Visualization and Imagery

Children develop the processes of generating, maintaining, and manipulating mental images of two- and three-dimensional objects, including moving, matching, and combining them. Such imagery and visualization are important across many topics in mathematics. Here, we emphasize the movement, especially mental movement, of shapes.

Age Range	Level Name	Level	Description
0–1	Intuitive Mover: Foundations	1	A child at this level explores the size and shape of objects by observing them as they move in space, using trial and error to discover how they fit into space, and eventually predicting what will fit inside a space without attempting all possible solutions. Such skills will eventually support future spatial visualization.
1–2	Concrete Slider, Flipper, Turner	2	Children at this level can move shapes to a location by physical trial and error.
3–4	Simple Slider and Turner	3	At this level, a child slides and turns objects accurately in easy tasks, guided by an early intuition that starts the motion and then adjusts (the motion, direction, or amount) in real time as the motion is carried out.
5	Beginning Slider, Flipper, Turner	4	Children at this level use the correct motions guided by more developed intuition, but not always accurate in direction and amount (adjusts these with trial and error).
6	Slider, Flipper, Turner	5	A child at this level performs slides and flips, often only horizontal and vertical, using manipulatives but guided by mental images of these motions (of turns of 45, 90, and 180° and flips over vertical and horizontal lines). That is, they can mentally imagine the motion and the result of it.
7	Diagonal Mover	6	At this level, a child performs diagonal slides and flips as well as all motions from previous levels.
8	Mental Mover	7	Children at this level predict results of moving shapes using mental images (any direction or amount).

Developmental Levels for Spatial Orientation

Children learn the relationships between different positions in space. For example: using coordinate grids and maps.

Age Range	Level Name	Level	Description
0	Foundations of Spatial Orientation	1	Children at this stage uses the earliest of two types of cognitive systems for spatial orientation—knowing where you are and how to get around in the world. Foundations includes the earliest of each of the following two types. Response Learning: Uses the first self-based system, related to the child's own position and movements. In Response Learning, the child notes a pattern of movements that have been associated with a goal, such as looking to the left when in a highchair, because that's where the food usually comes from. Cue Learning: Uses the first external-based systems, based on familiar landmarks in the child's world. In Cue Learning, the child associates a toy bear with a small chair on which it often sits.
0–2	Path Integrator	2	A child at this stage remembers and can repeat movements they have made including the approximate distances and directions. Children can crawl to a place of their choice, navigating an obstacle to arrive at a destination within sight.
1–2	Place Learner	3	Children at this stage, create "mental maps" by storing locations, distances, and directions to landmarks and solve spatial problems. They use the walls of a room as a frame of reference and use spatial vocabulary, such as "in," "on," and "under," along with vertical directionality terms as "up" and "down."
2–3	Local-Self Framework User	4	A child at this stage uses distant landmarks to find objects or locations near them, even after they have moved themselves relative to the landmarks, as long as the target object is specified ahead of time. The child can orient a horizontal or vertical line in space and uses spatial vocabulary to direct attention to spatial relations, including more difficult terms such as "beside" and "between."
4	Small Local Framework User	5	Children at this stage locate objects after movement, even if target is not specified ahead of time. They search a small area comprehensively, often using a circular search pattern. They can also use words referring to frames of reference such as "in front of" and "behind" or "left" and "right." In meaningful graphing contexts, children extrapolate lines from positions on both axes (like a coordinate grid) and determines where they intersect.
5	Local Framework User	6	A child at this stage locates objects after moving, maintaining the overall shape of the arrangement of objects. They represent an objects' positions relative to landmarks (e.g., about halfway in between two landmarks) and can keep track of their own location in open areas or mazes. The child uses spatial vocabulary to direct attention to spatial relations and can use coordinate labels in simple situations such as games.
6	Map User	7	Children at this stage locate objects using maps with pictorial cues. They extrapolate (extend) two coordinates, understanding the integration of them to one position, as well as use coordinate labels in simple situations.
7	Coordinate Plotter	8	A child at this stage reads and plots coordinates on a map.
8	Route Map Follower	9	Children at this stage follow a simple route map, with more accurate direction and distances.
8	Framework User	10	A child at this stage uses general frameworks that include the observer and landmarks. They may not use precise measurement, unless guided to do so. The child can follow and create maps, even if spatial relations are transformed.

Developmental Levels for Measurement: Length

Children learn to compare and then quantify the distance between the endpoints of an object.

Age Range	Level Name	Level	Description
1–2	Length Senser: Foundations	1	A child at this level makes simple comparisons of length intuitively (similar to what we saw in Subitizing) as young as six months of age. However, he or she may not recognize length as a distinct attribute (separate from general size, such as “small” and “big”).
3	Length Quantity Recognizer	2	At this level, children identify length/distance as an attribute. They may understand length as an absolute descriptor (e.g., all adults are tall), but not as a comparative (e.g., one person is taller than another). Children may also compare non-corresponding parts of shape in determining side length.
4	Length Direct Comparer	3	Children at this level physically align two objects to determine which is longer or if they are the same length. They use terms: long, longer, longest.
4–5	Length Indirect Comparer	4	A child at this level compares the length of two objects by representing them with a third object. He or she uses terms: long, longer, longest, short, shorter, shortest. When asked to measure, the child may assign a length by guessing or moving along a length while counting (without equal-length units). He or she may be able to measure with a ruler, but often lacks understanding or skill (e.g., ignores starting point).
4–5	Serial Orderer to 5 (Length)	5	Children at this level order lengths, marked in one to five units. They can also compare unmarked lengths that are clearly different using broad categories (“big” and “small”) and so can order three to five such objects but only by trial and error. With an increase in working memory, children begin to build a mental image of the final ordering in which the lengths increase “bit by bit” with each successive length the smallest increase. This leads to more accurate and somewhat more efficient ordering. (This level develops in parallel with “End-to-End Length Measurer”.)
4–5	End-to-End Length Measurer	6	At this level, children lay units end to end. They may not recognize the need for equal-length units or be able to measure if there are fewer units than needed. The ability to apply resulting measures to comparison situations develops later in this level. May use rulers with substantial guidance. (This level develops in parallel with “Serial Orderer to five (Length)”).
5–6	Serial Orderer to 6+ (Length)	7	A child at this level orders lengths, marked in one to six units. The child understands at least intuitively that any set of objects of different lengths can be placed into a series that always increases (or decreases) in length, so spontaneously seriates with few errors by selecting the shortest (or longest) object, then the next shortest (the one with the “smallest difference”), and so forth.
7	Length Unit Relater and Repeater	8	Children at this level measure by repeating (iterating) a single unit and understands the need for equal-length unit. They relate the size and number of units (inverse relationship). They can also add two lengths to obtain the length of a whole. Children often can use rulers with minimal guidance in straightforward situations.
8	Length Measurer	9	At this level, children measure, knowing need for identical units, relationship between different units, partitions of unit, zero point on rulers, and accumulation of distance. They also consider the length of a bent path as the sum of its parts (not the distance between the endpoints). These children begin to estimate.
8	Conceptual Ruler Measurer	10	A child at this level possesses an “internal” measurement tool. He or she mentally moves along an object, segmenting it and counting the segments. The child operates arithmetically on measures (“connected lengths”). He or she subdivides a unit at least into halves. The child can also estimate with accuracy.

9	Integrated Conceptual Path Measurer	11	Children at this level compute perimeter of a polygon, including complex cases. They can find several related cases of polygons with the same perimeter, or the same area, and relate those cases to one another by logical comparison to provide evidence of an underlying pattern. They also make claims about sets of complex paths based on perimeter and other linear aspects of the objects. The children can analyze length within two- and three-dimensional contexts. In selection of units, children show well-developed ideas of precision and accuracy.
9–10	Abstract Length Measurer	12	Children at this level organize and synthesize sets of objects based on perimeter or collections of complex bent paths based on overall length in two- or three-dimensional contexts to formulate and justify a valid argument. They can construct derived units with linear measures, such as miles per hour, and make appropriate unit conversions with derived measures. They also compute perimeter or path length, including units and divisions of units including measures of non-integer values. The children can explain that this subdivision process is potentially unlimited. They notice and are perturbed by geometric inconsistencies within polygons. They can measure to the degree of precision allowed by a tool by estimating to a fraction of the smallest calibration mark provided on the instrument.

Developmental Levels for Measurement: Area

Children learn to compare and then quantify an amount of two-dimensional surface that is contained within a boundary.

Age Range	Level Name	Level	Description
0–3	Area Senser: Foundations	1	Even children in their first year are sensitive to area. However, they may not explicitly recognize area as an attribute (separate from general size, such as "small" and "big") for some time. If asked to fill in a rectangle, preschoolers may just draw approximations of circles. Children at this level use side matching strategies in comparing areas.
4	Area Quantity Recognizer	2	At this level a child perceives the amount of two-dimensional space and can make intuitive comparisons. However, when asked to compare, the child may compare lengths more than areas because lengths are salient and familiar to them (e.g., compare one side of one piece of paper to the side of another) or make estimates based on a "length plus (not times) width" intuition. He or she may compare areas correctly if the task suggests superposition (putting one on top of the other). When a child is asked to partition a space into squares or copy an image of a rectangle partitioned into an array (rows and columns), he or she may simply draw squares inside the rectangle or other types of shapes or short paths on or around the rectangle.
4–5	Physical Coverer and Counter	3	Prompted to measure, children at this level attempt to cover a rectangular space with physical tiles. However, they do not organize or structure the 2-D space without considerable perceptual support, such as a grid that outlines each individual unit. In drawing (or imagining and pointing to count squares as units of area), children represent only certain aspects of that structure, such as approximately rectangular shapes next to one another. They make comparison areas based on simple, direct comparisons (e.g., a child places one sheet of paper over another piece of paper to select the sheet that covers more space).
5	Complete Coverer and Counter	4	At this level, a child draws a complete covering of a specific region without gaps or overlaps and in approximations of rows. When provided with more than the total number of physical tiles needed, he or she can build a region of specified area (e.g., build a rectangle with an area of twelve from a pile of twenty tiles).
5	Area Unit Relater and Repeater	5	A child at this level counts individual units, often trying to use the structure of rows. To cover a region with physical units, he or she repeats (iterates) an individual unit. The child can also draw a complete covering based on an intuitive notion of rows and columns, making equal sized units, but often draws them one at a time. He or she can relate the size and number of units to cover a region, recognizing that differently sized units will result in different measures and that the larger the unit, the fewer will be needed. The child also compares areas by accurately counting units in each and comparing the resulting measures.
6	Initial Composite Structurer	6	Children at this level identify a square unit as both a unit and a component of a larger unit of units (a row, column, or group) and uses those structures in counting or drawing. However, they need figural support to structure the space themselves (this may include physical motions of some of the tiles or drawing some collections of units rather than from using the dimensions). At this level, children usually do not coordinate the width and height and in measuring, may not use the dimensions of the rectangle to constrain the unit size. They can make reasonable estimates of areas.
7	Area Row and Column Structurer	7	At this level, children decompose and recompose partial units to make whole units. For example, children draw rows as rows making parallel horizontal lines and so forth. They begin conserving area and reasons about additive composition of areas (e.g., how regions that look different can have the same area measure) and recognizes the need for space-filling in most contexts.

Developmental Levels for Measurement: Volume

Children learn to compare and then quantify an amount that can be "packed" or "filled" into a 3-D space with iterations of a fluid unit that takes the shape of the container.

Age Range	Level Name	Level	Description
0–2	Volume Senser (Foundations)	1	Children are sensitive to volume even in the first year; however, they may not for some time explicitly recognize volume as an attribute (separate from general size, such as "small" and "big").
1–3	Volume Quantity Recognizer	2	At this level, a child identifies capacity or volume as attribute. He or she builds with blocks, associating more blocks with terms like "big" and fewer blocks with terms like "small."
3–5	Volume Filler	3	A child at this level can compare two containers by pouring one into the other (although can be confused at "which holds more" at first). He or she fills a container using another (smaller container) and counts the number needed to completely fill the larger container (but may not use accurately filled scoops and may not focus on quantifying the total volume or capacity). In packing situations, the child places cubes into a rectangular box to fill it. Eventually the child packs entire box with cubes in an organized way. He or she compares objects by physically or mentally aligning and refers to at least two dimensions of objects. The child may be able to compare two containers using a third container and transitive reasoning.
5–6	Volume Quantifier	4	At this level, children have partial understanding of cubes as filling a space. They are able to estimate number of scoops needed to fill and attend to both the portion of container filled and the portion remaining unfilled. Children also recognize when container is half full. They exhibit initial spatial structuring. They pack can pack a box neatly and completely with cubes; may count one cube at a time, while packing, to determine total. Children at this level compare objects by physically or mentally aligning and explicitly recognizing three dimensions
7	Volume Unit Relater and Repeater	5	Children at this level use simple units to fill containers, with accurate counting. They relate size and number of units explicitly, and understand that they will need fewer larger units, than smaller units, to fill or pack a given container. They can accurately convert units in 1:2 ratio.
7	Initial Composite 3-D Structurer	6	At this level, a child understands cubes as filling a space but does not use layers or multiplicative thinking. He or she moves to more accurate counting strategies. The child relates number of cubes to cubic units as measured by capacity. Given a graduated cylinder marked in cubic-inch units, the child understands that sand filled to the ten in the cylinder would fill a box that holds ten, one-inch cubes. He or she begins to visualize and operate on composite units such as rows or columns (what we call a $1 \times 1 \times n$ core). The child iterates to pack the space completely, accounting for "internal/ hidden" cubes. The child also decomposes space, allowing for accurate use of units and subunits. He or she recognizes when a box is half full, visualizes remaining rows or columns.
8	3-D Row and Column Structurer	7	Children at this level are able to coordinate flexibly filling, packing, and building aspects of volume. They show a propensity for additive comparisons (e.g., "this one has twelve more") but may show some nascent multiplicative comparisons (e.g., "this one is four times as big"). Initially children count or compute (e.g., number of rows times number of columns) the number of cubes in one layer, and then use addition or skip counting by layers to determine the total volume. Eventually they move to multiplication (e.g., number of cubes in a layer times number of layers).
8	3-D Array Structurer	8	At this level, a child has an abstract understanding of the rectangular prism volume formula. He or she shows a propensity for multiplicative comparisons, coordinates multiplicative and additive comparisons flexibly. With linear measures or other similar indications of the three dimensions, the child multiplicatively iterates cubes in a row, column, and/or layers to determine the area. (Constructions and drawings are not necessary.) In multiple contexts, the child can compute the volume of rectangular prisms from its dimensions and explain how that multiplication creates a measure of volume.

Developmental Levels for Measurement: Angle and Turn

Children learn to compare and then quantify angle and turn measure.

Age Range	Level Name	Level	Description
0–2	Angle and Turn Senser (Foundations)	1	Children are sensitive to angles as turning, both turning objects and their own body. (See more at the first three levels of the Spatial Orientation and the first level of Spatial Visualization.)
1–3	Intuitive Angle Builder	2	At this level, a child intuitively uses some angle measure notions in everyday settings, such as building with blocks, solving puzzles, and walking.
4–5	Implicit Angle User	3	A child at this level uses angles and some angle measure concepts, such as parallelism and perpendicularity (at least implicitly)—in physical alignment tasks, construction with blocks, or other everyday contexts. They may identify corresponding angles of a pair of congruent triangles using physical models. The child uses the word “angle” or other descriptive vocabulary to describe some of these situations
6	Angle Matcher	4	At this level, children match angles concretely. They can explicitly recognize parallels from non-parallels in specific contexts. They can also sort angles into “smaller” or “larger” (but may be misled by irrelevant features, such as length of line segments).
7	Angle Size Comparer	5	Children at this level differentiate angle and angle size from shapes and contexts and can also compare angle sizes. They first recognize right angles and then equal angles of other measures, in different orientations. Children at this level also compare simple turns. (Note that, without instruction, this and higher levels may not be achieved even by the end of the elementary grades.)
8	Angle Measurer	6	At this level, a child understands angle and angle measure in both primary aspects. They can represent multiple contexts in terms of the standard, generalizable concepts and procedures of angle and angle measure (e.g., two rays, the common endpoint, rotation of one ray to the other around that endpoint, and measure of that rotation).

Developmental Levels for Classification and Data Analysis

This learning trajectory develops children's ability to understand, gather, and use data. In the early years, data is an important context for solving problems but data analysis itself develops slowly. Children first learn the foundational ideas and processes of classification, and then learn to quantify the categories; that is, tell how many in each group. Eventually children learn to gather data to answer a question or decide an effective means to develop applied problem solving and number and/or spatial sense, as children simultaneously learn about data representations and analyses.

Age Range	Level Name	Level	Description
0–1	Foundations: Similarity Intuiter	1	Intuitively children recognize objects or situations as similar in some way (e.g., objects to suck or not, at around two weeks). Later, children place objects together that are different (six months) and then alike (12 months).
1–2	Similar/ Dissimilar Maker	2	By 18 months, children can form a set in which are identical and another in which objects are different. By two years of age, children intuitively form groups with objects that are similar on some attributes (may be mixed and inconsistent), but not necessarily identical. They may use functional relationships as a basis for sorting.
3	Simple Sorter	3	At this level, a child follows verbal rules for sorting scaffolded by an adult. (These may be made with shifting criteria; nevertheless, they play an essential role in number, through the unitizing process.) He or she can "fix" a simple sort with mistakes.
4	Sorter by Similar Attributes	4	Children at this level sort objects according to an explicit attribute (although they still may decide to switch attributes during the sorting). The result may appear to reflect adult categorizations, but often has a different basis, such as general resemblance.
4–6	Consistent, Flexible Sorter	5	At this level a child can sort consistently by a single attribute and re-classifies by different attributes. He or she sorts consistently and exhaustively by an attribute, given or created, and uses the terms "some" and "all."
4–6	Data Case Viewer	6	At this level children associate a value with an individual case. They use numeric data to identify largest/smallest cases and may graph by listing all cases. Before this level, children may be "pointers" in which data records point to the entire event ("We talked about favorite colors"). They use it like string tied around a finger, to remember that they did something.
5–6	Data Classifier	7	A child at this level treats cases with similar values as the same. He or she compares category frequencies (most and least popular case-types). The child visually compares two graphs and makes graphs by classifying and representing data in those categories.
5–7	Multiple Attribute Classifier	8	Children at this level classify objects by multiple attributes in a single sort.
7–8	Data Aggregator	9	At this level, children classify objects that may be perceptually different by more abstract attributes such as function or conceptual attributes. They focus on features of the data set as a whole. They describe relative frequency and density (shape), and location (centers). Children begin to understand the concepts of expectation (averages and probabilities) and variation ("spread" of values). Children at this level also understand ranges in data or the mode (the number that occurs most frequently). Eventually, they can focus on features of the data set as a whole, including the relative frequencies, density ("shape"), and location (centers, such as the mean).
8	Hierarchical Classifier	10	A child at this level classifies categories and subcategories using hierarchical inclusion. The child conscientiously classifies according to multiple attributes, naming and relating the attributes, understanding that objects could belong to more than one group.
8	Data Representer	11	At this level, a child shows an appreciation of the "center" of graphs and for their variation or spread. He or she compares graphs of data sets of the same size accurately. Similarly, the child can use other representations such as tables to detect variation in different sets, develop models of those variations, and apply them to make predictions.