During the early planning phase of *Everyday Mathematics 4*, a team of writers worked together to read and summarize the current body of research about fractions. The information that follows is the summary of that research. For more information about the planning phases of *Everyday Mathematics 4*, see the paper "*Everyday Mathematics* and the Writing Process."

Summary of the Examined Research

At the inception of our work in researching the conceptual development of mathematical concepts, we chose to focus our literature searches and reviews on learning trajectories and progressions for each strand. This focus stemmed from information in a document from the Consortium for Policy, Research, and Education that connected current research on learning trajectories with the Common Core State Standards (CCSS) (Daro, et al., 2011). As a result, for the rational number strand, we focused our reading on sources that included at least one component of a learning trajectory as defined by Confrey, et al.:

 "A researcher-conjectured, empirically-supported description of the ordered network of experiences a student encounters through instruction (i.e. activities, tasks, tools, forms of interaction and methods of evaluation), in order to move from informal ideas, through successive refinements of representation, articulation, and reflection, towards increasingly complex concepts over time" (Confrey, et al., 2008).

For the research summary that follows, we concentrated on those sources that included a sequence of concepts, instructional activities, or both, that covered a significant portion of the development of initial fraction concepts and operations. The narrative does not include a discussion of the research that supports the progressions or trajectories we reference. However, the authors of these sources represent long-standing and respected organizations in the field of elementary mathematics education. For example, Empson and Levi cite previous research by Cognitively Guided Instruction publications and extend those ideas (problem types and strategies) to include fraction and decimal ideas (Empson & Levi, 2011). The Rational Number Project (RNP) cites previous RNP research and work from two recently NSF-funded curriculum and research projects (RNP, 2009a; RNP, 2009b; Cramer, et al., 2002; Cramer & Wyberg, 2009). A dissertation by Peter Wilson under the direction of Jere Confrey provides a summary of relevant research on learning trajectories and rational number reasoning from the DELTA Project (Wilson, 2009; Myers, et al., 2009; Confrey, et al., 2008).

Research Summary

Wilson; Confrey, et.al; Empson and Levi; Fosnot and Dolk; and Lamon recommend developing fraction understanding through equal sharing of collections and/or partitioning regions into fair shares. The developmental timetable, the language, and the order in which concepts develop vary from author to author. However, they all advocate for students to actively decide how to partition sets and regions, in contrast to a more traditional development in which students identify or color fractional parts of already partitioned regions and then use standard notation to write numerators

and denominators. (Wilson, 2009; Confrey, et al., 2008; Empson & Levi, 2011; Fosnot & Dolk, 2002; Lamon, 2005; Lamon, 2007; RNP, 2009a; RNP 2009b).

Empson & Levi, Fosnot & Dolk, and, to some extent, Lamon and RNP recommend the development of meaning for fractions (e.g., concept of a whole) and fraction operations through the use of word problems in everyday contexts. In particular, Empson & Levi and Fosnot & Dolk rely on students' drawings and explanations of invented strategies for fraction representations. Wilson describes using counters for sharing collections and folding paper rectangles and circles to partition regions (Wilson, 2009; Empson & Levi, 2011; Fosnot & Dolk, 2002; Lamon, 2005; RNP, 2009a; RNP, 2009b).

The general progression for developing fraction concepts through equal sharing and partitioning begins with fair shares of collections and regions for two people (halving and doubling) with no remainders, moves to equal sharing in which students must reason about remainders that produce denominators of 2 and 4, extends to small odd numbers, and continues to larger denominators. For example, Wilson includes a progression with eight levels that culminates in solving problems that involve m objects shared among p people ($m<p$ for proper fractions and $m>p$ for improper fractions and mixed numbers). The values for p (the denominators) begin with 2 and 4 and progress to include odd numbers and larger denominators (Wilson, 2009; Myers, et al., 2009). For each level Wilson delineates a within-level framework for activities and language that grow in sophistication (Wilson, 2009):

1) Methods: Solve the task (How could you share?)
2) Multiple methods: Solve the task in multiple ways. (Is there another way to share?)
3) Justification: Justify solutions. (How do you know it's a fair share?)
4) Naming: Mathematically name the solution. (What would you call a share?)
5) Reversibility: Learn to reverse the process - often by multiplication. (If we put everyone's share back together, what would it be?)
6) Properties: Composition, compensation, transitivity, equivalence.

Empson and Levi begin using similar ideas with young children, but include development of concepts of equivalence and order through word problems. For example, students first solve problems that ask, "Who has more? A group of four students who share two apples or two students who share one apple?" By Grade 6, students solve similar problems that involve rates in which the denominators are not multiples of one another (e.g., If a machine can copy 18 pages in four seconds, how long will it take to copy 27 pages?). Equal sharing naturally leads to multiplicative reasoning, and they recommend including these types of problems in the early grades (e.g., How many whole sandwiches are needed if four children each get ½ sandwich?). Similarly, they recommend students as early as grade 1 solve problems that ask, "A snack is ½ slice of cheese, how many snacks in 2 slices?" Addition and subtraction are introduced later through invented strategies for solving word problems with no explicit instruction of the use of common denominators. The authors do recommend a series of problems that ask students to

solve open number sentences using reasoning about denominators and what they know about fractions (e.g., 2 x d = ½; ½ = a + ¼; 8 x 1/5 = n + 3/5) (Empson and Levi, 2011).

Fosnot and Dolk develop similar concepts through a series of big ideas using word problems, invented strategies, and student drawings. They recommend using sequences of word problems within one activity that increase in complexity while developing a big idea (Fosnot and Dolk, 2002).

RNP recommends using manipulatives to represent part/whole fractions for "initial fraction ideas". In particular, they recommend using fraction circle pieces and fraction strips (paper folding) as representations to develop visual images of fractions, so that students develop number sense for the size of fractions and can use the visual images to compare fractions and check the reasonableness of their answers. The introduction to RNP's *Fraction Operations & Initial Decimal Ideas Curriculum Module* states, "Our work over 20 years has shown that of all the manipulatives available for teaching about fractions, fraction circles are the most effective for building mental images that support students' understanding of order, equivalence and fraction operations" (RNP, 2009b). Results of a study in *Math Trailblazers* classrooms that used pattern blocks instead of fractions circles showed that, "Pattern blocks had limited value in aiding students' construction of mental images for the part-whole model as well as limited value in building meaning for adding and subtracting fractions" (Cramer & Wyberg, 2007).

RNP also uses chip representations in its curricula to develop fraction ideas for a discrete model and they use fractions strips as a transition from a part/whole model to the number line. These representations, along with pictures of other regions, are used to develop ideas of equivalence and to compare fractions. RNP strongly recommends having students spend a considerable amount of time using the representations to develop "initial fraction ideas" before moving to operations and to use activities that ask students to make connections among the representations. Addition and subtraction instruction begins with estimation activities, so that students can judge the reasonableness of their answers. Fraction circle pieces are used to develop methods for addition and subtraction with and without common denominators. Students explore multiplication and develop procedures using fraction circle pieces and paper folding of square regions. After students have learned how to add, subtract, and multiply using paper-and-pencil procedures, they deepen their understanding of the algorithms by explaining how to add, subtract, and multiply using number lines. Division is introduced through word problems along with student-invented strategies and the common-denominator algorithm (RNP, 2009a; RNP, 2009b; Cramer, et al., 2002; Cramer & Wyberg, 2009).

RNP makes a strong case for using consistent representations of fractions to develop initial fraction ideas as well as strategies and procedures for fraction operations. Using consistent representations supports the development of mental images that students will continue to use to estimate, compare, and perform operations.

In general, the researchers recommend delaying the use of fraction symbols and to allow for the development of conceptual knowledge. This delay helps students avoid errors from attending to superficial features of fraction notation (Empson and Levi, 2011) and from misapplying whole number ideas as they begin work with fractions. Students use language (such as halves, fourths, fourth of) before a/b notation. RNP recommends beginning with numeral-word notation, e. g, 1-half, 2-thirds (RNP, 2009a).

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