CHAPTER 7

Unit 2: Exponential and Logarithmic Functions

IN THIS CHAPTER

Summary: This chapter introduces arithmetic and geometric sequences and shows their relationship with linear and exponential functions. It also introduces the composition of functions and the inverse of a function. Using inverses, the logarithmic function is introduced, along with properties and graphs of exponential and logarithmic functions. Modeling aspects of contextual scenarios are also examined.

KEY IDEA

Key Ideas

✪ A sequence is an ordered list of numbers.
✪ Arithmetic sequences are sequences that have a common difference between terms.
✪ Geometric sequences are sequences that have a common ratio between terms.
✪ Arithmetic and geometric sequences are similar to linear and exponential functions.
✪ Functions can be combined using composition.
✪ Inverse functions are essential to solving equations and inequalities.
✪ Properties of exponential functions and their inverse function logarithms can be used to solve equations and inequalities.
✪ Exponential and logarithmic functions can be used to model many phenomena.

Sequences

A *sequence* is an ordered list of numbers that often follow a specific pattern or function. Each number in a sequence is called a *term*. Each term has a whole number position such as first, second, or third.

Example

The first 10 terms of a sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55. Find the next 3 terms.

Solution: The terms of the sequence are found by adding together the two preceding numbers in the sequence: $1 + 1 = 2$, $1 + 2 = 3$, $2 + 3 = 5$, and so on.

Term 11 = term 9 + term 10 = 34 + 55 = 89

Term 12 = term 10 + term 11 = 55 + 89 = 144

Term 13 = term 11 + term 12 = 89 + 144 = 233

Fun Fact: The preceding sequence is known as the Fibonacci Sequence, named after Leonardo of Pisa, later known as Fibonacci. Many real-world illustrations of the Fibonacci Sequence are found in nature. If you count the seed spirals of a sunflower in one direction, you will get the numbers in the Fibonacci Sequence.

Arithmetic Sequences

An *arithmetic sequence* is a sequence that has successive terms that have a constant rate of change or a common difference.

The general term of an arithmetic sequence is $a_n = a_0 + dn$, where a_0 is the initial value and d is the common difference. An alternate form is $a_n = a_k + d(n - k)$, where a_k is the kth term of the sequence.

Example

Write a formula for the sequence 6, 10, 14, 18, 22, . . . , then use that formula to find the 100th term of the sequence.

Solution: We first need to determine if the sequence is arithmetic by finding the difference between successive terms. The difference between term 2 and term 1 is $10 - 6 = 4$. The difference between term 3 and term 2 is $14 - 10 = 4$. The difference between term 4 and term 3 is $18 - 14 = 4$. The difference between term 5 and term 4 is $22 - 18 = 4$. Because the differences are the same, it is an arithmetic sequence.

Next, substitute values into the formula $a_n = a_0 + dn$. The initial value a_0 is found by subtracting the difference d from the first term in the sequence: $a_0 = a_1 - d = 6 - 4 = 2$.

Therefore, the formula is $a_n = 2 + 4n$. Term 100, or $a_{100} = 2 + 4(100) = 402$.

Geometric Sequences

A *geometric sequence* is a sequence that has successive terms that have a constant proportional change or a common ratio.

The general term of a geometric sequence is $g_n = g_0 r^n$, where g_0 is the initial value and r is the common ratio. An alternate form is $g_n = g_k r^{(n-k)}$, where g_k is the kth term of the sequence.

Example

Write a formula for the sequence 2, −6, 18, −54, 162, ... Use that formula to find the 12th term of the sequence.

Solution: We first need to determine if the sequence is geometric by finding the ratio between successive terms. The ratio between term 2 and term 1 is $\frac{-6}{2} = -3$. The ratio between term 3 and term 2 is $\frac{18}{-6} = -3$. The ratio between term 4 and term 3 is $\frac{-54}{18} = -3$. The ratio between term 5 and term 4 is $\frac{162}{-54} = -3$. Because the ratios are all the same, it is a geometric sequence.

Next, substitute values into the formula $g_n = g_0 r^n$. The initial value g_0 is found by dividing the ratio r by the first term in the sequence: $g_n = \frac{g_1}{r} = \frac{2}{-3} = -\frac{2}{3}$. Therefore, the formula is $g_n = -\frac{2}{3}(-3)^n$. The 12th term is $g_{12} = -\frac{2}{3}(-3)^{12} = -\frac{2}{3}(531{,}441) = 354{,}294$.

Arithmetic sequences are based on addition, while geometric sequences are based on multiplication.

Change in Linear and Exponential Functions

Linear functions of the form $f(x) = b + mx$ are similar to arithmetic sequences of the form $a_n = a_0 + dn$, because both can be expressed as an initial value (b or a_0) plus repeated addition of a constant rate of change, the slope (m or d). Similar to arithmetic sequences of the form $a_n = a_k + d(n - k)$, which are based on a known difference, d, and a kth term, linear functions can be expressed in the form $f(x) = y_i + m(x - x_i)$ based on a known slope, m, and a point (x_i, y_i).

Exponential functions of the form $f(x) = ab^x$ are similar to geometric sequences of the form $g_n = g_0 r^n$, as both can be expressed as an initial value (a or g_0) times repeated multiplication by a constant proportion (b or r). Similar to geometric sequences of the form $g_n = g_k r^{(n-k)}$, which are based on a known ratio, r, and a kth term, exponential functions can be expressed in the form $f(x) = y_i r^{(x-x_i)}$ based on a known ratio, r, and a point, (x_i, y_i). It should be noted that sequences and their corresponding functions may have different domains. Specifically, linear and exponential function domains are all real numbers, but sequences have domains of whole numbers.

Over equal-length input-value intervals, if the output values of a function change at a constant rate, then the function is linear; if the output values change proportionally, then the function is exponential. Also of note is that arithmetic sequences, linear functions, geometric sequences, and exponential functions all have the same property that they can be determined by two distinct sequence or function values.

Example 1

If the 6th term of an arithmetic sequence is 28 and the 15th term is 73, find the 30th term of the sequence.

Solution: Substituting $n = 6$ and $a_6 = 28$ into the formula $a_n = a_0 + dn$, we get the equation $28 = a_0 + 6d$ and substituting $n = 15$ and $a_{15} = 73$ into the same formula, we get the equation $73 = a_0 + 15d$. We can subtract the two equations to eliminate a_0.

$$\begin{array}{r} 73 = a_0 + 15d \\ -\ 28 = -(a_0 + 6d) \\ \hline 45 = 9d \end{array} \qquad \text{so, } d = 5$$

We can take the first (or second) equation and substitute $d = 5$ to find a_0:
$28 = a_0 + 6(5)$, so $a_0 = -2$. The formula for the arithmetic sequence is $a_n = -2 + 5n$.
Therefore, the 30th term is $a_{30} = -2 + 5(30) = 148$.

Example 2
If the 5th term of a geometric sequence is 4 and the 10th term is $\frac{1}{8}$, find the 20th term of the sequence.

Solution: Substituting $n = 5$ and $g_5 = 4$ into the formula $g_n = g_0 r^n$ we get the equation $4 = g_0 r^5$, and substituting $n = 10$ and $g_{10} = \frac{1}{8}$ into the formula we get the equation $\frac{1}{8} = g_0 r^{10}$.

We can divide the two equations to eliminate g_0. $\frac{\frac{1}{8}}{4} = \frac{g_0 r^{10}}{g_0 r^5} \rightarrow \frac{1}{32} = r^5 \rightarrow r = \sqrt[5]{\frac{1}{32}} \rightarrow r = \frac{1}{2}$.

We can take the first (or second) equation and substitute $r = \frac{1}{2}$ to find g_0.

$4 = g_0\left(\frac{1}{2}\right)^5 \rightarrow 4 = g_0\left(\frac{1}{32}\right) \rightarrow g_0 = 128$.

The formula for the geometric sequence is $g_n = 128\left(\frac{1}{2}\right)^n$.

Therefore, the 20th term is $g_{20} = 128\left(\frac{1}{2}\right)^{20}$.

Rewritten as powers of 2 results in $g_{20} = (2)^7\ (2^{-1})^{20} = (2)^7\ (2^{-20}) = 2^{-13} = \frac{1}{8{,}192}$.

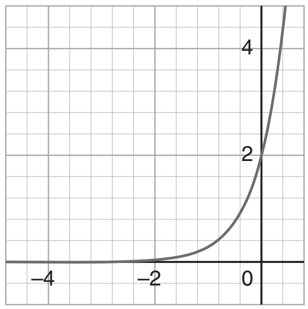
Exponential Functions

Exponential functions were introduced along with geometric sequences in the previous section. Let's take a closer look at them now.

> The general form of an *exponential function* is $f(x) = ab^x$, with initial value a, where $a \neq 0$, and base b, where $b > 0$ and $b \neq 1$. When $a > 0$ and $b > 1$, the exponential function is known as *exponential growth*. When $a > 0$ and $0 < b < 1$, the exponential function is known as *exponential decay*.

Graphs of Exponential Functions

Let's look at a graph of the exponential function $f(x) = 2(7)^x$ and identify some of the key characteristics.



CHARACTERISTIC	VALUE
Domain	All reals
Range	Positive reals
Intercept(s)	(0, 2)
Increasing/Decreasing	Always increasing
Concavity	Always concave up
Extrema	None
Point of Inflection	None
Asymptote	Horizontal at $y = 0$
End Behavior	$\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$

For an exponential function in general form, as the input values increase or decrease without bound, the output values will increase or decrease without bound or will get arbitrarily close to zero. That is, for an exponential function in general form, $\lim_{x\to\pm\infty} ab^x = \infty$, $\lim_{x\to\pm\infty} ab^x = -\infty$, or $\lim_{x\to\pm\infty} ab^x = 0$.

Example

When Brody entered kindergarten, his grandparents gave him a certificate of deposit (CD) for \$5,000 to help him pay for college. If the bank pays an annual rate of 3.5% compounded yearly, how much will Brody have when he starts college 13 years later?

Solution: Substituting the value 5,000 for a, 1.035 for b (3.5% must be converted to a decimal, which is 0.035. Then 1 must be added because each period of time we have an increase of the initial amount), and 13 for x results in the equation $y = 5{,}000(1.035)^{13}$.

Therefore, Brody will have \$7,819.78 when he starts college.

Properties of Exponential Functions

Exponential expressions can be rewritten using the properties of exponents.

PROPERTY	DEFINITION	EXAMPLE
Product Property	$b^m \cdot b^n = b^{m+n}$	$7^4 \cdot 7^5 = 7^{4+5} = 7^9$
Quotient Property	$\frac{b^m}{b^n} = b^{m-n}$	$\frac{4^8}{4^3} = 4^{8-3} = 4^5$
Power Property	$(b^m)^n = b^{mn}$	$(5^2)^3 = 5^{2\cdot 3} = 5^6$
Negative Exponent Property	$b^{-n} = \frac{1}{b^n}$	$8^{-3} = \frac{1}{8^3}$
Root Property	$b^{1/k} = \sqrt[k]{b}$, k is a natural number	$3^{1/4} = \sqrt[4]{3}$
Zero Power Property	$b^0 = 1,\ b \neq 0$	$2^0 = 1$

Example

Sketch the graphs of $f(x) = 10^x$ and $g(x) = \frac{1}{10^x}$ on the same axis. Analyze the graphs.

Solution: The graph of $g(x)$ is a reflection image of the graph of $f(x)$ over the y-axis because $g(x) = \frac{1}{10^x}$ can be rewritten as $g(x) = 10^{-x}$.

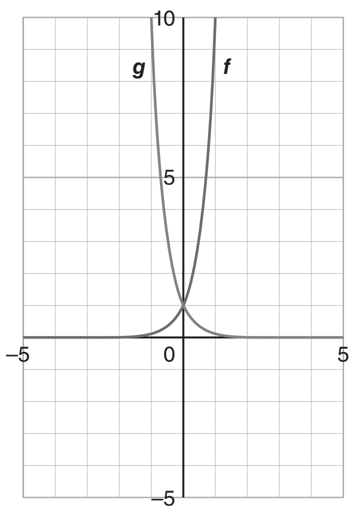
Domain: All real. Range: All positive real numbers.

Increasing/Decreasing: f is increasing over its entire domain, whereas g is decreasing over its entire domain.

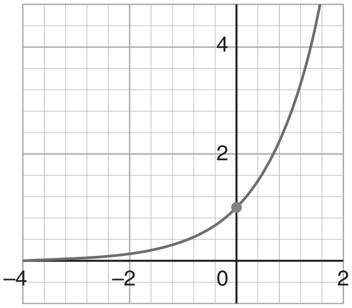
Maxima/minima: None.

End behavior: $\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} g(x) = \infty$, $\lim_{x\to\infty} g(x) = 0$.

Model: $f(x)$ is exponential growth, whereas $g(x)$ is exponential decay.

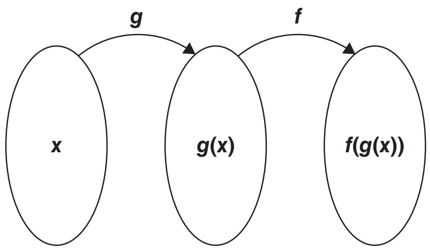


Like π for circles, an important exponential base that occurs naturally in higher order problems is base e, known as the natural number. The value of e is approximately 2.71828. The following is a graph of e^x.



Composition of Functions

The *composition of functions* $f(x)$ and $g(x)$ is the process of combining the two functions into a single function. If $g(x)$ is the first function and $f(x)$ is the second function, then it is represented as $f(g(x))$ or $(f \circ g)(x)$. The output values of g are used as input values of f. For this reason, the domain of the composite function is restricted to those input values of g for which the corresponding output value is in the domain of f.



Example 1
If $f(x) = \sqrt{x}$ and $g(x) = 2x - 1$, find $(f \circ g)(x)$ and $(g \circ f)(x)$. State the domain of each composition.

Solution: $f(g(x)) = f(2x-1) = \sqrt{2x-1}$. We need to restrict the domain to only nonnegative numbers under the radical, or $2x - 1 \geq 0$. This is equivalent to $\{x : x \geq ½\}$.
$g(f(x)) = g\left(\sqrt{x}\right) = 2\sqrt{x} - 1$. The domain is $\{x : x \geq 0\}$.
Notice that the composition of functions is generally not commutative, meaning $(f \circ g)(x) \neq (g \circ f)(x)$.

Example 2
Find $g(f(-2))$ using the following tables.

x	$f(x)$
–1	–2
–2	0
–3	2
–4	4

x	$g(x)$
0	5
1	8
2	11
3	14

Solution: From the table of $f(x)$, $f(-2) = 0$. So $g(f(-2)) = g(0)$. From the table of $g(x)$, $g(0) = 5$.
Therefore, $g(f(-2)) = 5$.

Example 3
For the given two functions $f(x) = kx - 4$ and $g(x) = kx + 6$, if the two composite functions $f(g(x))$ and $g(f(x))$ are equal, find k.

Solution: First let's find $f(g(x))$ and $g(f(x))$.
$f(g(x)) = f(kx + 6) = k(kx + 6) - 4 = k^2x + 6k - 4$
$g(f(x)) = g(kx - 4) = k(kx - 4) + 6 = k^2x - 4k + 6$
Because $f(g(x)) = g(f(x))$, then $k^2x + 6k - 4 = k^2x - 4k + 6$,
$6k - 4 = -4k + 6$,
$10k = 10$, which makes $k = 1$.

Inverse Functions

An *inverse function* can be thought of as a reverse mapping of the function. An inverse function, f^{-1}, maps the output values of a function, f, on its invertible domain to their corresponding input values; that is, if $f(a) = b$, then $f^{-1}(b) = a$. Alternately, on its invertible domain, if a function consists of input-output pairs (a, b), then the inverse function consists of input-output pairs (b, a). The domain may need to be restricted in order to make the function invertible.

The composition of a function f, and its inverse function, f^{-1}, is the identity function; that is, $f(f^{-1}(x)) = f^{-1}(f(x)) = x$.

On a function's invertible domain, the function's domain and range are the inverse function's range and domain. The inverse of the table of values of $y = f(x)$ can be found by reversing the input-output pairs; that is, (a, b) corresponds to (b, a).

The inverse of the graph of the function $y = f(x)$ can be found by reversing the roles of the x- and y-axes; that is, by reflecting the graph of the function over the graph of the identity function $h(x) = x$.

The inverse of the function can be found by determining the inverse operations to reverse the mapping. One method for finding the inverse of the function f is reversing the roles of x and y in the equation $y = f(x)$, then solving for $y = f^{-1}(x)$.

Example 1
If $f(x) = 7x + 1$, find $f^{-1}(x)$.

Solution:

Step 1: Rewrite the equation as $y = 7x + 1$.

Step 2: Reverse the roles of x and y. This results in $x = 7y + 1$.

Step 3: Solve for y. This results in $x - 1 = 7y$, so $y = \dfrac{x-1}{7}$.

Therefore, $f^{-1}(x) = \dfrac{x-1}{7}$.

Example 2

Show that $f(x) = 2x^2 + 1$ and $g(x) = \sqrt{\dfrac{x-1}{2}}$ are inverses.

Solution:

Step 1: Find $f(g(x)) = f\left(\sqrt{\dfrac{x-1}{2}}\right) = 2\left(\sqrt{\dfrac{x-1}{2}}\right)^2 + 1 = 2\left(\dfrac{x-1}{2}\right) + 1 = x - 1 + 1 = x,$

for $x \geq 1$ because of the domain of $g(x)$.

Step 2: Because $f(x)$ is not one-to-one, that is, $f(x)$ has multiple values of y from different values of x, the domain must be restricted in order to find the inverse. In this case, restricting the domain of $f(x)$ to $x \geq 0$ means we can find the inverse.

Find $g(f(x)) = g(2x^2 + 1) = \sqrt{\dfrac{(2x^2+1)-1}{2}} = \sqrt{\dfrac{2x^2}{2}} = \sqrt{x^2} = x.$

Because $f(g(x)) = g(f(x)) = x$, f and g are inverses on the appropriately restricted domains.

Logarithmic Expressions

Addition and subtraction are inverse operations. Multiplication and division are inverse operations. Exponentials and logarithms are inverse operations. The *logarithmic expression* $\log_b c$ is equal to the value that the base b must be exponentially raised to in order to obtain the value c.

$\log_b c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$.

When the base of a logarithmic expression is not specified, it is understood to be the common logarithm with base 10 and written $\log x$. When the base of a logarithm expression is e, it is referred to as natural logarithm and written $\ln x$.

On a logarithmic scale, each unit represents a multiplicative change of the base of the logarithm. For example, on a standard scale, the units might be 0, 1, 2, . . . , while on a logarithmic scale using logarithm base 10, the units might be 1, 10, 100, 1000, . . ., corresponding to 10^0, 10^1, 10^2, ...

Example 1

Find the value of $\log_2 32$.

Solution: Rewrite the logarithm as an exponential expression. $\log_2 32 = x$ can be written as $2^x = 32$. Because $2^5 = 32$, this means $x = 5$. Therefore, $\log_2 32 = 5$.

Example 2

Find the value of $\log_5 256$.

Solution: Rewrite the logarithm as an exponential expression $\log_5 256 = x$ can be written as $5^x = 256$. Because $5^3 = 125$ and $5^4 = 625$, the value for x must be between 3 and 4. Using a calculator, it can be approximated that $5^{3.445} \approx 256$. Therefore, $\log_5 256 \approx 3.445$.

Properties of Logarithms

The following table has several properties of logarithms that can be applied to solve logarithmic problems.

PROPERTY NAME	PROPERTY	GRAPHIC PROPERTY
Product Property	$\log_b(x \cdot y) = \log_b x + \log_b y$	Every horizontal dilation of a logarithmic function $f(x) = \log_b(kx)$, is equivalent to a vertical translation, $f(x) = \log_b(kx) = \log_b k + \log_b x = a + \log_b x$, where $a = \log_b k$.
Quotient Property	$\log_b \frac{x}{y} = \log_b x - \log_b y$	
Power Property	$\log_b x^n = n\log_b x$	Raising the input of a logarithmic function to a power, $f(x) = \log_b x^k$ results in a vertical dilation, $f(x) = \log_b x^k = k\log_b x$.
Change of Base Property	$\log_b x = \frac{\log_a x}{\log_a b}$, where $a > 0$ and $a \neq 1$	All logarithmic functions are vertical dilations of each other.

Example 1
Solve $15^x = 30$.

Solution: Let's rewrite the exponential equation as a logarithmic equation. $15^x = 30 \rightarrow \log_{15} 30 = x$. We can use the change of base property to rewrite $\log_{15} 30 = \frac{\log 30}{\log 15}$. Using a calculator $\frac{\log 30}{\log 15} = 1.256$. Therefore, $x = 1.256$. You can use a calculator to confirm $15^{1.256} \approx 30$.

Example 2
Use the properties to expand the expression $\log_b \frac{5x}{y^2}$.

Solution:

Step 1: Use the quotient property to rewrite: $\log_b \frac{5x}{y^2} = \log_b(5x) - \log_b y^2$

Step 2: Use the product and power properties: $= (\log_b 5 + \log_b x) - 2\log_b y$

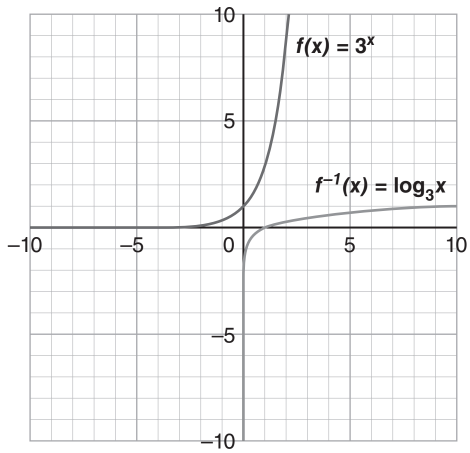
$$\log_b \frac{5x}{y^2} = \log_b 5 + \log_b x - 2\log_b y$$

Inverses of Exponential Functions

Because $f(x) = \log_b x$ and $g(x) = b^x$ are inverse functions, this means that when the functions are composed with one another the result is x. Meaning $f(g(x)) = g(f(x)) = x$. If (s, t) is an ordered pair of the exponential function, then (t, s) is an ordered pair of the logarithmic function. Graphically because the functions are inverses, the graph of the logarithmic function is a reflection of the graph of the exponential function over the graph of the identity function $h(x) = x$.

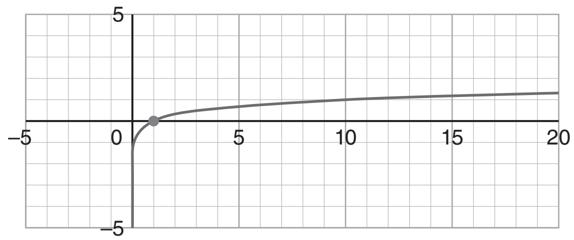
Example
Graph $f(x) = 3^x$ and its inverse.

Solution: The inverse, $f^{-1}(x) = \log_3 x$. Notice that the graphs are reflections across the line $y = x$. The y-intercept of $f(x) = 3^x$ is $(0, 1)$. The x-intercept of $f^{-1}(x) = \log_3 x$ is $(1, 0)$.



Graphs of Logarithmic Functions

Let's look at a graph of the logarithmic function $f(x) = \log x$ and identify some of the key characteristics.



CHARACTERISTIC	VALUE
Domain	All reals greater than 0
Range	All reals
Intercept(s)	(1, 0)
Increasing/Decreasing	Always increasing
Concavity	Always concave down
Extrema	None
Point of Inflection	None
Asymptote	Vertical at $x = 0$
End Behavior	$\lim_{x\to 0^+} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$

Exponential and Logarithmic Inequalities

Properties of exponents, properties of logarithms, and the inverse relationship between exponential and logarithmic functions can be used to solve equations and inequalities involving exponents and logarithms. When solving exponential and logarithmic equations found through analytical or graphical methods, the results should be examined for extraneous solutions precluded by the mathematical or contextual limitations.

Example 1
What are all values of x for which $2\log(x+1) = \log(x+13)$?

Solution:

Step 1: Set the equation equal to 0. $2\log(x+1) - \log(x+13) = 0$

Step 2: Rewrite using the power and quotient properties. $\log\frac{(x+1)^2}{(x+13)} = 0$

Step 3: Rewrite as an exponential equation. $10^0 = \frac{(x+1)^2}{(x+13)}$

Step 4: Rewrite 10^0 as 1. $1 = \frac{(x+1)^2}{(x+13)}$

Step 5: Multiply both sides by $(x + 13)$. $x+13 = (x+1)^2$

Step 6: Replace $(x + 1)^2$ with $x^2 + 2x + 1$ and set equal to 0. $x^2 + x - 12 = 0$

Step 7: Factor and set each factor equal to 0 and solve. $(x+4)(x-3) = 0$

$x = -4$ or $x = 3$

Step 8: Because the domain of a logarithmic function is all reals greater than 0, substituting $x = -4$ into $2\log(x+1)$ yields $2\log(-3)$, which is not defined; therefore $x = -4$ is an extraneous solution.

The value that solves the equation is $x = 3$.

Example 2
What are all values of x for which $2^x \geq 100$?

Solution:

Step 1: Rewrite the inequality as a logarithmic inequality. $\log_2 2^x \geq \log_2 100$

Step 2: Use the power property and change of base property to rewrite. $x \log_2 2 \geq \frac{\log 100}{\log 2}$

Step 3: Use a calculator to evaluate the logarithms. $x \geq 6.644$

The solution is $x \geq 6.644$.

Modeling

Two variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, quadratic, and exponential function models. Models can be compared based on contextual clues and applicability to determine which model is most appropriate. A model is justified as *appropriate* for a data set if the graph of the residuals of a regression, the residual plot, appear without pattern. The *error* in the model is the difference between the predicted and actual values. Depending on the data set and context, it may be more appropriate to have an underestimate or overestimate for any given interval.

Exponential Function Context and Data Modeling

For an exponential model in general form $f(x) = ab^x$, the base of the exponent, b, can be understood as a growth factor in successive unit changes in the input values and is related to a percent change in context. An exponential function model can be constructed from an appropriate ratio and initial value or from two input-output pairs. The initial value and the base can be found by solving a system of equations resulting from the two input-output pairs.

Exponential function models can be constructed by applying transformations to $f(x) = ab^x$ based on characteristics of a contextual scenario or data set. They can be used to predict values for the dependent variable, depending on the contextual constraints on the domain. A constant may need to be added to the dependent variable values of a data set to reveal a proportional growth pattern.

Example 1

A new car sells for \$38,500. The value of the car decreases by 17% annually. What will be the value of the car in 5 years?

Solution: The growth factor is 1 – 0.17, or 0.83. Using the general form of the exponential model yields, $f(x) = 38{,}500(0.83)^5$. Therefore, the car will be worth \$15,165.31 after 5 years.

Example 2

The number of bacteria on the fourth day of an experiment was 58. On the tenth day, the number increased to 368. Write an exponential model to represent the number of bacteria present d days after the experiment began.

Solution:

Step 1: Represent the model $f(d) = ab^d$ with the points (4, 58) and (10, 368). This results in $58 = ab^4$ and $368 = ab^{10}$.

Step 2: Divide the second equation by the first equation to eliminate a.
$$\frac{368}{58} = \frac{ab^{10}}{ab^4}$$
$$6.345 = b^6$$

Step 3: Solve for b.
$$b = \sqrt[6]{6.345} \approx 1.361$$

Step 4: Substitute b into one of the equations to solve for a.
$$58 = a(1.361)^4$$
$$\frac{58}{1.361^4} = a$$
$$16.923 \approx a$$

Step 5: Because a represents the starting amount of bacteria, it should be represented by the whole number 17.

Therefore, the exponential model is $f(d) = 17(1.361)^d$.

> Values that are used when solving an equation should be stored in a graphing calculator so as not to have a round-off error.

Example 3

The half-life of carbon-14 is known to be 5,720 years. If 400 grams of carbon-14 are stored for 1,000 years, how many grams will remain?

Solution: Half-life is the time required for a quantity to reduce to half of its value. When solving half-life formulas, we use the formula $A = A_0\left(\frac{1}{2}\right)^{t/H}$, where A = the amount remaining, A_0 = initial amount, t = time, H = half-life. Substituting the known information into the formula results in

$A = 400\left(\frac{1}{2}\right)^{1{,}000/5{,}720}$. After 1,000 years 354.350 grams will remain.

Logarithmic Function Context and Data Modeling

Logarithmic functions are inverses of exponential functions and can be used to model situations involving proportional growth, or repeated multiplication, where the input values change proportionally over equal-length output-value intervals. Alternately, if the output value is a whole number, it indicates how many times the initial value has been multiplied by the proportion.

Example

A concert starts at 7:00 p.m. and the doors to the concert venue open at 5:00 p.m. The number of patrons in the concert venue t minutes after 5:00 p.m. is listed in the following table. Using technology, estimate the logarithmic function $N(t) = a + b\ln t$ that models the data.

Minutes since 5:00 p.m.	15	30	45	60	75	90	105	120
Number of patrons in the venue	270	340	380	410	430	450	465	480

Solution:

Step 1: Enter the value from the table into a graphing calculator.

Step 2: Use the calculator's calculate function to determine the logarithmic function.

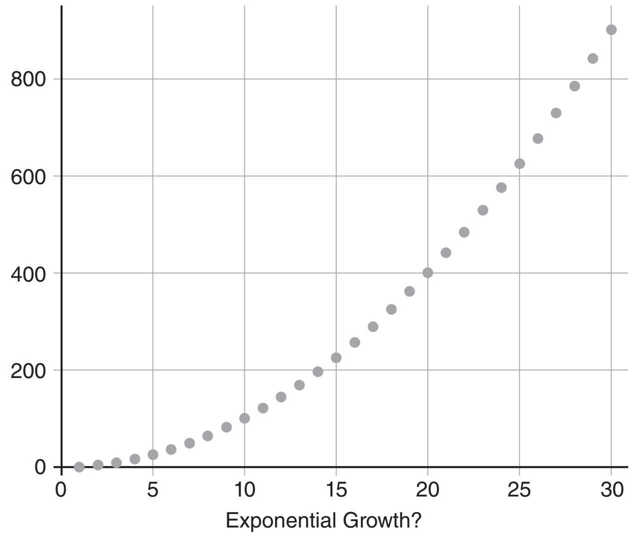
The function that models how many patrons are in the venue at time t minutes after 5:00 p.m. is $N(t) = -2.029 + 100.444(\ln t)$.

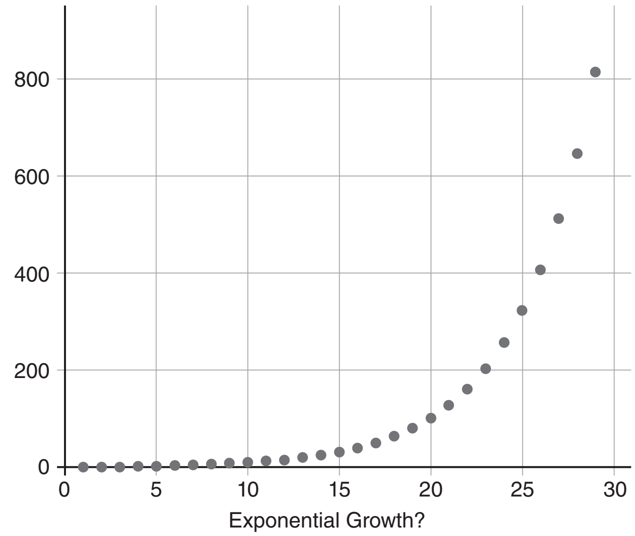
Semi-Log Plots

In a *semi-log plot*, one of the axes is logarithmically scaled. When the y-axis of a semi-log plot is logarithmically scaled, data or functions that demonstrate exponential characteristics will appear linear. An advantage of semi-log plots is that a constant never needs to be added to the dependent variable values to reveal that an exponential model is appropriate. Techniques used to model linear functions can be applied to a semi-log graph. For an exponential model of the form $y = ab^x$, the corresponding linear model for the semi-log plot is $y = (\log_n b)x + \log_n a$, where $n > 0$ and $n \neq 1$. Specifically, the linear rate of change is $\log_n b$, and the initial value is $\log_n a$.

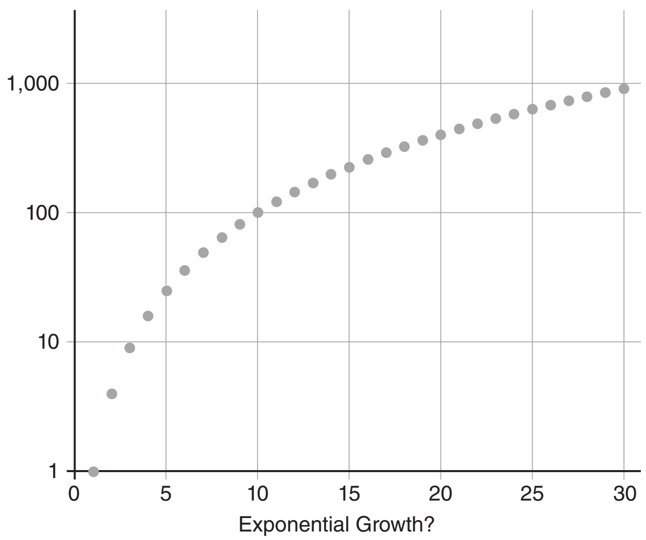
Example

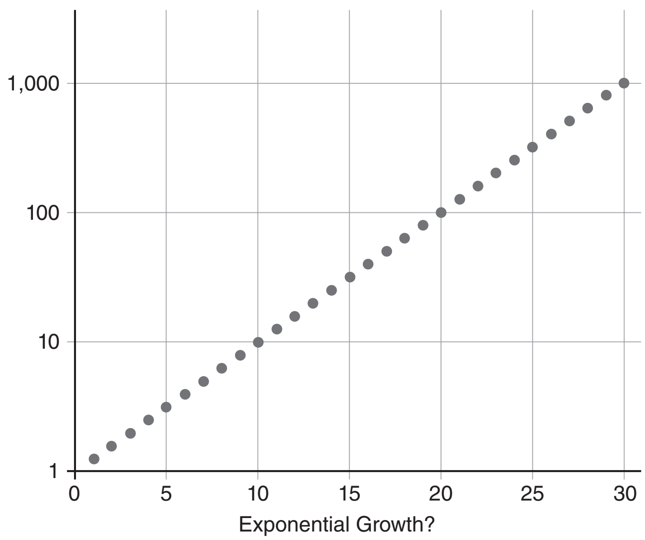
Consider the following graphs of two data sets. Both appear to have models that would be increasing and concave up. Which data set, if any, can be modeled by an exponential function? A quick check to determine if the model is exponential is to change the y-axis to a logarithmic scale.





The same data graphed with a logarithmic scale for the y-axis are shown in the following graphs. As you can see, the y-axis now is scaled 1, 10, 100, 1,000. This is equivalent to 10^0, 10^1, 10^2, and 10^3, which is a proportional scale.





Recall, exponential functions model growth patterns where successive output values over equal-length input-value intervals are proportional. The graph on the right appears to be linear, while the graph on the left does not appear to be linear. Because the graph on the right is linear on the new proportional scale, the original data for the right graph is exponential; the graph on the left does not appear linear, so the original data for the left graph is not exponential. In fact, the left graph can be modeled by $f(x) = x^2$ while the graph on the right can be modeled by $g(x) = (1.26)^x$.

› Rapid Review

Sequence

- A sequence is an ordered list of numbers that follow a specific pattern.
- An arithmetic sequence is a sequence that has successive terms that have a constant rate of change or common difference.
- The general term of an arithmetic sequence is $a_n = a_0 + dn$, where a_0 is the initial value and d is the common difference.
- A geometric sequence is a sequence that has successive terms that have a constant proportional change or a common ratio.
- The general term of a geometric sequence is $g_n = g_0 r^n$, where g_0 is the initial value and r is the common ratio.
- Linear functions are similar to arithmetic sequences because both have an initial value and a repeated addition of a constant.
- Exponential functions are similar to geometric sequences because both have an initial value and a repeated constant proportion.

Exponential Function

- Exponential functions arise from situations of constant growth.
- The general form of an *exponential function* is $f(x) = ab^x$, with initial value a, where $a \neq 0$, and base b, where $b > 0$ and $b \neq 1$. When $a > 0$ and $b > 1$, the exponential function is known as *exponential growth*. When $a > 0$ and $0 < b < 1$, the exponential function is known as *exponential decay*.
- When the base of an exponential function f is greater than 1, f is increasing, $\lim_{x \to -\infty} f(x) = 0$, $\lim_{x \to \infty} f(x) = \infty$.
- When the base of an exponential function f is less than 1, f is decreasing, $\lim_{x \to -\infty} f(x) = \infty$, and $\lim_{x \to \infty} f(x) = 0$.
- When $a > 0$, the graph is concave up, and when $a < 0$, the graph is concave down.
- Exponential expressions can be rewritten using the properties of exponents.

Logarithmic Function

- Logarithmic functions are inverses of exponential functions.
- $\log_b c = a$ if and only if $b^a = c$, where a and c are constants, $b > 0$, and $b \neq 1$.
- When the base of a logarithmic function g is greater than 1, g is increasing, $\lim_{x \to \infty} g(x) = \infty$. The graph is concave down.
- When the base of a logarithmic expression is not specified, it is understood to be the common logarithm with base 10 and written $\log x$.
- When the base of a logarithm expression is e, it is referred to as natural logarithm and written $\ln x$.
- Logarithm expressions can be rewritten using the properties of logarithms.

Modeling

- Two variables in a data set that demonstrate a slightly changing rate of change can be modeled by linear, quadratic, and exponential function models.

- Logarithmic functions are inverses of exponential functions and can be used to model situations involving proportional growth, or repeated multiplication, where the input values change proportionally over equal-length output-value intervals.
- Models can be used to predict values for the dependent variable, depending on the contextual constraints on the domain.

Miscellaneous

- Two functions f and g can be combined using the composition $(f \circ g)(x)$.
- An inverse function is a reverse mapping of a function, and is written f^{-1}
- If $(f \circ g)(x) = (g \circ f)(x) = x$, then the functions f and g are inverses.
- When solving exponential or logarithmic equations or inequalities, extraneous solutions might exist and should be excluded.
- For the AP exam, logarithmic scaling will only be applied to the y-axis to linearize exponential functions.

❯ Review Questions

Basic Level

1. Describe the graph of $y = 3\left(\frac{1}{5}\right)^x$ using limit notation.

2. Let $f(x) = 5^x$. If $g(x)$ is a transformation of $f(x)$ with a horizontal shift of 3 units right and a vertical shift of 4 units up, what is the equation of $g(x)$?

3. Using the following tables, what is the value of $f(g(2))$? What is the value of $g(f(-2))$?

x	−3	−2	−1	0	1	2
$f(x)$	6	0	3	5	1	−2

x	−4	−2	0	2	4	6
$g(x)$	3	1	−1	−3	−5	−7

4. Simplify completely using only positive exponents: $(2x^3y^2)^3(3x^7y^{-3})^2$.

5. If $f(x) = \log_3(x-1)$, find $f^{-1}(x)$.

6. If $f(x) = \log_4 x$, find $f^{-1}(2)$.

7. Solve for x: $\log_x 300 = 2$.

Advanced Level

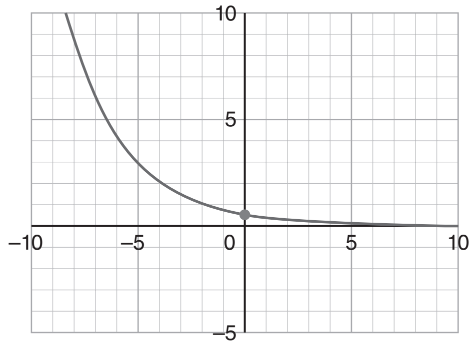
8. If the 2nd term of a geometric sequence is 36 and the 6th term is 2916, what is the 10th term?

9. An arithmetic sequence has the 7th term 41 and the 18th term 74. Find the formula for the nth term.

10. Simplify without using a calculator: $\log_4 \frac{1}{2}$.

11. Let $f(x) = 5x - 7$ and $g(x) = \frac{2x+1}{x-8}$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$.

12. $f(x)$ is graphed as follows. If $g(x) = 4f(x) - 10$, what is the value of $g(0)$?



13. Rewrite $2\log x - \log(4y) + \log(3x^4)$ as a single logarithm.

14. If $f^{-1}(x) = \dfrac{2x-1}{x+5}$, find $f(x)$.

15. Simplify completely using all positive exponents: $\dfrac{(5x^2y^4z^{-2})^3}{(2xy^3z^2)^{-2}}$

16. Solve for x: $\log(2x) + \log x^3 < 2$.

17. If Sam puts $10,000 into a retirement account at age 30 earning 2.6% compounded annually, how much will it grow to by retirement age 67 if no additional deposits or withdrawals are made?

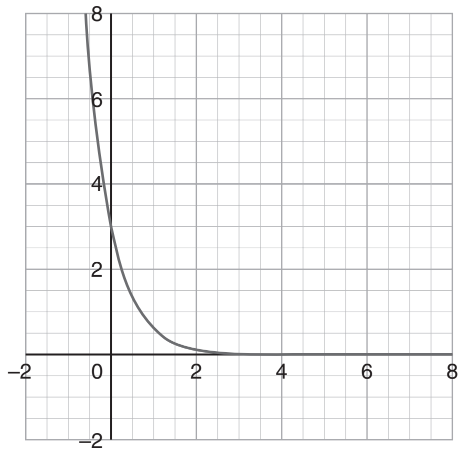
18. Express $\log_b \sqrt[4]{nw^3}$ in terms of $\log_b n$ and $\log_b w$.

19. A wise old ruler wanted to reward his friend for an act of extraordinary bravery. The friend said, "I ask you for just one thing. Take the chessboard and place on the first square one grain of rice. On the first day I will take this grain home to feed my family. On the second day, place on the second square two grains for me to take home. On the third day cover the third square with four grains for me to take. Each day double the number of grains you give me until you have placed rice on every square of the chessboard, then my reward will be complete." The wise old ruler replied, "This sounds like a small price to pay for your act of incredible bravery, I will do as you ask immediately." If the friend's request was granted, how much rice would be given to them on day 64, the final square of the chessboard?

20. Using the information from question 19, what is the rate of change between the sixth and seventh days? What is the rate of change between the seventh and eighth days? What does this illustrate about an exponential curve ab^x with $a > 0$?

❯ Answers and Explanations

1. • The following graph of $y=3\left(\frac{1}{5}\right)^x$ is an exponential decay function (the base b = 1/5, is between 0 and 1 or 0 < 1/5 < 1).
• When $a > 0$, exponential decay functions are always decreasing, always concave up, and have a horizontal asymptote at $y = 0$.
• $\lim_{x\to-\infty} f(x)=\infty$ and $\lim_{x\to\infty} f(x)=0$



2. • A horizontal shift of 3 replaces x with $x - 3$.
• A vertical shift of 4 adds 4 to the equation.
• $g(x) = 5^{x-3} + 4$

3. • To find $f(g(2))$, we first need to find $g(2)$. Using the bottom table, $g(2) = -3$.
• So $f(g(2)) = f(-3) = 6$.
• To find $g(f(-2))$, we first need to find $f(-2)$. Using the top table, $f(-2) = 0$.
• So $g(f(-2)) = g(0) = -1$.

4. • Let's simplify the first expression by using the power property: $(2x^3y^2)^3 = (2)^3(x^3)^3(y^2)^3 = 8x^9y^6$.
• Now the second: $(3x^7y^{-3})^2 = (3)^2(x^7)^2(y^{-3})^2 = 9x^{14}y^{-6}$.
• Now, apply the product property. $(8x^9y^6)(9x^{14}y^{-6}) = 72x^{23}y^0 = 72x^{23}$.

5. • Rewrite $f(x)=\log_3(x-1)$ as $y=\log_3(x-1)$.
• Interchange x and y: $x=\log_3(y-1)$.
• Rewrite as an exponential equation: $3^x = y-1$.
• Solve for y: $y=3^x+1$.
• So $f^{-1}(x)=3^x+1$.

6. • The inverse can be found by rewriting the logarithmic equation as the corresponding exponential equation. $f^{-1}(x)= 4^x$.
• $f^{-1}(2)= 4^2=16$.

7. • Solving the equation $\log_x 300=2$ involves rewriting as the equivalent exponential equation.
• $\log_x 300=2$ can be rewritten as $x^2 = 300 \to x=\pm\sqrt{300}=\pm\sqrt{100}\sqrt{3}=\pm10\sqrt{3}$.
• Because the base of a logarithm must be positive, there is only one solution, $x=10\sqrt{3}$.

8. • Substituting (2, 36) into the equation $g_n = g_0 r^n$ results in $36 = g_0 r^2$.
• Substituting (6, 2916) into the equation results in $2916 = g_0 r^6$.
• Divide the two equations to eliminate g_0. $\frac{2916}{36} = \frac{g_0 r^6}{g_0 r^2} \rightarrow 81 = r^4 \rightarrow r = \pm\sqrt[4]{81} \rightarrow r = \pm 3$.
• Using the first equation and substituting the known values results in $36 = g_0(3)^2 \rightarrow 36 = 9g_0 \rightarrow g_0 = 4$. Note: Substituting $r = -3$ into the equation yields the same result.
• $g_{10} = 4(3)^{10} = 236{,}196$.

9. • Substituting (7, 41) into the equation $a_n = a_0 + dn$ results in $41 = a_0 + 7d$.
• Substituting (18, 74) into the equation results in $74 = a_0 + 18d$.
• Subtract the two equations to eliminate a_0:
$$\begin{aligned} 74 &= a_0 + 18d \\ -\,41 &= -(a_0 + 7d) \\ \hline 33 &= 11d \\ d &= 3. \end{aligned}$$
• Using the first equation and substituting the known values results in $41 = a_0 + 7(3) \rightarrow a_0 = 20$.
• The arithmetic formula is $a_n = 20 + 3n$.

10. • Rewriting $\log_4 \frac{1}{2} = x$ as an exponential equation results in $4^x = \frac{1}{2}$.
• Rewriting both sides of the equation as a power of 2 results in $(2^2)^x = 2^{-1}$.
• Because the bases are equal, that means the exponents must be equal.
• $2x = -1$, so $\log_4 \frac{1}{2} = -\frac{1}{2}$.

11. • $(f \circ g)(x) = f(g(x)) = f\left(\frac{2x+1}{x-8}\right) = 5\left(\frac{2x+1}{x-8}\right) - 7 = \frac{10x+5}{x-8} - \frac{7(x-8)}{x-8} = \frac{10x+5-7x+56}{x-8} = \frac{3x+61}{x-8}$
• $(g \circ f)(x) = g(f(x)) = g(5x-7) = \frac{2(5x-7)+1}{(5x-7)-8} = \frac{10x-14+1}{5x-15} = \frac{10x-13}{5x-15}$

12. • From the graph it can been seen that $f(0) = \frac{1}{2}$.
• Because $g(x) = 4f(x)$, the point (0, ½) moves to 4[(0, ½)] or (0, 2).
• Applying the vertical shift −10, the point (0, 2) moves to (0, −8).
• Therefore, $g(0) = -8$.

13. • Applying the power property, $2\log x = \log x^2$.
• Applying the quotient and product properties gives $\log \frac{x^2(3x^4)}{4y} = \log \frac{3x^6}{4y}$.

14. • Finding the function given its inverse is the same procedure as finding the inverse given a function.
• Rewrite the equation replacing x with y and y with x. This results in $x = \frac{2y-1}{y+5}$.
• Solve for y. First cross-multiply: $x(y + 5) = 2y - 1$.
• Expand and isolate y: $xy + 5x = 2y - 1 \rightarrow xy - 2y = -5x - 1 \rightarrow y(x - 2) = -5x - 1$.
• $y = \frac{-5x-1}{x-2}$, so the function is $f(x) = \frac{-5x-1}{x-2}$.

15. • First, apply the power property (raise each term to the power) to the numerator and denominator: $\frac{(5x^2y^4z^{-2})^3}{(2xy^3z^2)^{-2}} = \frac{5^3(x^2)^3(y^4)^3(z^{-2})^3}{2^{-2}x^{-2}(y^3)^{-2}(z^2)^{-2}}$.

• Next, lets apply the power property (multiply the exponents) again to each individual term: $\frac{5^3(x^2)^3(y^4)^3(z^{-2})^3}{2^{-2}x^{-2}(y^3)^{-2}(z^2)^{-2}} = \frac{125x^{2\cdot 3}y^{4\cdot 3}z^{-2\cdot 3}}{2^{-2}x^{-2}y^{3\cdot -2}z^{2\cdot -2}}$.

• Next, apply the quotient property (subtract exponents): $\frac{125x^6y^{12}z^{-6}}{2^{-2}x^{-2}y^{-6}z^{-4}} = \frac{125x^{6-(-2)}y^{12-(-6)}z^{-6-(-4)}}{2^{-2}}$.

• Simplify: $\frac{125x^{6-(-2)}y^{12-(-6)}z^{-6-(-4)}}{2^{-2}} = (125)(4)x^8y^{18}z^{-2}$.

• Finally, use the negative exponent property: $\frac{500x^8y^{18}}{z^2}$.

16. • Rewrite using a single logarithm: $\log(2x) + \log x^3 = \log(2x \cdot x^3) = \log(2x^4)$.

• Rewrite the inequality as an exponential inequality: $\log(2x^4) < 2 \rightarrow 10^2 < 2x^4$.

• Solve the inequality: $100 < 2x^4 \rightarrow 50 < x^4 \rightarrow \pm\sqrt[4]{50} < x$.

• Substituting $x = -\sqrt[4]{50}$ would lead to an extraneous solution because the logarithm of a negative number does not exist.

17. • Using the compound interest formula $A = B(1 + r)^t$ where B is the initial amount, r is the interest rate, and t is years, we have $A = 10{,}000(1.026)^{37} = 25{,}849.512$.

• Sam will have \$25,849.51 after 37 years.

18. • Using the product property, $\log_b \sqrt[4]{nw^3} = \log_b n^{\frac{1}{4}} + \log_b w^{\frac{3}{4}}$.

• Using the power property, $\log_b n^{\frac{1}{4}} + \log_b w^{\frac{3}{4}} = \frac{1}{4}\log_n n + \frac{3}{4}\log_b w$.

19. • The values in the sequence are 1, 2, 4, 8, 16, ...

• Using the formula $g_n = g_1 r^{(n-1)}$, we get $g_{64} = 1(2)^{64-1}$.

• On the 64th day the friend would receive 9.223×10^{18} pieces of rice.

20. • Using the formula $g_n = g_1 r^{(n-1)}$, we get $g_6 = 1(2)^{6-1} = 32$, and $g_7 = 1(2)^{7-1} = 64$.

• The rate of change between the sixth and seventh day is $\frac{64-32}{7-6} = 32$ grains/day.

• Using the formula $g_n = g_1 r^{(n-1)}$, we get $g_8 = 1(2)^{8-1} = 128$.

• The rate of change between the seventh and eighth day is $\frac{128-64}{8-7} = 64$ grains/day.

• The graph of $f(x) = 1(2)^x$ is growing at a rate of $r = 2$.

• Because the rate of change is positive, exponential functions with $a > 0$ are concave up.