CHAPTER 9

Radicals and Rational Exponents

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OUTLINE

Math at Work:

Computer Support Specialist

As a computer support specialist, Pilar Benitez is confident in her technical and communication skills. She knows that she has to maintain a firm grasp on everything her company does and understand how it fits with changes in the larger tech industry. And, she has to be able to explain all of this to customers. "Our customers are small-business people and individuals," Pilar describes. "When they have a problem with one of our products, I'm the person who helps them resolve it."

But starting a new job at a large American software company requires that Pilar learn about how the company operates before she can do her best work. What products do they offer customers? Where can she find the technical specifications for all of the products? How are she and the rest of her team expected to share responsibilities? If she cannot answer a question, where can she find an answer? These are just some of the questions that need answers.

Pilar knows that if she can learn about her new company's policies and procedures, she will be off to a successful start. The same is true for students attending a new school; the more students know about their school's operation and culture, the more tools they have to succeed. In this chapter, we will learn about working with radicals and rational exponents and offer strategies for learning about your school.

POWER Study Strategies — Know Your School

Understanding how your college or university works and where to find resources are critical components of being successful in college. Where do you go if you have a question about financial aid? Where can you find out about clubs that are of interest to you? Where do you go if you have questions about which classes you still need to take in order to graduate? Let's use the P.O.W.E.R. framework to help you get to know your school.

P Prepare

- **I will learn about the different offices, resources, and services at my school as well as how my school works.**

O Organize

- Complete the emPOWERme survey that appears before the Chapter Summary to learn how well you know your school. Think about what you know and what you still need to learn.
- Know your school's website. Identify important places on the website such as where you can register for classes, pay your tuition online, how to make an appointment to see an adviser, and where you can find information about the tutoring center.
- Identify places you need to know at your school: important offices where you can get questions answered, services offered at your school, organizations or clubs that you can join, etc.
- Identify important deadlines you should know such as registering for classes, paying tuition, and submitting financial aid applications.
- Identify academic services that you may want to use such as the library, a tutoring center, and a testing center.
- Identify support services that may be helpful to you such as the campus health center, a veterans' support office, an office to help students with disabilities, and a child care center.
- Be sure you have your syllabus for each class with information about the class as well as the location of your instructors' offices and their office hours.

W Work

- It's time to learn about some of the offices, resources, and support services at your school and to learn about some of the procedures. Some important, common offices and resources are listed here, but different schools have different names for these services. Some apply to all students, and some may not apply to you. For the items that apply to you, fill in the blanks with the information for *your* school.

- *My school's website is* ______________________. Among other things, know how you can register for classes and pay your tuition online.
- *The office where I can go to register for classes is called the* ______________________ *office. Its location is* ______________________. Know the deadlines for registering for classes.
- When you are choosing your classes, know where they are located. If the campus is large, be sure you have enough time to get from one class to the next.
- *The office where I can go to ask questions about my tuition bill or to pay my bill in person is called the* ______________________ *office. Its location is* ______________________. Know when tuition is due.
- *The office where I can go to ask questions about financial aid, get financial aid forms, or turn in forms is called the* ______________________ *office. Its location is* ______________________. Know when to get the financial aid process started and when the forms are due.
- *The office where I can talk with someone about choosing a major or get help choosing classes is called the* ______________________ *office. Its location is* ______________________.

- *The office where I can go to talk with someone about personal problems that are interfering with my success in college is called the* ______________________ *office. Its location is* ______________________.
- Does your school have an emergency alert system? If so, know how to sign up for it. You may have options of receiving an email, text, or automated phone call.
- Most institutions have a Tutoring Center or a Math Lab that is available to students free of charge. *The place I can go to get free help with my math class is called the* ______________________. *Its location is* ______________________, *and its hours are* ______________________. What are the procedures for seeing a tutor?
- *My math instructor's office is located in* ______________________, *and his/her office hours are* ______________________.
- Most schools have a testing center where students can go if they are allowed extra time on their tests or if they need to take a make-up test. *At my school, this center is called the* ______________________. *Its hours are* ______________________. What are the procedures for taking a test in their center?
- *The office where I can learn about the resources available to me as a student with a disability is called the* ______________________ *office. Its location is* ______________________.
- Most campuses have an office that helps veterans. *On my campus, that office is called the* ______________________. *Its location is* ______________________.
- Many campuses have a child care center for students with children. *I can get information about the child care center at* ______________________.
- *The place I can go to find out about campus clubs, events, and organizations is called the* ______________________.
- Most institutions have a place where students can get help finding a job or preparing for an interview. *At my school, this office is called the* ______________________ *office. Its location is* ______________________.

E Evaluate

- Do you feel like you have a good understanding of how your school works and where to go for help? Did you learn the names and locations of the offices listed in the Work section?

R Rethink

- If you feel like you learned about the places and services that will be beneficial to you, be sure to use them when you need them! Also, guide classmates to the proper place when they need help.
- If you did not learn everything you think you need to know, ask one of your instructors or go to an information office at your school. Search the institution's website.
- Are there other services or resources that you feel like you need but did not learn about? Talk to one of your instructors and ask if there is such a place at your school.

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Chapter 9 POWER Plan

P Prepare | **O Organize**

What are your goals for Chapter 9?	How can you accomplish each goal?
1 Be prepared before and during class.	• ________ • ________ • ________ • ________
2 Understand the homework to the point where you could do it without needing any help or hints.	• ________ • ________ • ________
3 Use the P.O.W.E.R. framework to help you learn about your school.	• Complete the emPOWERme that appears before the Chapter Summary to find out how much you know about your school. • Read the Study Strategies to learn how to use the P.O.W.E.R. framework to learn about the resources at your school. • ________
4 Write your own goal. ________ ________	• ________ ________
What are your objectives for Chapter 9?	**How can you accomplish each objective?**
1 Be able to simplify square roots and higher roots. Be able to graph radical functions and determine the domain.	• Learn the different definitions for *square roots* and higher roots. • Be able to make a table of values for *radical functions* by choosing a proper domain.
2 Be able to perform operations with rational exponents.	• Learn the definition of $a^{1/n}$, $a^{m/n}$, and $a^{-m/n}$. • Review the rules of exponents found in Sections 6.1 and 6.2. • Know how to convert from radical form to exponential form.
3 Be able to simplify expressions containing square roots and higher roots including expressions with variables.	• Know the *product rules* and the *quotient rules.* • Learn how to tell when a *radical expression* is simplified. • Know how to simplify exponents with even and odd exponents.
4 Be able to add, subtract, and multiply radicals.	• Review the procedure for **Simplifying Radicals.** • Know the definition of *like radicals.* • Review the rules for multiplying polynomials so you can apply them to multiplying radicals. • Learn the procedure for **Multiplying Radicals.**
5 Be able to divide radical expressions and rationalize denominators.	• Learn the procedure for **Rationalizing Denominators,** including denominators with two terms. • Be able to find the *conjugate* of a radical expression, when appropriate. • Know how to determine when radical expressions are simplified.

6 Be able to solve radical equations, and check your solutions.	• Know the procedure for **Solving Radical Equations Containing Square Roots.** • Be able to determine when a solution is an *extraneous* solution. • Check all solutions.
7 Be able to simplify, add, subtract, multiply, and divide with complex numbers.	• Know the definition of *i*. • Know the procedure for **Adding and Subtracting Complex Numbers.** • Know the procedure for **Multiplying and Dividing Complex Numbers.**
8 Write your own goal. ____________________ ____________________	• ____________________ ____________________

W Work Read Sections 9.1 to 9.8, and complete the exercises.

E Evaluate Complete the Chapter Review and Chapter Test. How did you do?	**R Rethink** • Can you simplify, add, subtract, multiply, and divide radical expressions without looking at your notes or the book? Which do you need to practice more? • In your own words, explain the relationship between fractional exponents and radicals. • Do you understand how to solve the different types of radical equations, or do you need more practice? • Do you understand the operations with complex numbers? How are the operations similar to and different from operations with polynomials? • After doing the emPOWERme and Study Strategy, do you know more about your school than you did before? Is there anything else you would like to know?

9.1 Radical Expressions and Functions

P Prepare | **O Organize**

What are your objectives for Section 9.1?	How can you accomplish each objective?
1 Find Square Roots and Principal Square Roots	• Write the definitions of *square root, principal square root, negative square root, square root symbol* or *radical sign, radicand,* and *radical* in your notes with an example next to each. • Write the properties for **Radicands and Square Roots** in your notes, and learn them. • Complete the given examples on your own. • Complete You Trys 1 and 2.
2 Find Higher Roots	• Know the definition of an *index*. • Understand the definitions of the *nth root,* and write examples in your notes that match the definitions. • Complete the given examples on your own. • Complete You Trys 3 and 4.
3 Evaluate $\sqrt[n]{a^n}$	• Write the procedure for **Evaluating $\sqrt[n]{a^n}$** in your own words. • Complete the given example on your own. • Complete You Try 5.
4 Determine the Domains of Square Root and Cube Root Functions	• Know how to identify a *radical expression, radical function,* and a *square root function.* • Write the definition of the *domain of a square root function* and *domain of a cube root function* in your own words. • Complete the given examples on your own. • Complete You Trys 6–8.
5 Graph a Square Root Function	• Follow Example 9 to create your own procedure for **Graphing a Square Root Function.** • Complete You Try 9.
6 Graph a Cube Root Function	• Follow Example 10 to create your own procedure for **Graphing a Cube Root Function.** • Complete You Try 10.

W Work **Read the explanations, follow the examples, take notes, and complete the You Trys.**

Recall that exponential notation represents repeated multiplication. For example,

$$3^2 \text{ means } 3 \cdot 3, \text{ so } 3^2 = 9.$$
$$2^4 \text{ means } 2 \cdot 2 \cdot 2 \cdot 2, \text{ so } 2^4 = 16.$$

In this chapter we will study the opposite, or inverse, procedure, finding **roots** of numbers.

1 Find Square Roots and Principal Square Roots

EXAMPLE 1

In-Class Example 1

Find all square roots of 81.

Answer: $-9, 9$

Find all square roots of 25.

Solution

To find a *square* root of 25 ask yourself, "What number do I *square* to get 25?" Or, "What number *multiplied by itself* equals 25?" One number is 5 since $5^2 = 25$. Another number is -5 since $(-5)^2 = 25$. So, 5 and -5 are square roots of 25.

[YOU TRY 1] Find all square roots of 64.

The $\sqrt{}$ symbol represents the *positive* square root, or the **principal square root,** of a nonnegative number. For example, $\sqrt{25} = 5$.

BE CAREFUL Notice that $\sqrt{25} = 5$, but $\sqrt{25} \neq -5$. The $\sqrt{}$ symbol represents *only* the principal square root (positive square root).

To find the **negative square root** of a nonnegative number we must put a negative sign in front of the $\sqrt{}$. For example, $-\sqrt{25} = -5$.

Next we will define some terms associated with the $\sqrt{}$ symbol.

The symbol $\sqrt{}$ is the **square root symbol** or the **radical sign.** The number under the radical sign is the **radicand.** The entire expression, $\sqrt{25}$, is called a **radical.**

$$\text{Radical sign} \rightarrow \underbrace{\sqrt{25}}_{\text{Radical}} \leftarrow \text{Radicand} = 25$$

EXAMPLE 2

In-Class Example 2

Find each square root, if possible.

a) $\sqrt{16}$ b) $-\sqrt{9}$ c) $\sqrt{\frac{36}{49}}$ d) $-\sqrt{\frac{100}{9}}$ e) $\sqrt{-64}$

Answer: a) 4 b) -3 c) $\frac{6}{7}$ d) $-\frac{10}{3}$ e) not a real number

Hint

In Example 2, we are finding the **principal square roots** of 100 and $\frac{4}{25}$ and the **negative square roots** of 16 and $\frac{81}{49}$.

Find each square root, if possible.

a) $\sqrt{100}$ b) $-\sqrt{16}$ c) $\sqrt{\frac{4}{25}}$ d) $-\sqrt{\frac{81}{49}}$ e) $\sqrt{-9}$

Solution

a) $\sqrt{100} = 10$ since $(10)^2 = 100$.

b) $-\sqrt{16}$ means $-1 \cdot \sqrt{16}$. Therefore, $-\sqrt{16} = -1 \cdot \sqrt{16} = -1 \cdot 4 = -4$.

c) $\sqrt{\frac{4}{25}} = \frac{2}{5}$ since $\left(\frac{2}{5}\right)^2 = \frac{4}{25}$.

d) $-\sqrt{\frac{81}{49}}$ means $-1 \cdot \sqrt{\frac{81}{49}}$. So, $-\sqrt{\frac{81}{49}} = -1 \cdot \sqrt{\frac{81}{49}} = -1 \cdot \frac{9}{7} = -\frac{9}{7}$.

e) To find $\sqrt{-9}$, ask yourself, "What number do I *square* to get -9?" *There is no such real number* since $3^2 = 9$ and $(-3)^2 = 9$. Therefore, $\sqrt{-9}$ is not a real number.

YOU TRY 2

Find each square root.

a) $\sqrt{9}$ b) $-\sqrt{144}$ c) $\sqrt{\frac{25}{36}}$ d) $-\sqrt{\frac{1}{64}}$ e) $\sqrt{-49}$

Let's review what we know about radicands and add a third fact.

Property Radicands and Square Roots

1) If the radicand is a *perfect square,* the square root is a *rational* number.

Example: $\sqrt{16} = 4$ 16 is a perfect square. $\sqrt{\frac{100}{49}} = \frac{10}{7}$ $\frac{100}{49}$ is a perfect square.

2) If the radicand is a *negative number,* the square root is *not* a real number.

Example: $\sqrt{-25}$ is *not* a real number.

3) If the radicand is *positive and not a perfect square,* then the square root is an *irrational* number.

Example: $\sqrt{13}$ is irrational. 13 is not a perfect square.

The square root of such a number is a real number that is a nonrepeating, nonterminating decimal.

Sometimes, we must plot points containing radicals. For the purposes of graphing, approximating a radical to the nearest tenth is sufficient. A calculator with a $\sqrt{\ }$ key will give a better approximation of the radical.

W Hint

Do you think you could estimate $\sqrt{13}$? How could you do that?

2 Find Higher Roots

We saw in Example 2a) that $\sqrt{100} = 10$ since $(10)^2 = 100$. We can also find higher roots of numbers like $\sqrt[3]{a}$ (read as "the cube root of *a*"), $\sqrt[4]{a}$ (read as "the fourth root of *a*"), $\sqrt[5]{a}$, etc. We will look at a few roots of numbers before learning important rules.

EXAMPLE 3

In-Class Example 3

Find each root.

a) $\sqrt[3]{27}$ b) $\sqrt[3]{64}$

Answer: a) 3 b) 4

Find each root.

a) $\sqrt[3]{125}$ b) $\sqrt[5]{32}$

Solution

a) To find $\sqrt[3]{125}$ (the cube root of 125) ask yourself, "What number do I *cube* to get 125?" That number is 5.

$$\mathbf{\sqrt[3]{125} = 5} \text{ since } 5^3 = 125$$

Finding the cube root of a number is the *opposite,* or *inverse* procedure, of cubing a number.

W Hint

Do you remember the powers of whole numbers? If not, review the list in Section 1.3.

b) To find $\sqrt[5]{32}$ (the fifth root of 32) ask yourself, "What number do I raise to the *fifth power* to get 32?" That number is 2.

$$\mathbf{\sqrt[5]{32} = 2} \text{ since } 2^5 = 32$$

Finding the fifth root of a number and raising a number to the fifth power are *opposite,* or *inverse,* procedures.

YOU TRY 3

Find each root.

a) $\sqrt[3]{27}$ b) $\sqrt[3]{8}$

The symbol $\sqrt[n]{a}$ is read as "the *nth* root of *a*." If $\sqrt[n]{a} = b$, then $b^n = a$.

$$\text{Index} \rightarrow \underbrace{\sqrt[n]{a}}_{\text{Radical}} \leftarrow \text{Radicand} = a$$

We call n the **index** of the radical.

Note

When finding *square* roots we do not write $\sqrt[2]{a}$. The square root of *a* is written as $\sqrt{a}$, and the index is understood to be 2.

We know that a positive number, say 36, has a principal square root ($\sqrt{36}$, or 6) and a negative square root ($-\sqrt{36}$, or -6). This is true for all even roots of positive numbers: square roots, fourth roots, sixth roots, and so on. For example, 81 has a principal fourth root ($\sqrt[4]{81}$, or 3) and a negative fourth root ($-\sqrt[4]{81}$, or -3).

W Hint

Write examples in your notes to help you understand the different situations.

Definition *n*th Root

For any ***positive*** number a and any ***even*** index n,

the **principal *n*th root** of a is $\sqrt[n]{a}$. $\sqrt[\text{even}]{\text{positive}} = \text{principal (positive) root}$

the **negative *n*th root** of a is $-\sqrt[n]{a}$. $-\sqrt[\text{even}]{\text{positive}} = \text{negative root}$

For any ***negative*** number a and any ***even*** index n,

there is **no** real *n*th root of a. $\sqrt[\text{even}]{\text{negative}} = \text{no real root}$

For any number a and any ***odd*** index n,

there is **one** real *n*th root of a, $\sqrt[n]{a}$. $\sqrt[\text{odd}]{\text{any number}} = \text{exactly one root}$

BE CAREFUL

The definition means that $\sqrt[4]{81}$ *cannot* be -3 because $\sqrt[4]{81}$ is *defined* as the principal fourth root of 81, which must be positive. $\sqrt[4]{81} = 3$

EXAMPLE 4

In-Class Example 4

Find each root, if possible.
a) $\sqrt[3]{-27}$ b) $-\sqrt[4]{81}$
c) $\sqrt[4]{-81}$ d) $\sqrt[3]{-125}$
e) $\sqrt[6]{64}$

Answer: a) -3 b) -3
c) not a real number
d) -5 e) 2

Find each root, if possible.

a) $\sqrt[4]{16}$ b) $-\sqrt[4]{16}$ c) $\sqrt[4]{-16}$ d) $\sqrt[3]{64}$ e) $\sqrt[3]{-64}$

Solution

a) To find $\sqrt[4]{16}$ ask yourself, "What *positive* number do I raise to the *fourth power* to get 16?" Since $2^4 = 16$ and 2 is positive, $\mathbf{\sqrt[4]{16} = 2}$.

b) In part a) we found that $\sqrt[4]{16} = 2$, so $\mathbf{-\sqrt[4]{16} = -(\sqrt[4]{16}) = -2}$.

c) To find $\sqrt[4]{-16}$ ask yourself, "What number do I raise to the *fourth power* to get -16?" There is no such real number since $2^4 = 16$ and $(-2)^4 = 16$. Therefore, $\mathbf{\sqrt[4]{-16}}$ **has *no real root.*** (Recall from the definition that $\sqrt[\text{even}]{\text{negative}}$ has no real root.)

d) To find $\sqrt[3]{64}$ ask yourself, "What number do I *cube* to get 64?" Since $4^3 = 64$ and since we know that $\sqrt[\text{odd}]{\text{any number}}$ gives exactly one root, $\mathbf{\sqrt[3]{64} = 4}$.

e) To find $\sqrt[3]{-64}$ ask yourself, "What number do I *cube* to get -64?" Since $(-4)^3 = -64$ and since we know that $\sqrt[\text{odd}]{\text{any number}}$ gives exactly one root, $\mathbf{\sqrt[3]{-64} = -4}$.

[YOU TRY 4]

Find each root, if possible.

a) $\sqrt[6]{64}$ b) $-\sqrt[6]{64}$ c) $\sqrt[6]{-64}$ d) $\sqrt[3]{125}$ e) $\sqrt[3]{-125}$

3 Evaluate $\sqrt[n]{a^n}$

Earlier we said that the $\sqrt{}$ symbol represents only the *positive* square root of a number. For example, $\sqrt{9} = 3$. It is also true that $\sqrt{(-3)^2} = \sqrt{9} = 3$.

If a variable is in the radicand and we do not know whether the variable represents a positive number, then we must use the absolute value symbol to evaluate the radical. Then we know that the result will be a positive number. For example, $\sqrt{a^2} = |a|$.

What if the index is greater than 2? Let's look at how to find the following roots:

$$\sqrt[4]{(-2)^4} = \sqrt[4]{16} = 2 \qquad \sqrt[3]{(-4)^3} = \sqrt[3]{-64} = -4$$

When the index on the radical is any positive, even integer and we do not know whether the variable in the radicand represents a positive number, we must use the absolute value symbol to write the root. However, when the index is a positive, odd integer, we do not need to use the absolute value symbol.

Procedure Evaluating $\sqrt[n]{a^n}$

1) If n is a positive, *even* integer, then $\sqrt[n]{a^n} = |a|$.
2) If n is a positive, *odd* integer, then $\sqrt[n]{a^n} = a$.

EXAMPLE 5

In-Class Example 5

Simplify.
a) $\sqrt{(-9)^2}$ b) $\sqrt{r^2}$
c) $\sqrt[3]{(-2)^3}$ d) $\sqrt[7]{p^7}$
e) $\sqrt[4]{(z-6)^4}$
f) $\sqrt[5]{(7m+4)^5}$

Answer: a) 9 b) $|r|$ c) -2
d) p e) $|z-6|$ f) $7m+4$

W Hint
In your notes, summarize when you need the absolute value symbol and when you do not.

Simplify.

a) $\sqrt{(-7)^2}$ b) $\sqrt{k^2}$ c) $\sqrt[3]{(-5)^3}$ d) $\sqrt[7]{n^7}$

e) $\sqrt[4]{(y-9)^4}$ f) $\sqrt[5]{(8p+1)^5}$

Solution

a) $\sqrt{(-7)^2} = |-7| = 7$ When the index is even, use the absolute value symbol to be certain that the result is not negative.

b) $\sqrt{k^2} = |k|$ When the index is even, use the absolute value symbol to be certain that the result is not negative.

c) $\sqrt[3]{(-5)^3} = -5$ The index is odd, so the absolute value symbol is not necessary.

d) $\sqrt[7]{n^7} = n$ The index is odd, so the absolute value symbol is not necessary.

e) $\sqrt[4]{(y-9)^4} = |y-9|$ Even index: use the absolute value symbol to be certain that the result is not negative.

f) $\sqrt[5]{(8p+1)^5} = 8p+1$ Odd index: the absolute value symbol is not necessary.

YOU TRY 5

Simplify.

a) $\sqrt{(-12)^2}$ b) $\sqrt{w^2}$ c) $\sqrt[3]{(-3)^3}$ d) $\sqrt[5]{r^5}$

e) $\sqrt[6]{(t+4)^6}$ f) $\sqrt[7]{(4h-3)^7}$

4 Determine the Domains of Square Root and Cube Root Functions

An algebraic expression containing a radical is called a **radical expression.** When real numbers are substituted for the variable in radical expressions like $\sqrt{x}$, $\sqrt{4t+1}$, and $\sqrt[3]{p}$ so that the expression is defined, each value that is substituted will produce *only one* value for the expression. Therefore, function notation can be used to represent radical expressions.

Radical functions are functions of the form $f(x) = \sqrt[n]{x}$. Let's look at some square root and cube root functions.

Two examples of **square root functions** are $f(x) = \sqrt{x}$ and $g(r) = \sqrt{2r-9}$.

EXAMPLE 6

In-Class Example 6

Let $f(x) = \sqrt{x}$ and $g(a) = \sqrt{2a-7}$. Find the function values, if possible.
a) $f(49)$ b) $g(5)$ c) $f(-4)$
d) $g(3)$

Answer: a) 7 b) $\sqrt{3}$
c) not a real number
d) not a real number

Let $f(x) = \sqrt{x}$ and $g(r) = \sqrt{2r-9}$. Find the function values, if possible.

a) $f(64)$ b) $g(7)$ c) $f(-25)$ d) $g(3)$

Solution

a) $f(64) = \sqrt{64} = 8$

b) $g(7) = \sqrt{2(7)-9} = \sqrt{14-9} = \sqrt{5}$

c) $f(-25) = \sqrt{-25}$; not a real number

d) $g(3) = \sqrt{2(3)-9} = \sqrt{-3}$; not a real number

YOU TRY 6

Let $f(x) = \sqrt{x}$ and $h(t) = \sqrt{3t-10}$. Find the function values, if possible.

a) $f(25)$ b) $h(9)$ c) $f(-11)$ d) $h(2)$

Parts c) and d) of Example 6 illustrate that when the radicand of a square root function is negative, the function is undefined. Therefore, *any value that makes the radicand negative is not in the domain of a square root function.*

Definition

The **domain of a square root function** consists of all of the real numbers that can be substituted for the variable so that radicand is nonnegative.

To determine the domain of a square root function, set up an inequality so that the radicand ≥ 0. Solve for the variable. These are the real numbers in the domain of the function.

EXAMPLE 7

In-Class Example 7

Determine the domain of each square root function.
a) $f(x) = \sqrt{x}$
b) $g(a) = \sqrt{2a - 7}$

Answers: a) $[0, \infty)$
b) $\left[\frac{7}{2}, \infty\right)$

Determine the domain of each square root function.

a) $f(x) = \sqrt{x}$ b) $g(r) = \sqrt{2r - 9}$

Solution

a) The radicand, x, must be greater than or equal to zero. We write that as the inequality $x \geq 0$. In interval notation, we write the domain as $[0, \infty)$.

b) In the square root function $g(r) = \sqrt{2r - 9}$, the radicand, $2r - 9$, must be nonnegative. We write this as $2r - 9 \geq 0$. To determine the domain of the function, solve the inequality $2r - 9 \geq 0$.

$$2r - 9 \geq 0 \quad \text{The value of the radicand must be } \geq 0.$$
$$2r \geq 9$$
$$r \geq \frac{9}{2} \quad \text{Solve.}$$

Any value of r that satisfies $r \geq \frac{9}{2}$ will make the radicand greater than or equal to zero. The domain of $g(r) = \sqrt{2r - 9}$ is $\left[\frac{9}{2}, \infty\right)$.

YOU TRY 7

Determine the domain of each square root function.

a) $h(x) = \sqrt{x - 9}$ b) $k(t) = \sqrt{7t + 2}$

Two examples of cube root functions are $f(x) = \sqrt[3]{x}$ and $h(a) = \sqrt[3]{a - 5}$. Let's look at these next.

EXAMPLE 8

In-Class Example 8

Let $f(x) = \sqrt[3]{x}$ and $h(t) = \sqrt[3]{t - 9}$. Find the function values, if possible.
a) $f(27)$ b) $f(-10)$ c) $h(1)$
d) $h(13)$

Answers: a) 3 b) $\sqrt[3]{-10}$
c) -2 d) $\sqrt[3]{4}$

Let $f(x) = \sqrt[3]{x}$ and $h(a) = \sqrt[3]{a - 5}$. Find the function values, if possible.

a) $f(125)$ b) $f(-7)$ c) $h(10)$ d) $h(-3)$

Solution

a) $f(125) = \sqrt[3]{125} = 5$

b) $f(-7) = \sqrt[3]{-7}$

c) $h(10) = \sqrt[3]{10 - 5} = \sqrt[3]{5}$

d) $h(-3) = \sqrt[3]{-3 - 5} = \sqrt[3]{-8} = -2$

YOU TRY 8

Let $f(x)=\sqrt[3]{x}$ and $g(c)=\sqrt[3]{2c+3}$. Find the function values, if possible.

a) $f(25)$ b) $f(-27)$ c) $g(-2)$ d) $g(2)$

Unlike square root functions, it is possible to evaluate cube root functions when the radicand is negative. Therefore, *any real number may be substituted into a cube root function and the function will be defined.*

Definition

The **domain of a cube root function** is the set of all real numbers. We can write this in interval notation as $(-\infty, \infty)$.

In fact we can say that when n is an odd, positive number, the domain of $f(x)=\sqrt[n]{x}$ is all real numbers, or $(-\infty, \infty)$. This is because the odd root of any real number is, itself, a real number.

5 Graph a Square Root Function

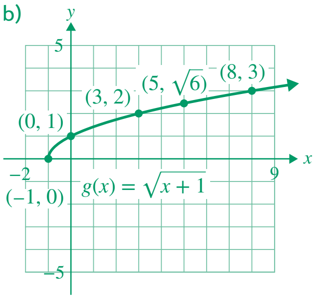
We need to know the domain of a square root function in order to sketch its graph.

EXAMPLE 9

In-Class Example 9

Graph each function.
a) $f(x)=\sqrt{x}$
b) $g(x)=\sqrt{x+1}$

Answers:
a) See graph in the solution to Example 9a.
b)



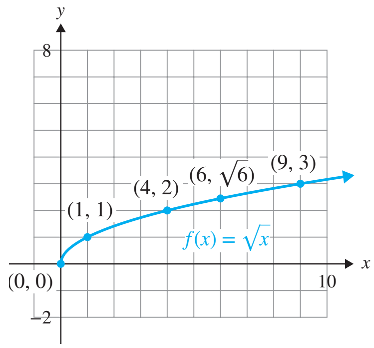
Graph each function.

a) $f(x)=\sqrt{x}$ b) $g(x)=\sqrt{x+4}$

Solution

a) In Example 7 we found that the domain of $f(x)=\sqrt{x}$ is $[0, \infty)$. When we make a table of values, we will start by letting $x = 0$, the smallest number in the domain, and then choose real numbers greater than 0. Usually it is easiest to choose values for x that are perfect squares so that it will be easier to plot the points. We will also plot the point $(6, \sqrt{6})$ so that you can see where it lies on the graph. Connect the points with a smooth curve.

$f(x)=\sqrt{x}$	
x	$f(x)$
0	0
1	1
4	2
6	$\sqrt{6}\approx 2.4$
9	3



The graph reinforces the fact that this is a function. It passes the vertical line test.

Hint

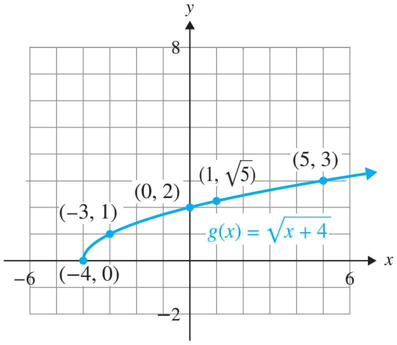
In your notes, write an explanation in your own words about how to graph a square root function.

b) To graph $g(x) = \sqrt{x+4}$ we will begin by determining its domain. Solve $x + 4 \geq 0$.

$$x + 4 \geq 0 \qquad \text{The value of the radicand must be} \geq 0.$$
$$x \geq -4 \qquad \text{Solve.}$$

The domain of $g(x)$ is $[-4, \infty)$. When we make a table of values, we will start by letting $x = -4$, the smallest number in the domain, and then choose real numbers greater than -4. We will choose values for x so that the radicand will be a perfect square. This will make it easier to plot the points. We will also plot the point $(1, \sqrt{5})$ so that you can see where it lies on the graph. Connect the points with a smooth curve.

$g(x) = \sqrt{x+4}$	
x	$g(x)$
-4	0
-3	1
0	2
1	$\sqrt{5} \approx 2.2$
5	3



Since this graph represents a function, it passes the vertical line test.

YOU TRY 9

Graph $f(x) = \sqrt{x+2}$.

6 Graph a Cube Root Function

The domain of a cube root function consists of all real numbers. Therefore, we can substitute any real number into the function and it will be defined. However, we want to choose our numbers carefully. *To make the table of values, pick values in the domain so that the radicand will be a perfect cube, and choose values for the variable that will give us positive numbers, negative numbers, and zero for the value of the radicand.* This will help us to graph the function correctly.

EXAMPLE 10

Graph each function.

a) $f(x) = \sqrt[3]{x}$ b) $g(x) = \sqrt[3]{x-1}$

In-Class Example 10

Graph each function.

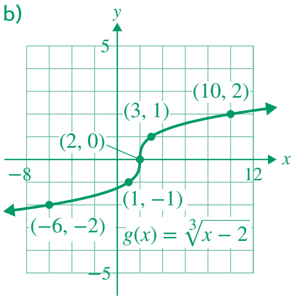
a) $f(x) = \sqrt[3]{x}$

b) $g(x) = \sqrt[3]{x-2}$

Answers:

a) See graph in the solution to Example 10a.

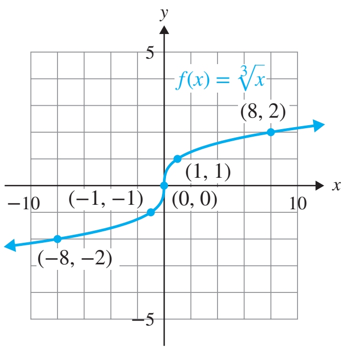
b)



Solution

a) Make a table of values. Choose x-values that are perfect cubes. Also, remember to choose x-values that are positive, negative, and zero. Plot the points, and connect them with a smooth curve.

$f(x) = \sqrt[3]{x}$	
x	$f(x)$
0	0
1	1
8	2
-1	-1
-8	-2



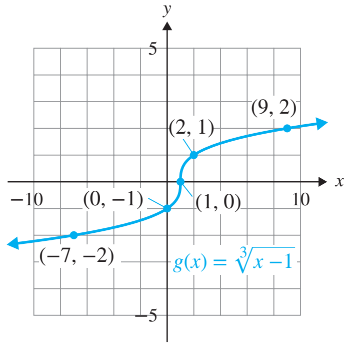
The graph passes the vertical line test for functions.

b) Remember, for the table of values we want to choose values for x that will give us positive numbers, negative numbers, and zero *in the radicand.* First we will determine what value of x will make the radicand in $g(x) = \sqrt[3]{x-1}$ equal to zero.

$$\begin{aligned} x - 1 &= 0 \\ x &= 1 \end{aligned}$$

If $x = 1$, the radicand equals zero. Therefore, the first value we will put in the table of values is $x = 1$. Then, choose a couple of numbers *greater than* 1 and a couple that are *less than* 1 so that we get positive and negative numbers in the radicand. Also, we will choose our x-values so that the radicand will be a perfect cube. Plot the points, and connect them with a smooth curve.

x	$g(x) = \sqrt[3]{x-1}$
1	$\sqrt[3]{1-1} = \sqrt[3]{0} = 0$
2	$\sqrt[3]{2-1} = \sqrt[3]{1} = 1$
9	$\sqrt[3]{9-1} = \sqrt[3]{8} = 2$
0	$\sqrt[3]{0-1} = \sqrt[3]{-1} = -1$
-7	$\sqrt[3]{-7-1} = \sqrt[3]{-8} = -2$



Since this graph represents a function, it passes the vertical line test.

YOU TRY 10

Graph $f(x) = \sqrt[3]{x-3}$.

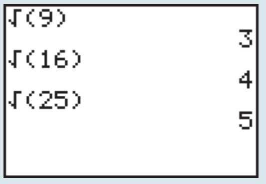
Using Technology

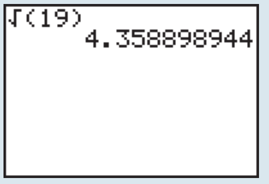
We can evaluate square roots, cube roots, or even higher roots using a graphing calculator. A radical sometimes evaluates to an integer and sometimes must be approximated using a decimal.

To evaluate a square root:

For example, to evaluate $\sqrt{9}$ press $\boxed{2^{nd}}$ $\boxed{x^2}$, enter the radicand $\boxed{9}$, and then press $\boxed{)}$ $\boxed{\text{ENTER}}$. The result is 3 as shown on the screen on the left below. When the radicand is a perfect square such as 9, 16, or 25, then the square root evaluates to a whole number. For example, $\sqrt{16}$ evaluates to 4 and $\sqrt{25}$ evaluates to 5 as shown.

If the radicand of a square root is not a perfect square, then the result is a decimal approximation. For example, to evaluate $\sqrt{19}$ press $\boxed{2^{nd}}$ $\boxed{x^2}$, enter the radicand $\boxed{1}$ $\boxed{9}$, and then press $\boxed{)}$ $\boxed{\text{ENTER}}$. The result is approximately 4.3589, rounded to four decimal places.





To evaluate a cube root:

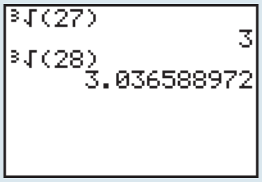
For example, to evaluate $\sqrt[3]{27}$ press MATH 4, enter the radicand 2 7, and then press) ENTER. The result is 3 as shown.

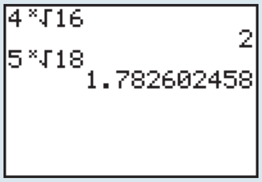
If the radicand is a perfect cube such as 27, then the cube root evaluates to an integer. Since 28 is not a perfect cube, the cube root evaluates to approximately 3.0366.

To evaluate radicals with an index greater than 3:

For example, to evaluate $\sqrt[4]{16}$ enter the index 4, press MATH 5, enter the radicand 1 6, and press ENTER. The result is 2.

Since the fifth root of 18 evaluates to a decimal, the result is an approximation of 1.7826, rounded to four decimal places as shown.





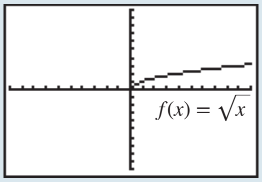
Evaluate each root using a graphing calculator. If necessary, approximate to the nearest tenth.

1) $\sqrt{25}$ 2) $\sqrt[3]{216}$ 3) $\sqrt{29}$ 4) $\sqrt{324}$ 5) $\sqrt[5]{1024}$ 6) $\sqrt[3]{343}$

Using Technology

We can use a graphing calculator to find the domain of a square root function or cube root function visually. The domain consists of the x-values of the points shown on the graph.

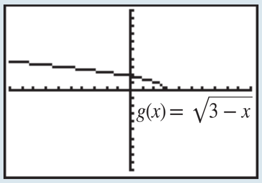
We first consider the basic shape of a square root function. To graph the equation $f(x) = \sqrt{x}$, press 2nd x^2 X,T,Θ,n) to the right of \Y$_1$ =. Press ZOOM and select 6:ZStandard to graph the equation.



The left side of the graph begins at the point (0, 0), and the right side of the graph continues up and to the right forever. The x-values of the graph consist of all x-values greater than or equal to 0. In interval notation, the domain is $[0, \infty)$.

The domain of any square root function can be found using a similar approach. First graph the function, and then look at the x-values of the points on the graph. The graph of a square root function will always start at a number and extend to positive or negative infinity.

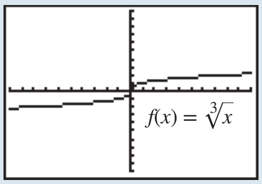
For example, consider the graph of the function $g(x) = \sqrt{3 - x}$ as shown.



The largest x-value on the graph is 3. The x-values of the graph consist of all x-values less than or equal to 3. In interval notation, the domain is $(-\infty, 3]$.

Next consider the basic shape of a cube root function.

To graph the equation $f(x) = \sqrt[3]{x}$ press MATH, select 4:$\sqrt[3]{}$ (, and press X,T,Θ,n) to the right of \Y$_1$=. Press ZOOM and select 6:ZStandard to graph the equation as shown on the graph at right.



The left side of the graph extends down and to the left forever, and the right side of the graph extends up and to the right forever. In interval notation, the domain is $(-\infty, \infty)$. This is true for any cube root function, so the domain is always $(-\infty, \infty)$.

Determine the domain using a graphing calculator. Use interval notation in your answer.

7) $f(x) = \sqrt{x-2}$ 8) $g(x) = \sqrt{x+3}$ 9) $h(x) = \sqrt{2-x}$

10) $f(x) = -\sqrt{x+1}$ 11) $f(x) = \sqrt[3]{x+5}$ 12) $g(x) = \sqrt[3]{4-x}$

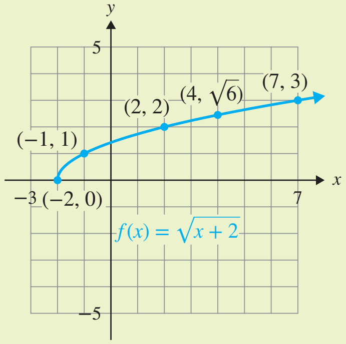
ANSWERS TO [YOU TRY] EXERCISES

1) 8, −8 2) a) 3 b) −12 c) $\frac{5}{6}$ d) $-\frac{1}{8}$ e) not a real number 3) a) 3 b) 2

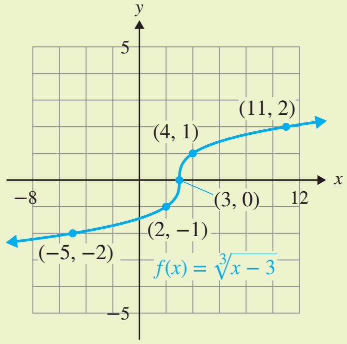
4) a) 2 b) −2 c) not a real number d) 5 e) −5 5) a) 12 b) $|w|$ c) −3 d) r e) $|t+4|$ f) $4h-3$ 6) a) 5 b) $\sqrt{17}$ c) not a real number d) not a real number

7) a) $[9, \infty)$ b) $\left[-\frac{2}{7}, \infty\right)$ 8) a) $\sqrt[3]{25}$ b) −3 c) −1 d) $\sqrt[3]{7}$

9)



10)



ANSWERS TO TECHNOLOGY EXERCISES

1) 5 2) 6 3) 5.4 4) 18 5) 4 6) 7 7) $[2, \infty)$ 8) $[-3, \infty)$
9) $(-\infty, 2]$ 10) $[-1, \infty)$ 11) $(-\infty, \infty)$ 12) $(-\infty, \infty)$

E Evaluate 9.1 Exercises

Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Objective 1: Find Square Roots and Principal Square Roots

Decide whether each statement is true or false. If it is false, explain why.

1) $\sqrt{121} = 11$ and -11. False; $\sqrt{121} = 11$ because the $\sqrt{\ }$ symbol means principal square root.

2) $\sqrt{81} = 9$ True

3) The square root of a negative number is a negative number.
False; the square root of a negative number is not a real number.

4) The even root of a negative number is a negative number. False; it is not a real number.

Find all square roots of each number.

5) 144 12 and −12 6) 2500 50 and −50

7) $\frac{36}{25}$ $\frac{6}{5}$ and $-\frac{6}{5}$ 8) 0.01 0.1 and −0.1

Find each square root, if possible.

9) $\sqrt{49}$ 7 10) $\sqrt{169}$ 13

11) $\sqrt{-4}$ not real 12) $\sqrt{-100}$ not real

13) $\sqrt{\frac{81}{25}}$ $\frac{9}{5}$ 14) $\sqrt{\frac{121}{4}}$ $\frac{11}{2}$

15) $-\sqrt{36}$ −6 16) $-\sqrt{0.04}$ −0.2

Objective 2: Find Higher Roots

Decide if each statement is true or false. If it is false, explain why.

17) The cube root of a negative number is a negative number. True

18) The odd root of a negative number is not a real number. False; the odd root of a negative number is a negative number.

19) Every nonnegative real number has two real, even roots. False; the only even root of zero is zero.

20) $-\sqrt[4]{10,000} = -10$ True

21) Explain how to find $\sqrt[3]{64}$.

22) Explain how to find $\sqrt[4]{16}$.

23) Does $\sqrt[4]{-81} = -3$? Why or why not?

24) Does $\sqrt[3]{-8} = -2$? Why or why not?

Find each root, if possible.

25) $\sqrt[3]{125}$ 5

26) $\sqrt[3]{27}$ 3

27) $\sqrt[3]{-1}$ -1

28) $\sqrt[3]{-8}$ -2

29) $\sqrt[4]{81}$ 3

30) $\sqrt[4]{16}$ 2

31) $\sqrt[4]{-1}$ not real

32) $\sqrt[4]{-81}$ not real

33) $-\sqrt[4]{16}$ -2

34) $-\sqrt[4]{1}$ -1

35) $\sqrt[5]{-32}$ -2

36) $-\sqrt[6]{64}$ -2

37) $-\sqrt[3]{-1000}$ 10

38) $-\sqrt[3]{-27}$ 3

39) $\sqrt[6]{-64}$ not real

40) $\sqrt[4]{-16}$ not real

41) $\sqrt[3]{\dfrac{8}{125}}$ $\frac{2}{5}$

42) $\sqrt[4]{\dfrac{81}{16}}$ $\frac{3}{2}$

43) $\sqrt{60-11}$ 7

44) $\sqrt{100+21}$ 11

45) $\sqrt[3]{9-36}$ -3

46) $\sqrt{1-9}$ not real

47) $\sqrt{5^2+12^2}$ 13

48) $\sqrt{3^2+4^2}$ 5

Objective 3: Evaluate $\sqrt[n]{a^n}$

49) If n is a positive, even integer and we are not certain that $a \geq 0$, then we must use the absolute value symbol to evaluate $\sqrt[n]{a^n}$. That is, $\sqrt[n]{a^n} = |a|$. Why must we use the absolute value symbol?

50) If n is a positive, odd integer then $\sqrt[n]{a^n} = a$ for any value of a. Why don't we need to use the absolute value symbol?

Simplify.

51) $\sqrt{8^2}$ 8

52) $\sqrt{5^2}$ 5

53) $\sqrt{(-6)^2}$ 6

54) $\sqrt{(-11)^2}$ 11

55) $\sqrt{y^2}$ $|y|$

56) $\sqrt{d^2}$ $|d|$

57) $\sqrt[3]{5^3}$ 5

58) $\sqrt[3]{(-4)^3}$ -4

59) $\sqrt[3]{z^3}$ z

60) $\sqrt[7]{t^7}$ t

61) $\sqrt[4]{h^4}$ $|h|$

62) $\sqrt[6]{m^6}$ $|m|$

63) $\sqrt{(x+7)^2}$ $|x+7|$

64) $\sqrt{(a-9)^2}$ $|a-9|$

65) $\sqrt[3]{(2t-1)^3}$ $2t-1$

66) $\sqrt[5]{(6r+7)^5}$ $6r+7$

67) $\sqrt[4]{(3n+2)^4}$ $|3n+2|$

68) $\sqrt[3]{(x-6)^3}$ $x-6$

69) $\sqrt[7]{(d-8)^7}$ $d-8$

70) $\sqrt[6]{(4y+3)^6}$ $|4y+3|$

Objective 4: Determine the Domains of Square Root and Cube Root Functions

71) Is -1 in the domain of $f(x) = \sqrt{x}$? Explain your answer. No, because $\sqrt{-1}$ is not a real number.

72) Is -1 in the domain of $f(x) = \sqrt[3]{x}$? Explain your answer. Yes, because $\sqrt[3]{-1} = -1$.

73) How do you find the domain of a square root function?

74) What is the domain of a cube root function? $(-\infty, \infty)$

Let $f(x) = \sqrt{x}$ and $g(t) = \sqrt{3t+4}$. Find each of the following, if possible, and simplify.

75) $f(100)$ 10

76) $f(9)$ 3

77) $f(-49)$ not a real number

78) $f(-64)$ not a real number

79) $g(-1)$ 1

80) $g(3)$ $\sqrt{13}$

81) $g(-5)$ not a real number

82) $g(-2)$ not a real number

83) $f(a)$ $\sqrt{a}$

84) $g(w)$ $\sqrt{3w+4}$

85) $f(t+4)$ $\sqrt{t+4}$

86) $f(5p-9)$ $\sqrt{5p-9}$

87) $g(2n-1)$ $\sqrt{6n+1}$

88) $g(m+10)$ $\sqrt{3m+34}$

Let $f(a) = \sqrt[3]{a}$ and $g(x) = \sqrt[3]{4x-1}$. Find each of the following, and simplify.

89) $f(64)$ 4

90) $f(-125)$ -5

91) $f(-27)$ -3

92) $g(3)$ $\sqrt[3]{11}$

93) $g(-4)$ $\sqrt[3]{17}$

94) $g(0)$ -1

95) $g(r)$ $\sqrt[3]{4r-1}$

96) $f(z)$ $\sqrt[3]{z}$

97) $f(c+8)$ $\sqrt[3]{c+8}$

98) $f(5k-2)$ $\sqrt[3]{5k-2}$

99) $g(2a-3)$ $\sqrt[3]{8a-13}$

100) $g(6-w)$ $\sqrt[3]{24-4w}$

Determine the domain of each function.

101) $h(n) = \sqrt{n+2}$ $[-2, \infty)$

102) $g(c) = \sqrt{c+10}$ $[-10, \infty)$

103) $p(a) = \sqrt{a-8}$ $[8, \infty)$

104) $f(a) = \sqrt{a-1}$ $[1, \infty)$

105) $f(a) = \sqrt[3]{a-7}$ $(-\infty, \infty)$

106) $h(t) = \sqrt[3]{t}$ $(-\infty, \infty)$

107) $r(k) = \sqrt{3k+7}$ $\left[-\frac{7}{3}, \infty\right)$

108) $k(x) = \sqrt{2x-5}$ $\left[\frac{5}{2}, \infty\right)$

109) $g(x) = \sqrt[3]{2x-5}$ $(-\infty, \infty)$

110) $h(c) = \sqrt[3]{-c}$ $(-\infty, \infty)$

111) $g(t) = \sqrt{-t}$ $(-\infty, 0]$

112) $h(x) = \sqrt{3-x}$ $(-\infty, 3]$

113) $r(a) = \sqrt{9-7a}$ $\left(-\infty, \frac{9}{7}\right]$

114) $g(c) = \sqrt{8-5c}$ $\left(-\infty, \frac{8}{5}\right]$

Objective 5: Graph a Square Root Function

Determine the domain, and then graph each function.

115) $f(x) = \sqrt{x-1}$

116) $g(x) = \sqrt{x-4}$

117) $g(x) = \sqrt{x+3}$

118) $h(x) = \sqrt{x+1}$

119) $h(x) = \sqrt{x} - 2$

120) $f(x) = \sqrt{x} + 2$

121) $f(x) = \sqrt{-x}$

122) $g(x) = \sqrt{-x} - 3$

Objective 6: Graph a Cube Root Function

Determine the domain, and then graph each function.

123) $f(x) = \sqrt[3]{x+1}$

124) $g(x) = \sqrt[3]{x+2}$

125) $h(x) = \sqrt[3]{x-2}$

126) $g(x) = \sqrt[3]{-x}$

127) $g(x) = \sqrt[3]{x} + 1$

128) $h(x) = \sqrt[3]{x} - 1$

129) A stackable storage cube has a storage capacity of 8 ft^3. The length of a side of the cube, s, is given by $s = \sqrt[3]{V}$, where V is the volume of the cube. How long is each side of the cube? 2 ft

130) A die (singular of dice) has a surface area of 13.5 cm^3. Since the die is in the shape of a cube the length of a side of the die, s, is given by $s = \sqrt{\frac{A}{6}}$, where A is the surface area of the cube. How long is each side of the cube? 1.5 cm

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131) If an object is dropped from the top of a building 160 ft tall, then the formula $t = \sqrt{\frac{160-h}{16}}$ describes how many seconds, t, it takes for the object to reach a height of h ft above the ground. If a piece of ice falls off the top of this building, how long would it take to reach the ground? Give an exact answer and an approximation to two decimal places. $\sqrt{10}$ sec; 3.16 sec

132) A circular flower garden has an area of 51 ft^2. The radius, r, of a circle in terms of its area is given by $r = \sqrt{\frac{A}{\pi}}$, where A is the area of the circle. What is the radius of the garden? Give an exact answer and an approximation to two decimal places. $\sqrt{\frac{51}{\pi}}$ ft; 4.03 ft

133) The speed limit on a street in a residential neighborhood is 25 mph. A car involved in an accident on this road left skid marks 40 ft long. Accident investigators determine that the speed of the car can be described by the function $S(d) = \sqrt{22.5d}$, where $S(d)$ is the speed of the car, in miles per hour, at the time of impact and d is the length of the skid marks, in feet. Was the driver speeding at the time of the accident? Show all work to support your answer.
Yes. The car was traveling at 30 mph.

134) A car involved in an accident on a wet highway leaves skid marks 170 ft long. Accident investigators determine that the speed of the car, $S(d)$ in miles per hour, can be described by the function $S(d) = \sqrt{19.8d}$, where d is the length of the skid marks, in feet. The speed limit on the highway is 65 mph. Was the car speeding when the accident occurred? Show all work to support your answer.
No. The car was traveling about 58 mph.

Use the following information for Exercises 135–140.

The period of a pendulum is the time it takes for the pendulum to make one complete swing back and forth. The period, $T(L)$ in seconds, can be described by the function $T(L) = 2\pi\sqrt{\frac{L}{32}}$, where L is the length of the pendulum, in feet. For each exercise give an *exact answer and an answer rounded to two decimal places.* Use 3.14 for π.

135) Find the period of the pendulum whose length is 8 ft.
π sec; 3.14 sec

136) Find the period of the pendulum whose length is 2 ft. $\frac{\pi}{2}$ sec; 1.57 sec

137) Find $T\left(\frac{1}{2}\right)$, and explain what it means in the context of the problem.

138) Find $T(1.5)$, and explain what it means in the context of the problem.

139) Find the period of a 30-inch-long pendulum.

140) Find the period of a 54-inch-long pendulum.

R Rethink

R1) How well do you remember the powers of whole numbers? Refer to the list in Section 1.3 if you need to review them.

R2) Explain, in your own words, how to determine the domain of a square root function.

9.2 Rational Exponents

Prepare | **Organize**

What are your objectives for Section 9.2?	How can you accomplish each objective?
1 Evaluate Expressions of the Form $a^{1/n}$	• Write the definition of $a^{1/n}$ in your own words. • Complete the given example on your own. • Complete You Try 1.
2 Evaluate Expressions of the Form $a^{m/n}$	• Write the definition of $a^{m/n}$ in your own words. • Complete the given example on your own. • Complete You Try 2.
3 Evaluate Expression of the Form $a^{-m/n}$	• Write the definition of $a^{-m/n}$ in your own words. • Complete the given example on your own. • Complete You Try 3.
4 Combine the Rules of Exponents	• Review the rules of exponents found in Sections 6.1 and 6.2 to help follow the examples. • Complete the given examples on your own. • Complete You Trys 4 and 5.
5 Convert a Radical Expression to Exponential Form and Simplify	• Use the same rules of exponents to convert between radical and exponential forms. • Complete the given examples on your own. • Complete You Trys 6 and 7.

Work

Read the explanations, follow the examples, take notes, and complete the You Trys.

1 Evaluate Expressions of the Form $a^{1/n}$

In this section, we will explain the relationship between radicals and rational exponents (fractional exponents). Sometimes, converting between these two forms makes it easier to simplify expressions.

Definition

If n is a positive integer greater than 1 and $\sqrt[n]{a}$ is a real number, then

$$a^{1/n} = \sqrt[n]{a}$$

(The denominator of the fractional exponent is the index of the radical.)

EXAMPLE 1

In-Class Example 1

Write in radical form, and evaluate.
a) $9^{1/2}$ b) $27^{1/3}$ c) $32^{1/5}$
d) $(-144)^{1/2}$ e) $-16^{1/4}$
f) $(-32)^{1/5}$

Answer: a) 3 b) 3 c) 2
d) not a real number
e) −2 f) −2

Write in radical form, and evaluate.

a) $8^{1/3}$ b) $49^{1/2}$ c) $81^{1/4}$

d) $-64^{1/6}$ e) $(-16)^{1/4}$ f) $(-125)^{1/3}$

Solution

a) The denominator of the fractional exponent is the index of the radical. Therefore, $8^{1/3} = \sqrt[3]{8} = 2$.

b) The denominator in the exponent of $49^{1/2}$ is 2, so the index on the radical is 2, meaning *square* root.

$$49^{1/2} = \sqrt{49} = 7$$

c) $81^{1/4} = \sqrt[4]{81} = 3$

d) $-64^{1/6} = -(64^{1/6}) = -\sqrt[6]{64} = -2$

e) $(-16)^{1/4} = \sqrt[4]{-16}$, which is not a real number. Remember, the even root of a negative number is not a real number.

f) $(-125)^{1/3} = \sqrt[3]{-125} = -5$ The odd root of a negative number is a negative number.

YOU TRY 1

Write in radical form, and evaluate.

a) $16^{1/4}$ b) $121^{1/2}$ c) $1000^{1/3}$ d) $-81^{1/4}$ e) $(-49)^{1/2}$ f) $(-125)^{1/3}$

2 Evaluate Expressions of the Form $a^{m/n}$

We can add another relationship between rational exponents and radicals.

Definition

If m and n are positive integers and $\frac{m}{n}$ is in lowest terms, then

$$\boldsymbol{a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m}$$

provided that $a^{1/n}$ is a real number.

(The denominator of the fractional exponent is the index of the radical, and the numerator is the power to which we raise the radical expression.) We can also think of $a^{m/n}$ this way: $a^{m/n} = (a^m)^{1/n} = \sqrt[n]{a^m}$.

EXAMPLE 2

In-Class Example 2

Write in radical form, and evaluate.
a) $8^{2/3}$ b) $-16^{3/2}$
c) $(-16)^{3/2}$ d) $-32^{4/5}$
e) $(-125)^{2/3}$

Answer: a) 4 b) −64
c) not a real number
d) −16 e) 25

Write in radical form, and evaluate.

a) $25^{3/2}$ b) $-64^{2/3}$ c) $(-81)^{3/2}$ d) $-81^{3/2}$ e) $(-1000)^{2/3}$

Solution

a) The *denominator* of the fractional exponent is the *index* of the radical, and the *numerator* is the *power* to which we raise the radical expression.

$$\begin{aligned} 25^{3/2} &= (25^{1/2})^3 && \text{Use the definition to rewrite the exponent.} \\ &= (\sqrt{25})^3 && \text{Rewrite as a radical.} \\ &= 5^3 && \sqrt{25} = 5 \\ &= 125 \end{aligned}$$

W Hint

In your notes, write an explanation in your own words about how to convert from exponential form to radical form. Include examples.

b) To evaluate $-64^{2/3}$, *first* evaluate $64^{2/3}$, *then* take the negative of that result.

$$\begin{aligned} -64^{2/3} = -(64^{2/3}) &= -(64^{1/3})^2 && \text{Use the definition to rewrite the exponent.} \\ &= -(\sqrt[3]{64})^2 && \text{Rewrite as a radical.} \\ &= -(4)^2 && \sqrt[3]{64} = 4 \\ &= -16 \end{aligned}$$

c) $(-81)^{3/2} = [(-81)^{1/2}]^3$
$= (\sqrt{-81})^3$ Not a real number. The even root of a negative number is not a real number.

d) $-81^{3/2} = -(81^{1/2})^3 = -(\sqrt{81})^3 = -(9)^3 = -729$

e) $(-1000)^{2/3} = [(-1000)^{1/3}]^2 = (\sqrt[3]{-1000})^2 = (-10)^2 = 100$

[YOU TRY 2]

Write in radical form, and evaluate.

a) $32^{2/5}$ b) $-100^{3/2}$ c) $(-100)^{3/2}$ d) $(-1)^{4/5}$ e) $-1^{5/3}$

BE CAREFUL

In Example 2, notice how the parentheses affect how we evaluate an expression. The base of the expression $(-81)^{3/2}$ is -81, while the base of $-81^{3/2}$ is 81.

3 Evaluate Expressions of the Form $a^{-m/n}$

Recall that if n is any integer and $a \neq 0$, then $a^{-n} = \left(\frac{1}{a}\right)^n = \frac{1}{a^n}$.

That is, to rewrite the expression with a *positive* exponent, take the reciprocal of the base. For example,

$$2^{-4} = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

We can extend this idea to rational exponents.

Definition

If $a^{m/n}$ is a nonzero real number, then

$$a^{-m/n} = \left(\frac{1}{a}\right)^{m/n} = \frac{1}{a^{m/n}}$$

(To rewrite the expression with a *positive* exponent, take the reciprocal of the base.)

EXAMPLE 3

Rewrite with a positive exponent, and evaluate.

a) $36^{-1/2}$ b) $32^{-2/5}$ c) $\left(\frac{125}{64}\right)^{-2/3}$

In-Class Example 3

Rewrite with a positive exponent, and evaluate.

a) $9^{-3/2}$ b) $81^{-3/4}$ c) $\left(\frac{8}{27}\right)^{-4/3}$

Answer: a) $\frac{1}{27}$ b) $\frac{1}{27}$ c) $\frac{81}{16}$

W Hint

Write out the steps as you are reading the example!

Solution

a) To write $36^{-1/2}$ with a positive exponent, take the reciprocal of the base.

$$36^{-1/2} = \left(\frac{1}{36}\right)^{1/2}$$ The reciprocal of 36 is $\frac{1}{36}$.

$$= \sqrt{\frac{1}{36}}$$ The denominator of the fractional exponent is the index of the radical.

$$= \frac{1}{6}$$

b) $$32^{-2/5} = \left(\frac{1}{32}\right)^{2/5}$$ The reciprocal of 32 is $\frac{1}{32}$.

$$= \left(\sqrt[5]{\frac{1}{32}}\right)^2$$ The denominator of the fractional exponent is the index of the radical.

$$= \left(\frac{1}{2}\right)^2$$ $\sqrt[5]{\frac{1}{32}} = \frac{1}{2}$

$$= \frac{1}{4}$$

W Hint

The negative exponent does not make the expression negative!

c) $$\left(\frac{125}{64}\right)^{-2/3} = \left(\frac{64}{125}\right)^{2/3}$$ The reciprocal of $\frac{125}{64}$ is $\frac{64}{125}$.

$$= \left(\sqrt[3]{\frac{64}{125}}\right)^2$$ The denominator of the fractional exponent is the index of the radical.

$$= \left(\frac{4}{5}\right)^2$$ $\sqrt[3]{\frac{64}{125}} = \frac{4}{5}$

$$= \frac{16}{25}$$

[YOU TRY 3]

Rewrite with a positive exponent, and evaluate.

a) $144^{-1/2}$ b) $16^{-3/4}$ c) $\left(\frac{8}{27}\right)^{-2/3}$

4 Combine the Rules of Exponents

We can combine the rules presented in this section with the rules of exponents we learned in Chapter 6 to simplify expressions containing numbers or variables.

EXAMPLE 4

Simplify completely. The answer should contain only positive exponents.

a) $(6^{1/5})^2$ b) $25^{3/4} \cdot 25^{-1/4}$ c) $\dfrac{8^{2/9}}{8^{11/9}}$

In-Class Example 4

Simplify completely. The answer should contain only positive exponents.

a) $(3^{2/5})^6$ b) $27^{2/3} \cdot 27^{-1/3}$

c) $\dfrac{4^{13/3}}{4^{7/3}}$

Answer: a) $3^{12/5}$ b) 3 c) 16

Solution

a) $(6^{1/5})^2 = 6^{2/5}$ Multiply exponents.

b) $25^{3/4} \cdot 25^{-1/4} = 25^{\frac{3}{4}+\left(-\frac{1}{4}\right)}$ Add exponents.

$= 25^{2/4} = 25^{1/2} = 5$

c) $\dfrac{8^{2/9}}{8^{11/9}} = 8^{\frac{2}{9} - \frac{11}{9}}$ Subtract exponents.

$= 8^{-9/9}$ Subtract $\frac{2}{9} - \frac{11}{9}$.

$= 8^{-1}$ Reduce $-\frac{9}{9}$.

$= \left(\dfrac{1}{8}\right)^1 = \dfrac{1}{8}$

YOU TRY 4

Simplify completely. The answer should contain only positive exponents.

a) $(16^{1/12})^3$ b) $49^{3/8} \cdot 49^{1/8}$ c) $\dfrac{27^{4/9}}{27^{16/9}}$

EXAMPLE 5

Simplify completely. Assume the variables represent positive real numbers. The answer should contain only positive exponents.

a) $r^{1/8} \cdot r^{3/8}$ b) $\left(\dfrac{x^{2/3}}{y^{1/4}}\right)^6$ c) $\dfrac{n^{-5/6} \cdot n^{1/3}}{n^{-1/6}}$ d) $\left(\dfrac{a^{-7}b^{1/2}}{a^5c^{1/3}}\right)^{-3/4}$

In-Class Example 5

Simplify completely. Assume the variables represent positive real numbers. The answer should contain only positive exponents.

a) $x^{1/4} \cdot x^{5/4}$ b) $\left(\dfrac{p^{1/6}}{q^{3/2}}\right)^8$

c) $\dfrac{h^{2/3} \cdot h^{-3/4}}{h^{5/3}}$

d) $\left(\dfrac{m^{1/2}n^{-3}}{m^{3/5}n^{-5}}\right)^{-3/2}$

Answer: a) $x^{3/2}$ b) $\dfrac{p^{4/3}}{q^{12}}$

c) $\dfrac{1}{h^{7/4}}$ d) $\dfrac{m^{3/20}}{n^3}$

Solution

a) $r^{1/8} \cdot r^{3/8} = r^{\frac{1}{8} + \frac{3}{8}}$ Add exponents.

$= r^{4/8} = r^{1/2}$

b) $\left(\dfrac{x^{2/3}}{y^{1/4}}\right)^6 = \dfrac{x^{\frac{2}{3}\cdot 6}}{y^{\frac{1}{4}\cdot 6}}$ Multiply exponents.

$= \dfrac{x^4}{y^{3/2}}$ Reduce.

c) $\dfrac{n^{-5/6} \cdot n^{1/3}}{n^{-1/6}} = \dfrac{n^{-\frac{5}{6} + \frac{1}{3}}}{n^{-1/6}} = \dfrac{n^{-\frac{5}{6} + \frac{2}{6}}}{n^{-1/6}} = \dfrac{n^{-3/6}}{n^{-1/6}}$ Add exponents.

$= n^{-\frac{3}{6} - \left(-\frac{1}{6}\right)} = n^{-2/6} = n^{-1/3} = \dfrac{1}{n^{1/3}}$ Subtract exponents.

d) $\left(\dfrac{a^{-7}b^{1/2}}{a^5b^{1/3}}\right)^{-3/4} = \left(\dfrac{a^5b^{1/3}}{a^{-7}b^{1/2}}\right)^{3/4}$ Eliminate the negative from the outermost exponent by taking the reciprocal of the base.

Simplify the expression inside the parentheses by subtracting the exponents.

$$= (a^{5-(-7)}b^{1/3 - 1/2})^{3/4} = (a^{5+7}b^{2/6 - 3/6})^{3/4} = (a^{12}b^{-1/6})^{3/4}$$

Apply the power rule, and simplify.

$$= (a^{12})^{3/4}(b^{-1/6})^{3/4} = a^9b^{-1/8} = \frac{a^9}{b^{1/8}}$$

YOU TRY 5

Simplify completely. Assume the variables represent positive real numbers. The answer should contain only positive exponents.

a) $(a^3b^{1/5})^{10}$ b) $\dfrac{t^{3/10}}{t^{7/10}}$ c) $\dfrac{s^{3/4}}{s^{1/2} \cdot s^{-5/4}}$ d) $\left(\dfrac{x^4y^{3/8}}{x^9y^{1/4}}\right)^{-2/5}$

5 Convert a Radical Expression to Exponential Form and Simplify

Some radicals can be simplified by first putting them into rational exponent form and then converting them back to radicals.

EXAMPLE 6

In-Class Example 6

Follow the instructions for Example 6.

a) $\sqrt[10]{36^5}$ b) $\sqrt[8]{x^6}$

Answer: a) 6 b) $\sqrt[4]{x^3}$

Hint

In your notes, write an explanation in your own words that describes how to convert from radical form to exponential form. Include examples.

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form. Assume the variable represents a nonnegative real number.

a) $\sqrt[8]{9^4}$ b) $\sqrt[6]{s^4}$

Solution

a) Because the index of the radical is the denominator of the exponent and the power is the numerator, we can write

$$\sqrt[8]{9^4} = 9^{4/8} \quad \text{Write with a rational exponent.}$$
$$= 9^{1/2} = 3$$

b) $$\sqrt[6]{s^4} = s^{4/6} \quad \text{Write with a rational exponent.}$$
$$= s^{2/3} = \sqrt[3]{s^2}$$

The expression $\sqrt[6]{s^4}$ is not in simplest form because the 4 and the 6 contain a common factor of 2, but $\sqrt[3]{s^2}$ *is* in simplest form because 2 and 3 do not have any common factors besides 1.

YOU TRY 6

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form. Assume the variable represents a nonnegative real number.

a) $\sqrt[6]{125^2}$ b) $\sqrt[10]{p^4}$

In Section 9.1 we said that if a is negative and n is a positive, even number, then $\sqrt[n]{a^n} = |a|$. For example, if we are *not* told that k is positive, then $\sqrt{k^2} = |k|$. However, if we assume that k is positive, then $\sqrt{k^2} = k$. **In the rest of this chapter, we will assume that all variables represent positive, real numbers unless otherwise stated.** When we make this assumption, we do not need to use absolute values when simplifying even roots. And if we consider this together with the relationship between radicals and rational exponents we have another way to explain why $\sqrt[n]{a^n} = a$.

EXAMPLE 7

In-Class Example 7

Simplify.

a) $\sqrt[3]{11^3}$ b) $(\sqrt[4]{3})^4$ c) $\sqrt{p^2}$

Answer: a) 11 b) 3 c) p

Simplify.

a) $\sqrt[3]{5^3}$ b) $(\sqrt[4]{9})^4$ c) $\sqrt{k^2}$

Solution

a) $\sqrt[3]{5^3} = (5^3)^{1/3} = 5^{3 \cdot \frac{1}{3}} = 5^1 = 5$

b) $(\sqrt[4]{9})^4 = (9^{1/4})^4 = 9^{\frac{1}{4} \cdot 4} = 9^1 = 9$

c) $\sqrt{k^2} = (k^2)^{1/2} = k^{2 \cdot \frac{1}{2}} = k^1 = k$

YOU TRY 7

Simplify.

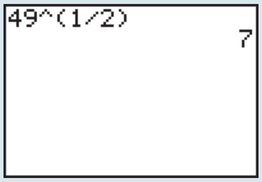
a) $(\sqrt{10})^2$ b) $\sqrt[3]{7^3}$ c) $\sqrt[4]{t^4}$

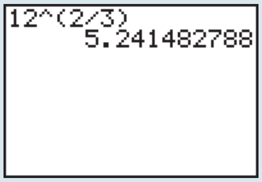
Using Technology

We can evaluate square roots, cube roots, or even higher roots by first rewriting the radical in exponential form and then using a graphing calculator.

For example, to evaluate $\sqrt{49}$, first rewrite the radical as $49^{1/2}$, then enter $\boxed{4}$ $\boxed{9}$, press $\boxed{\wedge}$ $\boxed{(}$, enter $\boxed{1}$ $\boxed{\div}$ $\boxed{2}$, and press $\boxed{)}$ $\boxed{\text{ENTER}}$. The result is 7, as shown on the screen on the left below.

To approximate $\sqrt[3]{12^2}$ rounded to the nearest tenth, first rewrite the radical as $12^{2/3}$, then enter $\boxed{1}$ $\boxed{2}$, press $\boxed{\wedge}$ $\boxed{(}$, enter $\boxed{2}$ $\boxed{\div}$ $\boxed{3}$, and press $\boxed{)}$ $\boxed{\text{ENTER}}$. The result is 5.241482788 as shown on the screen on the right below. The result rounded to the nearest tenth is then 5.2.





To evaluate radicals with an index greater than 3, follow the same procedure explained above. Evaluate by rewriting in exponential form if necessary and then using a graphing calculator. If necessary, approximate to the nearest tenth.

1) $16^{1/2}$ 2) $\sqrt[3]{512}$ 3) $\sqrt{37}$ 4) $361^{1/2}$ 5) $4096^{2/3}$ 6) $2401^{1/4}$

ANSWERS TO [YOU TRY] EXERCISES

1) a) 2 b) 11 c) 10 d) -3 e) not a real number f) -5 2) a) 4 b) -1000 c) not a real number d) 1 e) -1 3) a) $\frac{1}{12}$ b) $\frac{1}{8}$ c) $\frac{9}{4}$ 4) a) 2 b) 7 c) $\frac{1}{81}$ 5) a) $a^{30}b^2$ b) $\frac{1}{t^{2/5}}$ c) $s^{3/2}$ d) $\frac{x^2}{y^{1/20}}$ 6) a) 5 b) $\sqrt[5]{p^2}$ 7) a) 10 b) 7 c) t

ANSWERS TO TECHNOLOGY EXERCISES

1) 4 2) 8 3) 6.1 4) 19 5) 256 6) 7

E Evaluate 9.2 Exercises

Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Objective 1: Evaluate Expressions of the Form $a^{1/n}$

1) Explain how to write $25^{1/2}$ in radical form.

2) Explain how to write $1^{1/3}$ in radical form.

Write in radical form, and evaluate.

3) $9^{1/2}$ 3

4) $64^{1/2}$ 8

5) $1000^{1/3}$ 10

6) $27^{1/3}$ 3

7) $32^{1/5}$ 2

8) $81^{1/4}$ 3

9) $-64^{1/6}$ -2

10) $-125^{1/3}$ -5

11) $\left(\frac{4}{121}\right)^{1/2}$ $\frac{2}{11}$

12) $\left(\frac{4}{9}\right)^{1/2}$ $\frac{2}{3}$

13) $\left(\frac{125}{64}\right)^{1/3}$ $\frac{5}{4}$

14) $\left(\frac{16}{81}\right)^{1/4}$ $\frac{2}{3}$

15) $-\left(\frac{36}{169}\right)^{1/2}$ $-\frac{6}{13}$

16) $-\left(\frac{1000}{27}\right)^{1/3}$ $-\frac{10}{3}$

17) $(-81)^{1/4}$ not a real number

18) $(-169)^{1/2}$ not a real number

19) $(-1)^{1/7}$ -1

20) $(-8)^{1/3}$ -2

Objective 2: Evaluate Expressions of the Form $a^{m/n}$

21) Explain how to write $16^{3/4}$ in radical form.

22) Explain how to write $100^{3/2}$ in radical form.

Write in radical form, and evaluate.

23) $8^{4/3}$ 16

24) $81^{3/4}$ 27

25) $64^{5/6}$ 32

26) $32^{3/5}$ 8

27) $(-125)^{2/3}$ 25

28) $(-1000)^{2/3}$ 100

29) $-36^{3/2}$ -216

30) $-27^{4/3}$ -81

31) $(-81)^{3/4}$ not a real number

32) $(-25)^{3/2}$ not a real number

33) $\left(\frac{16}{81}\right)^{3/4}$ $\frac{8}{27}$

34) $-16^{5/4}$ -32

35) $-\left(\frac{1000}{27}\right)^{2/3}$ $-\frac{100}{9}$

36) $-\left(\frac{8}{27}\right)^{4/3}$ $-\frac{16}{81}$

Objective 3: Evaluate Expressions of the Form $a^{-m/n}$

Decide whether each statement is true or false. Explain your answer.

37) $81^{-1/2} = -9$

38) $\left(\frac{1}{100}\right)^{-3/2} = \left(\frac{1}{100}\right)^{2/3}$

Rewrite with a positive exponent, and evaluate.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

39) $64^{-1/2} = \left(\frac{1}{64}\right)^{1/2}$ The reciprocal of 64 is $\frac{1}{64}$.

$= \sqrt{\frac{1}{64}}$ The denominator of the fractional exponent is the index of the radical.

$= \frac{1}{8}$ Simplify.

40) $\left(\frac{1}{1000}\right)^{-1/3} = (1000)^{1/3}$ The reciprocal of $\frac{1}{1000}$ is 1000.

$= \sqrt[3]{1000}$ The denominator of the fractional exponent is the index of the radical.

$= 10$ Simplify.

41) $49^{-1/2}$ $\frac{1}{7}$

42) $100^{-1/2}$ $\frac{1}{10}$

43) $1000^{-1/3}$ $\frac{1}{10}$

44) $27^{-1/3}$ $\frac{1}{3}$

45) $\left(\frac{1}{81}\right)^{-1/4}$ 3

46) $\left(\frac{1}{32}\right)^{-1/5}$ 2

47) $-\left(\frac{1}{64}\right)^{-1/3}$ -4

48) $-\left(\frac{1}{125}\right)^{-1/3}$ -5

49) $64^{-5/6}$ $\frac{1}{32}$

50) $81^{-3/4}$ $\frac{1}{27}$

51) $125^{-2/3}$ $\frac{1}{25}$

52) $64^{-2/3}$ $\frac{1}{16}$

53) $\left(\frac{25}{4}\right)^{-3/2}$ $\frac{8}{125}$

54) $\left(\frac{9}{100}\right)^{-3/2}$ $\frac{1000}{27}$

55) $\left(\frac{64}{125}\right)^{-2/3}$ $\frac{25}{16}$

56) $\left(\frac{81}{16}\right)^{-3/4}$ $\frac{8}{27}$

Objective 4: Combine the Rules of Exponents

Simplify completely. The answer should contain only positive exponents.

57) $2^{2/3} \cdot 2^{7/3}$ 8

58) $5^{3/4} \cdot 5^{5/4}$ 25

59) $(9^{1/4})^2$ 3

60) $(7^{2/3})^3$ 49

61) $8^{7/5} \cdot 8^{-3/5}$ $8^{4/5}$

62) $6^{-4/3} \cdot 6^{5/3}$ $6^{1/3}$

63) $\frac{2^{23/4}}{2^{3/4}}$ 32

64) $\frac{5^{3/2}}{5^{9/2}}$ $\frac{1}{125}$

65) $\frac{4^{2/5}}{4^{6/5} \cdot 4^{3/5}}$ $\frac{1}{4^{7/5}}$

66) $\frac{6^{-1}}{6^{1/2} \cdot 6^{-5/2}}$ 6

Simplify completely. The answer should contain only positive exponents.

67) $z^{1/6} \cdot z^{5/6}$ z

68) $h^{1/6} \cdot h^{-3/4}$ $\frac{1}{h^{7/12}}$

69) $(-9v^{5/8})(8v^{3/4})$ $-72v^{11/8}$

70) $(-3x^{-1/3})(8x^{4/9})$ $-24x^{1/9}$

71) $\frac{a^{5/9}}{a^{4/9}}$ $a^{1/9}$

72) $\frac{x^{1/6}}{x^{5/6}}$ $\frac{1}{x^{2/3}}$

73) $\frac{48w^{3/10}}{10w^{2/5}}$ $\frac{24}{5w^{1/10}}$

74) $\frac{20c^{-2/3}}{72c^{5/6}}$ $\frac{5}{18c^{3/2}}$

75) $(x^{-2/9})^3$ $\frac{1}{x^{2/3}}$

76) $(n^{-2/7})^3$ $\frac{1}{n^{6/7}}$

77) $(z^{1/5})^{2/3}$ $z^{2/15}$

78) $(r^{4/3})^{5/2}$ $r^{10/3}$

79) $(81u^{8/3}v^4)^{3/4}$ $27u^2v^3$

80) $(64x^6y^{12/5})^{5/6}$ $32x^5y^2$

81) $(32r^{1/3}s^{4/9})^{3/5}$ $8r^{1/5}s^{4/15}$

82) $(125a^9b^{1/4})^{2/3}$ $25a^6b^{1/6}$

83) $\left(\frac{f^{6/7}}{27g^{-5/3}}\right)^{1/3}$ $\frac{f^{2/7}g^{5/9}}{3}$

84) $\left(\frac{16c^{-8}}{b^{-11/3}}\right)^{3/4}$ $\frac{8b^{11/4}}{c^6}$

85) $\left(\frac{x^{-5/3}}{w^{3/2}}\right)^{-6}$ $x^{10}w^9$

86) $\left(\frac{t^{-3/2}}{u^{1/4}}\right)^{-4}$ t^6u

87) $\frac{y^{1/2} \cdot y^{-1/3}}{y^{5/6}}$ $\frac{1}{y^{2/3}}$

88) $\frac{t^5}{t^{1/2} \cdot t^{3/4}}$ $t^{15/4}$

89) $\left(\frac{a^4b^3}{32a^{-2}b^4}\right)^{2/5}$ $\frac{a^{12/5}}{4b^{2/5}}$

90) $\left(\frac{16c^{-8}d^3}{c^4d^5}\right)^{3/2}$ $\frac{64}{c^{18}d^3}$

91) $\left(\frac{r^{4/5}t^{-2}}{r^{2/3}t^5}\right)^{-3/2}$ $\frac{t^{21/2}}{r^{1/5}}$

92) $\left(\frac{x^{10}y^{1/6}}{x^{-8}y^{2/3}}\right)^{-2/3}$ $\frac{y^{1/3}}{x^{12}}$

93) $\left(\frac{h^{-2}k^{5/2}}{h^{-8}k^{5/6}}\right)^{-5/6}$ $\frac{1}{h^5k^{25/18}}$

94) $\left(\frac{c^{1/8}d^{-4}}{c^{3/4}d}\right)^{-8/5}$ cd^8

95) $p^{1/2}(p^{2/3} + p^{1/2})$ $p^{7/6} + p$

96) $w^{4/3}(w^{1/2} - w^3)$ $w^{11/6} - w^{13/3}$

Objective 5: Convert a Radical Expression to Exponential Form and Simplify

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

97) $\sqrt[12]{25^6} = 25^{6/12}$ Write with a rational exponent.
$= 25^{1/2}$ Reduce the exponent.
$= 5$ Evaluate.

98) $\sqrt[10]{c^4} = c^{4/10}$ Write with a rational exponent.
$= c^{2/5}$ Reduce the exponent.
$= \sqrt[5]{c^2}$ Write in radical form.

99) $\sqrt[6]{49^3}$ 7

100) $\sqrt[9]{8^3}$ 2

101) $\sqrt[4]{81^2}$ 9

102) $\sqrt{3^2}$ 3

103) $(\sqrt{5})^2$ 5

104) $(\sqrt[3]{10})^3$ 10

105) $(\sqrt[3]{12})^3$ 12

106) $(\sqrt[4]{15})^4$ 15

107) $\sqrt[3]{x^{12}}$ x^4

108) $\sqrt[4]{t^8}$ t^2

109) $\sqrt[6]{k^2}$ $\sqrt[3]{k}$

110) $\sqrt[9]{w^6}$ $\sqrt[3]{w^2}$

111) $\sqrt[4]{z^2}$ $\sqrt{z}$

112) $\sqrt[8]{m^4}$ $\sqrt{m}$

113) $\sqrt{d^4}$ d^2

114) $\sqrt{s^6}$ s^3

The wind chill temperature, WC, measures how cold it feels outside (for temperatures under 50 degrees F) when the velocity of the wind, V, is considered along with the air temperature, T. The stronger the wind at a given air temperature, the colder it feels.

The formula for calculating wind chill is

$$\text{WC} = 35.74 + 0.6215T - 35.75V^{4/25} + 0.4275TV^{4/25}$$

where WC and T are in degrees Fahrenheit and V is in miles per hour. (http://www.nws.noaa.gov/om/windchill/windchillglossary.shtml)

Use this information for Exercises 115 and 116, and round all answers to the nearest degree.

115) Determine the wind chill when the air temperature is 20 degrees and the wind is blowing at the given speed.

a) 5 mph 13 degrees F

b) 15 mph 6 degrees F

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116) Determine the wind chill when the air temperature is 10 degrees and the wind is blowing at the given speed. Round your answer to the nearest degree.

a) 12 mph −5 degrees F

b) 20 mph −9 degrees F

Rethink

R1) Describe how having a good understanding of the rules of exponents has helped you do these exercises.

R2) What do you understand well, and what do you need to practice more? What can you do to increase your understanding?

9.3 Simplifying Expressions Containing Square Roots

P Prepare | **O Organize**

What are your objectives for Section 9.3?	How can you accomplish each objective?
1 Multiply Square Roots	• Write the definition of the *product rule for square roots* in your own words. • Complete the given example on your own. • Complete You Try 1.
2 Simplify the Square Root of a Whole Number	• Write the property for **When Is a Square Root Simplified?** • Complete the given example on your own. • Complete You Try 2.
3 Use the Quotient Rule for Square Roots	• Write the definition of the *quotient rule for square roots* in your own words. • Complete the given example on your own. • Complete You Try 3.
4 Simplify Square Root Expressions Containing Variables with Even Exponents	• Write the property for $\sqrt{a^m}$ in your notes. • Complete the given example on your own. • Complete You Try 4.
5 Simplify Square Root Expressions Containing Variables with Odd Exponents	• Write the procedure for **Simplifying a Radical Containing Variables** in your own words. • Complete the given examples on your own. • Complete You Trys 5–7.
6 Simplify More Square Root Expressions Containing Variables	• Apply the rules from previous objectives, and follow the example. • Complete You Try 8.

W Work

Read the explanations, follow the examples, take notes, and complete the You Trys.

In this section, we will introduce rules for finding the product and quotient of square roots as well as for simplifying expressions containing square roots.

1 Multiply Square Roots

Let's begin with the product $\sqrt{4} \cdot \sqrt{9}$. We can find the product like this: $\sqrt{4} \cdot \sqrt{9} = 2 \cdot 3 = 6$. Also notice that $\sqrt{4} \cdot \sqrt{9} = \sqrt{4 \cdot 9} = \sqrt{36} = 6$.

We obtain the same result. This leads us to the product rule for multiplying expressions containing square roots.

Definition Product Rule for Square Roots

Let a and b be nonnegative real numbers. Then,

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$$

In other words, the product of two square roots equals the square root of the product.

EXAMPLE 1

In-Class Example 1

Multiply.
a) $\sqrt{7} \cdot \sqrt{3}$ b) $\sqrt{5} \cdot \sqrt{g}$

Answer: a) $\sqrt{21}$ b) $\sqrt{5g}$

Multiply. a) $\sqrt{5} \cdot \sqrt{2}$ b) $\sqrt{3} \cdot \sqrt{x}$

Solution

a) $\sqrt{5} \cdot \sqrt{2} = \sqrt{5 \cdot 2} = \sqrt{10}$ b) $\sqrt{3} \cdot \sqrt{x} = \sqrt{3 \cdot x} = \sqrt{3x}$

BE CAREFUL

We can multiply radicals this way *only if* the indices are the same. We will see later how to multiply radicals with different indices such as $\sqrt{5} \cdot \sqrt[3]{t}$.

[YOU TRY 1]

Multiply.

a) $\sqrt{6} \cdot \sqrt{5}$ b) $\sqrt{10} \cdot \sqrt{r}$

2 Simplify the Square Root of a Whole Number

Knowing how to simplify radicals is very important in the study of algebra. We begin by discussing how to simplify expressions containing square roots.

How do we know when a square root is simplified?

Property When Is a Square Root Simplified?

An expression containing a square root is simplified when all of the following conditions are met:

1) The radicand does not contain any factors (other than 1) that are perfect squares.
2) The radicand does not contain any fractions.
3) There are no radicals in the denominator of a fraction.

Note: Condition 1) implies that the radical cannot contain variables with exponents greater than or equal to 2, the index of the square root.

We will discuss higher roots in Section 9.4.

To simplify expressions containing square roots we reverse the process of multiplying. That is, we use the product rule that says $\sqrt{a \cdot b} = \sqrt{a} \cdot \sqrt{b}$ where a or b are perfect squares.

EXAMPLE 2

Simplify completely.

a) $\sqrt{18}$ b) $\sqrt{500}$ c) $\sqrt{21}$ d) $\sqrt{48}$

In-Class Example 2

Simplify completely.
a) $\sqrt{45}$ b) $\sqrt{72}$
c) $\sqrt{12}$ d) $\sqrt{15}$

Answer: a) $3\sqrt{5}$ b) $6\sqrt{2}$
c) $2\sqrt{3}$ d) $\sqrt{15}$

Solution

a) The radical $\sqrt{18}$ is not in simplest form since 18 contains a factor (other than 1) that is a perfect square. Think of two numbers that multiply to 18 so that at least one of the numbers is a perfect square: $18 = 9 \cdot 2$.

(While it is true that $18 = 6 \cdot 3$, neither 6 nor 3 is a perfect square.)
Rewrite $\sqrt{18}$:

$$\begin{aligned} \sqrt{18} &= \sqrt{9 \cdot 2} && \text{9 is a perfect square.} \\ &= \sqrt{9} \cdot \sqrt{2} && \text{Product rule} \\ &= 3\sqrt{2} && \sqrt{9} = 3 \end{aligned}$$

$3\sqrt{2}$ is completely simplified because 2 does not have any factors that are perfect squares.

W Hint

What is the first question you should ask when simplifying a square root expression?

b) Does 500 have a factor that is a perfect square? *Yes!* $500 = 100 \cdot 5$. To simplify $\sqrt{500}$, rewrite it as

$$\begin{aligned} \sqrt{500} &= \sqrt{100 \cdot 5} && \text{100 is a perfect square.} \\ &= \sqrt{100} \cdot \sqrt{5} && \text{Product rule} \\ &= 10\sqrt{5} && \sqrt{100} = 10 \end{aligned}$$

$10\sqrt{5}$ is completely simplified because 5 does not have any factors that are perfect squares.

c) $21 = 3 \cdot 7$ Neither 3 nor 7 is a perfect square.
$21 = 1 \cdot 21$ Although 1 is a perfect square, it will not help us simplify $\sqrt{21}$.

$\sqrt{21}$ is in simplest form.

d) There are different ways to simplify $\sqrt{48}$. We will look at two of them.

i) Two numbers that multiply to 48 are 16 and 3 with 16 being a perfect square. We can write

$$\sqrt{48} = \sqrt{16 \cdot 3} = \sqrt{16} \cdot \sqrt{3} = 4\sqrt{3}$$

ii) We can also think of 48 as $4 \cdot 12$ since 4 is a perfect square. We can write

$$\sqrt{48} = \sqrt{4 \cdot 12} = \sqrt{4} \cdot \sqrt{12} = 2\sqrt{12}$$

Therefore, $\sqrt{48} = 2\sqrt{12}$. Is $\sqrt{12}$ in simplest form? *No, because* $12 = 4 \cdot 3$ *and 4 is a perfect square*. We must continue to simplify.

$$\begin{aligned} \sqrt{48} &= 2\sqrt{12} \\ &= 2\sqrt{4 \cdot 3} = 2\sqrt{4} \cdot \sqrt{3} = 2 \cdot 2 \cdot \sqrt{3} = 4\sqrt{3} \end{aligned}$$

$4\sqrt{3}$ is completely simplified because 3 does not have any factors that are perfect squares.

Example 2(d) shows that using either $\sqrt{48} = \sqrt{16 \cdot 3}$ or $\sqrt{48} = \sqrt{4 \cdot 12}$ leads us to the same result. Furthermore, this example illustrates that a radical is not always *completely* simplified after just one iteration of the simplification process. It is necessary to always examine the radical to determine whether or not it can be simplified more.

Note
After simplifying a radical, look at the result and ask yourself, *"Is the radical in simplest form?"* If it is not, simplify again. Asking yourself this question will help you to be sure that the radical *is* completely simplified.

YOU TRY 2

Simplify completely.

a) $\sqrt{28}$ b) $\sqrt{75}$ c) $\sqrt{35}$ d) $\sqrt{72}$

3 Use the Quotient Rule for Square Roots

Let's simplify $\frac{\sqrt{36}}{\sqrt{9}}$. We can say $\frac{\sqrt{36}}{\sqrt{9}} = \frac{6}{3} = 2$. It is also true that $\frac{\sqrt{36}}{\sqrt{9}} = \sqrt{\frac{36}{9}} = \sqrt{4} = 2$. This leads us to the quotient rule for dividing expressions containing square roots.

Definition Quotient Rule for Square Roots

Let a and b be nonnegative real numbers such that $b \neq 0$. Then,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

The square root of a quotient equals the quotient of the square roots.

EXAMPLE 3

In-Class Example 3

Simplify completely.

a) $\sqrt{\frac{8}{49}}$ b) $\sqrt{\frac{25}{5}}$ c) $\frac{\sqrt{120}}{\sqrt{10}}$ d) $\sqrt{\frac{5}{36}}$

Answer: a) $\frac{2\sqrt{2}}{7}$ b) $\sqrt{5}$ c) $2\sqrt{3}$ d) $\frac{\sqrt{5}}{6}$

Simplify completely.

a) $\sqrt{\frac{9}{49}}$ b) $\sqrt{\frac{200}{2}}$ c) $\frac{\sqrt{72}}{\sqrt{6}}$ d) $\sqrt{\frac{5}{81}}$

Solution

a) Because 9 and 49 are each perfect squares, find the square root of each separately.

$$\sqrt{\frac{9}{49}} = \frac{\sqrt{9}}{\sqrt{49}} \quad \text{Quotient rule}$$
$$= \frac{3}{7} \quad \sqrt{9} = 3 \text{ and } \sqrt{49} = 7$$

b) Neither 200 nor 2 is a perfect square, but if we simplify $\frac{200}{2}$ we get 100, which *is* a perfect square.

$$\sqrt{\frac{200}{2}} = \sqrt{100} \quad \text{Simplify } \frac{200}{2}.$$
$$= 10$$

Hint
In part c), which method do you prefer? Why?

c) We can simplify $\dfrac{\sqrt{72}}{\sqrt{6}}$ using two different methods.

i) Begin by applying the quotient rule to obtain a fraction under *one* radical and simplify the fraction.

$$\frac{\sqrt{72}}{\sqrt{6}} = \sqrt{\frac{72}{6}} \qquad \text{Quotient rule}$$
$$= \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$$

ii) We can apply the product rule to rewrite $\sqrt{72}$ then simplify the fraction.

$$\frac{\sqrt{72}}{\sqrt{6}} = \frac{\sqrt{6} \cdot \sqrt{12}}{\sqrt{6}} \qquad \text{Product rule}$$
$$= \frac{\overset{1}{\cancel{\sqrt{6}}} \cdot \sqrt{12}}{\underset{1}{\cancel{\sqrt{6}}}} \qquad \text{Divide out the common factor.}$$
$$= \sqrt{12} = \sqrt{4 \cdot 3} = \sqrt{4} \cdot \sqrt{3} = 2\sqrt{3}$$

Either method will produce the same result.

d) The fraction $\dfrac{5}{81}$ is in simplest form, and 81 *is* a perfect square. Begin by applying the quotient rule.

$$\sqrt{\frac{5}{81}} = \frac{\sqrt{5}}{\sqrt{81}} \qquad \text{Quotient rule}$$
$$= \frac{\sqrt{5}}{9} \qquad \sqrt{81} = 9$$

[YOU TRY 3]

Simplify completely.

a) $\sqrt{\dfrac{100}{169}}$ b) $\sqrt{\dfrac{27}{3}}$ c) $\dfrac{\sqrt{250}}{\sqrt{5}}$ d) $\sqrt{\dfrac{11}{36}}$

4 Simplify Square Root Expressions Containing Variables with Even Exponents

Recall that a square root is not simplified if it contains any factors that are perfect squares. *This means that a square root containing variables is simplified if the power on each variable is less than 2.* For example, $\sqrt{r^6}$ is not in simplified form. If r represents a nonnegative real number, then we can use rational exponents to simplify $\sqrt{r^6}$.

$$\sqrt{r^6} = (r^6)^{1/2} = r^{6 \cdot \frac{1}{2}} = r^{6/2} = r^3$$

Multiplying $6 \cdot \dfrac{1}{2}$ is the same as dividing 6 by 2. We can generalize this result with the following statement.

Property $\sqrt{a^m}$

If a is a nonnegative real number and m is an integer, then

$$\sqrt{a^m} = a^{m/2}$$

We can combine this property with the product and quotient rules to simplify radical expressions.

EXAMPLE 4

In-Class Example 4

Simplify completely.

a) $\sqrt{b^2}$ b) $\sqrt{100a^4}$

c) $\sqrt{24p^8}$ d) $\sqrt{\dfrac{27}{g^6}}$

Answer: a) b b) $10a^2$

c) $2p^4\sqrt{6}$ d) $\dfrac{3\sqrt{3}}{g^3}$

Hint

Are you writing out the examples as you are reading them?

Simplify completely.

a) $\sqrt{z^2}$ b) $\sqrt{49t^2}$ c) $\sqrt{18b^{14}}$ d) $\sqrt{\dfrac{32}{n^{20}}}$

Solution

a) $\sqrt{z^2} = z^{2/2} = z^1 = z$

b) $\sqrt{49t^2} = \sqrt{49} \cdot \sqrt{t^2} = 7 \cdot t^{2/2} = 7t$

c) $\sqrt{18b^{14}} = \sqrt{18} \cdot \sqrt{b^{14}}$ Product rule

$= \sqrt{9} \cdot \sqrt{2} \cdot b^{14/2}$ 9 is a perfect square.

$= 3\sqrt{2} \cdot b^7$ Simplify.

$= 3b^7\sqrt{2}$ Rewrite using the commutative property.

d) We begin by using the quotient rule.

$$\sqrt{\frac{32}{n^{20}}} = \frac{\sqrt{32}}{\sqrt{n^{20}}} = \frac{\sqrt{16} \cdot \sqrt{2}}{n^{20/2}} = \frac{4\sqrt{2}}{n^{10}}$$

[YOU TRY 4]

Simplify completely.

a) $\sqrt{y^2}$ b) $\sqrt{144p^{16}}$ c) $\sqrt{54c^{10}}$ d) $\sqrt{\dfrac{45}{w^4}}$

5 Simplify Square Root Expressions Containing Variables with Odd Exponents

How do we simplify an expression containing a square root if the power under the square root is odd? We can use the product rule for radicals and fractional exponents to help us understand how to simplify such expressions.

EXAMPLE 5

In-Class Example 5

Simplify completely.

a) $\sqrt{m^9}$ b) $\sqrt{t^{13}}$

Answer: a) $m^4\sqrt{m}$ b) $t^6\sqrt{t}$

Simplify completely.

a) $\sqrt{x^7}$ b) $\sqrt{c^{11}}$

Solution

a) To simplify $\sqrt{x^7}$, write x^7 as the product of two factors so that the exponent of one of the factors is the *largest* number less than 7 that is divisible by 2 (the index of the radical).

$\sqrt{x^7} = \sqrt{x^6 \cdot x^1}$ 6 is the largest number less than 7 that is divisible by 2.

$= \sqrt{x^6} \cdot \sqrt{x}$ Product rule

$= x^{6/2} \cdot \sqrt{x}$ Use a fractional exponent to simplify.

$= x^3\sqrt{x}$ $6 \div 2 = 3$

b) To simplify $\sqrt{c^{11}}$, write c^{11} as the product of two factors so that the exponent of one of the factors is the *largest* number less than 11 that is divisible by 2 (the index of the radical).

$\sqrt{c^{11}} = \sqrt{c^{10} \cdot c^1}$ 10 is the largest number less than 11 that is divisible by 2.

$= \sqrt{c^{10}} \cdot \sqrt{c}$ Product rule

$= c^{10/2} \cdot \sqrt{c}$ Use a fractional exponent to simplify.

$= c^5\sqrt{c}$ $10 \div 2 = 5$

YOU TRY 5

Simplify completely.

a) $\sqrt{m^5}$ b) $\sqrt{z^{19}}$

We used the product rule to simplify each radical in Example 5. During the simplification, however, we always divided an exponent by 2. This idea of division gives us another way to simplify radical expressions. Once again, let's look at the radicals and their simplified forms in Example 5 to see how we can simplify radical expressions using division.

$$\sqrt{x^7} = x^3\sqrt{x^1} = x^3\sqrt{x} \qquad \sqrt{c^{11}} = c^5\sqrt{c^1} = c^5\sqrt{c}$$

Index of radical → $2\overline{)\,7}$, 3 → Quotient, -6, 1 → Remainder

Index of radical → $2\overline{)\,11}$, 5 → Quotient, -10, 1 → Remainder

Hint

Write a sample expression that follows these steps in your notes.

Procedure Simplifying a Radical Containing Variables

To simplify a radical expression containing variables:

1) Divide the original exponent in the radicand by the index of the radical.
2) The exponent on the variable *outside* of the radical will be the *quotient* of the division problem.
3) The exponent on the variable *inside* of the radical will be the *remainder* of the division problem.

EXAMPLE 6

Simplify completely.

a) $\sqrt{t^9}$ b) $\sqrt{16b^5}$ c) $\sqrt{45y^{21}}$

In-Class Example 6

Simplify completely.

a) $\sqrt{k^5}$ b) $\sqrt{49v^3}$ c) $\sqrt{32m^{19}}$

Answer: a) $k^2\sqrt{k}$ b) $7v\sqrt{v}$ c) $4m^9\sqrt{2m}$

Solution

a) To simplify $\sqrt{t^9}$, divide: $2\overline{)\,9}$, 4 → Quotient, -8, 1 → Remainder

$$\sqrt{t^9} = t^4\sqrt{t^1} = t^4\sqrt{t}$$

b) $\sqrt{16b^5} = \sqrt{16}\cdot\sqrt{b^5}$ Product rule

$= 4\cdot b^2\sqrt{b^1}$ $5 \div 2$ gives a quotient of 2 and a remainder of 1.

$= 4b^2\sqrt{b}$

c) $\sqrt{45y^{21}} = \sqrt{45}\cdot\sqrt{y^{21}}$ Product rule

$= \sqrt{9}\cdot\sqrt{5}\cdot y^{10}\sqrt{y^1}$

Product rule; $21 \div 2$ gives a quotient of 10 and a remainder of 1.

$= 3\sqrt{5}\cdot y^{10}\sqrt{y}$ $\sqrt{9} = 3$

$= 3y^{10}\cdot\sqrt{5}\cdot\sqrt{y}$ Use the commutative property to rewrite the expression.

$= 3y^{10}\sqrt{5y}$ Use the product rule to write the expression with one radical.

Hint

Do you prefer the method in Example 5 or Example 6 for simplifying these types of expressions?

YOU TRY 6

Simplify completely.

a) $\sqrt{c^{13}}$ b) $\sqrt{100v^7}$ c) $\sqrt{32a^3}$

If a radical contains more than one variable, apply the product or quotient rule.

EXAMPLE 7

In-Class Example 7

Simplify completely.

a) $\sqrt{18m^7n^9}$ b) $\sqrt{\dfrac{3f^9}{g^{12}}}$

Answer: a) $3m^3n^4\sqrt{2mn}$ b) $\dfrac{f^4\sqrt{3f}}{g^6}$

Hint

It is easier to simplify expressions like these when you begin by writing them as the product or quotient of separate square roots.

Simplify completely.

a) $\sqrt{8a^{15}b^3}$ b) $\sqrt{\dfrac{5r^{27}}{s^8}}$

Solution

a) $\sqrt{8a^{15}b^3} = \sqrt{8} \cdot \sqrt{a^{15}} \cdot \sqrt{b^3}$

$= \sqrt{4} \cdot \sqrt{2} \cdot a^7\sqrt{a^1} \cdot b^1\sqrt{b^1}$

Product rule; $15 \div 2$ gives a quotient of 7 and a remainder of 1. $3 \div 2$ gives a quotient of 1 and a remainder of 1.

$= 2\sqrt{2} \cdot a^7\sqrt{a} \cdot b\sqrt{b}$ $\quad\sqrt{4} = 2$

$= 2a^7b \cdot \sqrt{2} \cdot \sqrt{a} \cdot \sqrt{b}$ Use the commutative property to rewrite the expression.

$= 2a^7b\sqrt{2ab}$ Use the product rule to write the expression with one radical.

b) $\sqrt{\dfrac{5r^{27}}{s^8}} = \dfrac{\sqrt{5r^{27}}}{\sqrt{s^8}}$ Quotient rule

$= \dfrac{\sqrt{5} \cdot \sqrt{r^{27}}}{s^4}$ ⟵ Product rule; ⟵ $8 \div 2 = 4$

$= \dfrac{\sqrt{5} \cdot r^{13}\sqrt{r^1}}{s^4}$ $27 \div 2$ gives a quotient of 13 and a remainder of 1.

$= \dfrac{r^{13} \cdot \sqrt{5} \cdot \sqrt{r}}{s^4}$ Use the commutative property to rewrite the expression.

$= \dfrac{r^{13}\sqrt{5r}}{s^4}$ Use the product rule to write the expression with one radical.

YOU TRY 7

Simplify completely.

a) $\sqrt{27x^{10}y^9}$ b) $\sqrt{\dfrac{40u^{13}}{v^{20}}}$

6 Simplify More Square Root Expressions Containing Variables

Next we will look at some examples of multiplying and dividing radical expressions that contain variables. Remember to always look at the result and ask yourself, *"Is the radical in simplest form?"* If it is not, simplify completely.

EXAMPLE 8

In-Class Example 8

Perform the indicated operation, and simplify completely.

a) $\sqrt{8b} \cdot \sqrt{2b}$

b) $\sqrt{3x^5y} \cdot \sqrt{6xy^4}$

c) $\dfrac{\sqrt{36r^7}}{\sqrt{2r^3}}$

Answer: a) $4b$ b) $3x^3y^2\sqrt{2y}$ c) $3r^2\sqrt{2}$

Perform the indicated operation, and simplify completely.

a) $\sqrt{6t} \cdot \sqrt{3t}$ b) $\sqrt{2a^3b} \cdot \sqrt{8a^2b^5}$ c) $\dfrac{\sqrt{20x^5}}{\sqrt{5x}}$

Solution

a) $\sqrt{6t} \cdot \sqrt{3t} = \sqrt{6t \cdot 3t}$ Product rule

$= \sqrt{18t^2}$

$= \sqrt{18} \cdot \sqrt{t^2}$ Product rule

$= \sqrt{9 \cdot 2} \cdot t = \sqrt{9} \cdot \sqrt{2} \cdot t = 3\sqrt{2} \cdot t = 3t\sqrt{2}$

Hint

In parts b) and c), think about which method you prefer and why.

b) $\sqrt{2a^3b} \cdot \sqrt{8a^2b^5}$

There are two good methods for multiplying these radicals.

i) Multiply the radicands to obtain one radical.

$$\sqrt{2a^3b} \cdot \sqrt{8a^2b^5} = \sqrt{2a^3b \cdot 8a^2b^5} \quad \text{Product rule}$$
$$= \sqrt{16a^5b^6} \quad \text{Multiply.}$$

Is the radical in simplest form? *No*.

$$= \sqrt{16} \cdot \sqrt{a^5} \cdot \sqrt{b^6} \quad \text{Product rule}$$
$$= 4 \cdot a^2\sqrt{a} \cdot b^3 \quad \text{Evaluate.}$$
$$= 4a^2b^3\sqrt{a} \quad \text{Commutative property}$$

ii) Simplify each radical, then multiply.

$$\sqrt{2a^3b} = \sqrt{2} \cdot \sqrt{a^3} \cdot \sqrt{b} \qquad \sqrt{8a^2b^5} = \sqrt{8} \cdot \sqrt{a^2} \cdot \sqrt{b^5}$$
$$= \sqrt{2} \cdot a\sqrt{a} \cdot \sqrt{b} \qquad = 2\sqrt{2} \cdot a \cdot b^2\sqrt{b}$$
$$= a\sqrt{2ab} \qquad = 2ab^2\sqrt{2b}$$

Then, $\sqrt{2a^3b} \cdot \sqrt{8a^2b^5} = a\sqrt{2ab} \cdot 2ab^2\sqrt{2b}$

$$= a \cdot 2ab^2 \cdot \sqrt{ab} \cdot \sqrt{2b} \quad \text{Commutative property}$$
$$= 2a^2b^2\sqrt{4ab^2} \quad \text{Multiply.}$$
$$= 2a^2b^2 \cdot 2 \cdot b \cdot \sqrt{a} \quad \sqrt{4ab^2} = 2b\sqrt{a}$$
$$= 4a^2b^3\sqrt{a} \quad \text{Multiply.}$$

Both methods give the same result.

c) We can use the quotient rule first or simplify first.

i) $$\frac{\sqrt{20x^5}}{\sqrt{5x}} = \sqrt{\frac{20x^5}{5x}} \quad \text{Use the quotient rule first.}$$
$$= \sqrt{4x^4} = \sqrt{4} \cdot \sqrt{x^4} = 2x^2$$

ii) $$\frac{\sqrt{20x^5}}{\sqrt{5x}} = \frac{\sqrt{20} \cdot \sqrt{x^5}}{\sqrt{5x}} \quad \text{Simplify first by using the product rule.}$$
$$= \frac{\sqrt{4} \cdot \sqrt{5} \cdot x^2\sqrt{x}}{\sqrt{5x}} \quad \text{Product rule; simplify } \sqrt{x^5}.$$
$$= \frac{2\sqrt{5} \cdot x^2\sqrt{x}}{\sqrt{5x}} \quad \sqrt{4} = 2$$
$$= \frac{2x^2\sqrt{5x}}{\sqrt{5x}} \quad \text{Product rule}$$
$$= 2x^2 \quad \text{Divide out the common factor.}$$

Both methods give the same result. In this case, the second method was longer. Sometimes, however, this method *can* be more efficient.

[YOU TRY 8]

Perform the indicated operation, and simplify completely.

a) $\sqrt{2n^3} \cdot \sqrt{6n}$ b) $\sqrt{15cd^5} \cdot \sqrt{3c^2d}$ c) $\dfrac{\sqrt{128k^9}}{\sqrt{2k}}$

ANSWERS TO [YOU TRY] EXERCISES

1) a) $\sqrt{30}$ b) $\sqrt{10r}$ 2) a) $2\sqrt{7}$ b) $5\sqrt{3}$ c) $\sqrt{35}$ d) $6\sqrt{2}$ 3) a) $\frac{10}{13}$ b) 3 c) $5\sqrt{2}$ d) $\frac{\sqrt{11}}{6}$ 4) a) y b) $12p^8$ c) $3c^5\sqrt{6}$ d) $\frac{3\sqrt{5}}{w^2}$ 5) a) $m^2\sqrt{m}$ b) $z^9\sqrt{z}$ 6) a) $c^6\sqrt{c}$ b) $10v^3\sqrt{v}$ c) $4a\sqrt{2a}$ 7) a) $3x^5y^4\sqrt{3y}$ b) $\frac{2u^6\sqrt{10u}}{v^{10}}$ 8) a) $2n^2\sqrt{3}$ b) $3cd^3\sqrt{5c}$ c) $8k^4$

Evaluate 9.3 Exercises

Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Unless otherwise stated, assume all variables represent nonnegative real numbers.

Objective 1: Multiply Square Roots

Multiply and simplify.

1) $\sqrt{3}\cdot\sqrt{7}$ $\sqrt{21}$ 2) $\sqrt{11}\cdot\sqrt{5}$ $\sqrt{55}$

3) $\sqrt{10}\cdot\sqrt{3}$ $\sqrt{30}$ 4) $\sqrt{7}\cdot\sqrt{2}$ $\sqrt{14}$

5) $\sqrt{6}\cdot\sqrt{y}$ $\sqrt{6y}$ 6) $\sqrt{5}\cdot\sqrt{p}$ $\sqrt{5p}$

Objective 2: Simplify the Square Root of a Whole Number

Label each statement as true or false. Give a reason for your answer.

7) $\sqrt{20}$ is in simplest form.
False; 20 contains a factor of 4 which is a perfect square.

8) $\sqrt{35}$ is in simplest form.

9) $\sqrt{42}$ is in simplest form.

10) $\sqrt{63}$ is in simplest form.
False; 63 contains a factor of 9 which is a perfect square.

Simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

11) $\sqrt{60} = \sqrt{4\cdot 15}$ Factor.
$= \sqrt{4}\cdot\sqrt{15}$ Product rule
$= 2\sqrt{15}$ Simplify.

12) $\sqrt{200} = \sqrt{100\cdot 2}$ Factor.
$= \sqrt{100}\cdot\sqrt{2}$ Product rule
$= 10\sqrt{2}$ Simplify.

Simplify completely. If the radical is already simplified, then say so.

13) $\sqrt{20}$ $2\sqrt{5}$ 14) $\sqrt{12}$ $2\sqrt{3}$

15) $\sqrt{54}$ $3\sqrt{6}$ 16) $\sqrt{63}$ $3\sqrt{7}$

17) $\sqrt{33}$ simplified 18) $\sqrt{15}$ simplified

19) $\sqrt{108}$ $6\sqrt{3}$ 20) $\sqrt{80}$ $4\sqrt{5}$

21) $\sqrt{98}$ $7\sqrt{2}$ 22) $\sqrt{96}$ $4\sqrt{6}$

23) $\sqrt{38}$ simplified 24) $\sqrt{46}$ simplified

25) $\sqrt{400}$ 20 26) $\sqrt{900}$ 30

27) $\sqrt{750}$ $5\sqrt{30}$ 28) $\sqrt{420}$ $2\sqrt{105}$

Objective 3: Use the Quotient Rule for Square Roots

Simplify completely.

29) $\sqrt{\frac{144}{25}}$ $\frac{12}{5}$ 30) $\sqrt{\frac{16}{81}}$ $\frac{4}{9}$

31) $\frac{\sqrt{4}}{\sqrt{49}}$ $\frac{2}{7}$ 32) $\frac{\sqrt{64}}{\sqrt{121}}$ $\frac{8}{11}$

33) $\frac{\sqrt{54}}{\sqrt{6}}$ 3 34) $\frac{\sqrt{48}}{\sqrt{3}}$ 4

35) $\sqrt{\frac{60}{5}}$ $2\sqrt{3}$ 36) $\sqrt{\frac{40}{5}}$ $2\sqrt{2}$

37) $\frac{\sqrt{120}}{\sqrt{6}}$ $2\sqrt{5}$ 38) $\frac{\sqrt{54}}{\sqrt{3}}$ $3\sqrt{2}$

39) $\frac{\sqrt{35}}{\sqrt{5}}$ $\sqrt{7}$ 40) $\frac{\sqrt{30}}{\sqrt{2}}$ $\sqrt{15}$

41) $\sqrt{\frac{6}{49}}$ $\frac{\sqrt{6}}{7}$ 42) $\sqrt{\frac{2}{81}}$ $\frac{\sqrt{2}}{9}$

43) $\sqrt{\frac{45}{16}}$ $\frac{3\sqrt{5}}{4}$ 44) $\sqrt{\frac{60}{49}}$ $\frac{2\sqrt{15}}{7}$

Objective 4: Simplify Square Root Expressions Containing Variables with Even Exponents

Simplify completely.

45) $\sqrt{x^8}$ x^4 46) $\sqrt{q^6}$ q^3

47) $\sqrt{w^{14}}$ w^7 48) $\sqrt{t^{16}}$ t^8

49) $\sqrt{100c^2}$ $10c$ 50) $\sqrt{9z^8}$ $3z^4$

51) $\sqrt{64k^6m^{10}}$ $8k^3m^5$

52) $\sqrt{25p^{20}q^{14}}$ $5p^{10}q^7$

53) $\sqrt{28r^4}$ $2r^2\sqrt{7}$

54) $\sqrt{27z^{12}}$ $3z^6\sqrt{3}$

55) $\sqrt{300q^{22}t^{16}}$ $10q^{11}t^8\sqrt{3}$

56) $\sqrt{50n^4y^4}$ $5n^2y^2\sqrt{2}$

57) $\sqrt{\dfrac{81}{c^6}}$ $\dfrac{9}{c^3}$

58) $\sqrt{\dfrac{h^2}{169}}$ $\dfrac{h}{13}$

59) $\dfrac{\sqrt{40}}{\sqrt{t^8}}$ $\dfrac{2\sqrt{10}}{t^4}$

60) $\dfrac{\sqrt{18}}{\sqrt{m^{30}}}$ $\dfrac{3\sqrt{2}}{m^{15}}$

61) $\sqrt{\dfrac{75x^2}{y^{12}}}$ $\dfrac{5x\sqrt{3}}{y^6}$

62) $\sqrt{\dfrac{44}{w^2z^{18}}}$ $\dfrac{2\sqrt{11}}{wz^9}$

Objective 5: Simplify Square Root Expressions Containing Variables with Odd Exponents

Simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

63) $\sqrt{w^9} = \sqrt{w^8 \cdot w^1}$ Factor.
$= \sqrt{w^8} \cdot \sqrt{w^1}$ Product rule
$= w^4\sqrt{w}$ Simplify.

64) $\sqrt{z^{19}} = \sqrt{z^{18} \cdot z^1}$ Factor.
$= \sqrt{z^{18}} \cdot \sqrt{z^1}$ Product rule
$= z^9\sqrt{z}$ Simplify.

65) $\sqrt{a^5}$ $a^2\sqrt{a}$

66) $\sqrt{c^7}$ $c^3\sqrt{c}$

67) $\sqrt{g^{13}}$ $g^6\sqrt{g}$

68) $\sqrt{k^{15}}$ $k^7\sqrt{k}$

69) $\sqrt{h^{31}}$ $h^{15}\sqrt{h}$

70) $\sqrt{b^{25}}$ $b^{12}\sqrt{b}$

71) $\sqrt{72x^3}$ $6x\sqrt{2x}$

72) $\sqrt{100a^5}$ $10a^2\sqrt{a}$

73) $\sqrt{13q^7}$ $q^3\sqrt{13q}$

74) $\sqrt{20c^9}$ $2c^4\sqrt{5c}$

75) $\sqrt{75t^{11}}$ $5t^5\sqrt{3t}$

76) $\sqrt{45p^{17}}$ $3p^8\sqrt{5p}$

77) $\sqrt{c^8d^2}$ c^4d

78) $\sqrt{r^4s^{12}}$ r^2s^6

79) $\sqrt{a^4b^3}$ $a^2b\sqrt{b}$

80) $\sqrt{x^2y^9}$ $xy^4\sqrt{y}$

81) $\sqrt{u^5v^7}$ $u^2v^3\sqrt{uv}$

82) $\sqrt{f^3g^9}$ $fg^4\sqrt{fg}$

83) $\sqrt{36m^9n^4}$ $6m^4n^2\sqrt{m}$

84) $\sqrt{4t^6u^5}$ $2t^3u^2\sqrt{u}$

85) $\sqrt{44x^{12}y^5}$ $2x^6y^2\sqrt{11y}$

86) $\sqrt{63c^7d^4}$ $3c^3d^2\sqrt{7c}$

87) $\sqrt{32t^5u^7}$ $4t^2u^3\sqrt{2tu}$

88) $\sqrt{125k^3l^9}$ $5kl^4\sqrt{5kl}$

89) $\sqrt{\dfrac{a^7}{81b^6}}$ $\dfrac{a^3\sqrt{a}}{9b^3}$

90) $\sqrt{\dfrac{x^5}{49y^6}}$ $\dfrac{x^2\sqrt{x}}{7y^3}$

91) $\sqrt{\dfrac{3r^9}{s^2}}$ $\dfrac{r^4\sqrt{3r}}{s}$

92) $\sqrt{\dfrac{17h^{11}}{k^8}}$ $\dfrac{h^5\sqrt{17h}}{k^4}$

Objective 6: Simplify More Square Root Expressions Containing Variables

Perform the indicated operation, and simplify. Assume all variables represent positive real numbers.

93) $\sqrt{5} \cdot \sqrt{10}$ $5\sqrt{2}$

94) $\sqrt{8} \cdot \sqrt{6}$ $4\sqrt{3}$

95) $\sqrt{21} \cdot \sqrt{3}$ $3\sqrt{7}$

96) $\sqrt{2} \cdot \sqrt{14}$ $2\sqrt{7}$

97) $\sqrt{w} \cdot \sqrt{w^5}$ w^3

98) $\sqrt{d^3} \cdot \sqrt{d^{11}}$ d^7

99) $\sqrt{n^3} \cdot \sqrt{n^4}$ $n^3\sqrt{n}$

100) $\sqrt{a^{10}} \cdot \sqrt{a^3}$ $a^6\sqrt{a}$

101) $\sqrt{2k} \cdot \sqrt{8k^5}$ $4k^3$

102) $\sqrt{5z^9} \cdot \sqrt{5z^3}$ $5z^6$

103) $\sqrt{5a^6b^5} \cdot \sqrt{10ab^4}$

104) $\sqrt{6x^4y^3} \cdot \sqrt{2x^5y^2}$

105) $\sqrt{8c^9d^2} \cdot \sqrt{5cd^7}$

106) $\sqrt{6t^3u^3} \cdot \sqrt{3t^7u^4}$

107) $\dfrac{\sqrt{18k^{11}}}{\sqrt{2k^3}}$ $3k^4$

108) $\dfrac{\sqrt{48m^{15}}}{\sqrt{3m^9}}$ $4m^3$

109) $\dfrac{\sqrt{120h^8}}{\sqrt{3h^2}}$ $2h^3\sqrt{10}$

110) $\dfrac{\sqrt{72c^{10}}}{\sqrt{6c^2}}$ $2c^4\sqrt{3}$

111) $\dfrac{\sqrt{50a^{16}b^9}}{\sqrt{5a^7b^4}}$ $a^4b^2\sqrt{10ab}$

112) $\dfrac{\sqrt{21y^8z^{18}}}{\sqrt{3yz^{13}}}$ $y^3z^2\sqrt{7yz}$

113) The velocity v of a moving object can be determined from its mass m and its kinetic energy KE using the formula $v = \sqrt{\dfrac{2\text{KE}}{m}}$, where the velocity is in meters/second, the mass is in kilograms, and the KE is measured in joules. A 600-kg roller coaster car is moving along a track and has kinetic energy of 120,000 joules. What is the velocity of the car? 20 m/sec

114) The length of a side s of an equilateral triangle is a function of its area A and can be described by $s(A) = \sqrt{\dfrac{4\sqrt{3}A}{3}}$. If an equilateral triangle has an area of $6\sqrt{3}$ cm^2, how long is each side of the triangle? $2\sqrt{6}$ cm

Rethink

R1) Explain how you know when a square root expression is simplified and when it is not.

R2) After completing the exercises, do you feel that you have learned the definitions, properties, and procedures by practicing them? Explain.

R3) Look at your homework and circle the types of problems that were difficult for you. Then, identify the step where you had trouble understanding what to do. What do these problems have in common?

9.4 Simplifying Expressions Containing Higher Roots

P Prepare | **O Organize**

What are your objectives for Section 9.4?	How can you accomplish each objective?
1 Multiply Higher Roots	• Write the *product rule for higher roots* in your own words. • Complete the given example on your own. • Complete You Try 1.
2 Simplify Higher Roots of Integers	• Write the property for **When Is a Radical Simplified?** in your own words. • Know the two different methods for simplifying higher roots. • Complete the given example on your own. • Complete You Try 2.
3 Use the Quotient Rule for Higher Roots	• Write the *quotient rule for higher roots* in your own words. • Complete the given example on your own. • Complete You Try 3.
4 Simplify Radicals Containing Variables	• Write the property for $\sqrt[n]{a^m}$ in your notes. • Complete the given examples on your own. • Complete You Trys 4–6.
5 Multiply and Divide Radicals with Different Indices	• Create a procedure by following Example 7, and write it in your notes. • Complete You Try 7.

W Work

Read the explanations, follow the examples, take notes, and complete the You Trys.

In Section 9.1 we first discussed finding higher roots like $\sqrt[4]{16} = 2$ and $\sqrt[3]{-27} = -3$. In this section, we will extend what we learned about multiplying, dividing, and simplifying *square* roots to doing the same with higher roots.

1 Multiply Higher Roots

Definition Product Rule for Higher Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, then

$$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$$

This rule enables us to multiply and simplify radicals with any index in a way that is similar to multiplying and simplifying square roots.

EXAMPLE 1

In-Class Example 1

Multiply.
a) $\sqrt[4]{3}\cdot\sqrt[4]{8}$ b) $\sqrt[3]{n}\cdot\sqrt[3]{21}$

Answer: a) $\sqrt[4]{24}$ b) $\sqrt[3]{21n}$

Multiply.

a) $\sqrt[3]{2}\cdot\sqrt[3]{7}$ b) $\sqrt[4]{t}\cdot\sqrt[4]{10}$

Solution

a) $\sqrt[3]{2}\cdot\sqrt[3]{7}=\sqrt[3]{2\cdot 7}=\sqrt[3]{14}$ b) $\sqrt[4]{t}\cdot\sqrt[4]{10}=\sqrt[4]{t\cdot 10}=\sqrt[4]{10t}$

[YOU TRY 1]

Multiply.

a) $\sqrt[4]{6}\cdot\sqrt[4]{5}$ b) $\sqrt[5]{8}\cdot\sqrt[5]{k^2}$

BE CAREFUL

Remember that we can apply the product rule in this way *only* if the indices of the radicals are the same. Later in this section we will discuss how to multiply radicals with different indices.

2 Simplify Higher Roots of Integers

In Section 9.3 we said that a simplified *square root* cannot contain any *perfect squares*. Next we list the conditions that determine when a radical with *any* index is in simplest form.

W Hint

Summarize this property in your notes.

Property When Is a Radical Simplified?

Let P be an expression and let n be an integer greater than 1. Then $\sqrt[n]{P}$ is completely simplified when all of the following conditions are met:

1) The radicand does not contain any factors (other than 1) that are perfect nth powers.
2) The exponents in the radicand and the index of the radical do not have any common factors (other than 1).
3) The radicand does not contain any fractions.
4) There are no radicals in the denominator of a fraction.

Note

Condition 1) implies that the radical cannot contain variables with exponents greater than or equal to n, the index of the radical.

To simplify radicals with any index, use the product rule $\sqrt[n]{a\cdot b}=\sqrt[n]{a}\cdot\sqrt[n]{b}$, where a or b is an nth power.

Remember, to be certain that a radical is simplified completely, always look at the radical carefully and ask yourself, *"Is the radical in simplest form?"*

EXAMPLE 2

Simplify completely.

a) $\sqrt[3]{250}$ b) $\sqrt[4]{48}$

In-Class Example 2

Simplify completely.
a) $\sqrt[3]{81}$ b) $\sqrt[4]{144}$

Answer: a) $3\sqrt[3]{3}$ b) $2\sqrt[4]{9}$

Solution

a) We will look at two methods for simplifying $\sqrt[3]{250}$.

i) Since we must simplify the *cube* root of 250, think of two numbers that multiply to 250 so that at least one of the numbers is a *perfect cube.*

$$250 = 125 \cdot 2$$

$$\begin{aligned} \sqrt[3]{250} &= \sqrt[3]{125 \cdot 2} && \text{125 is a perfect cube.} \\ &= \sqrt[3]{125} \cdot \sqrt[3]{2} && \text{Product rule} \\ &= 5\sqrt[3]{2} && \sqrt[3]{125} = 5 \end{aligned}$$

Is $5\sqrt[3]{2}$ *in simplest form?* Yes, because 2 does not have any factors that are perfect cubes.

ii) Use a factor tree to find the prime factorization of 250: $250 = 2 \cdot 5^3$.

$$\begin{aligned} \sqrt[3]{250} &= \sqrt[3]{2 \cdot 5^3} && 2 \cdot 5^3 \text{ is the prime factorization of 250.} \\ &= \sqrt[3]{2} \cdot \sqrt[3]{5^3} && \text{product rule} \\ &= \sqrt[3]{2} \cdot 5 && \sqrt[3]{5^3} = 5 \\ &= 5\sqrt[3]{2} && \text{Commutative property} \end{aligned}$$

We obtain the same result using either method.

Hint

In your notes and in your own words, explain how you know that a radical with any index is in simplest form.

b) We will use two methods for simplifying $\sqrt[4]{48}$.

i) Since we must simplify the *fourth* root of 48, think of two numbers that multiply to 48 so that at least one of the numbers is a *perfect fourth power.*

$$48 = 16 \cdot 3$$

$$\begin{aligned} \sqrt[4]{48} &= \sqrt[4]{16 \cdot 3} && \text{16 is a perfect fourth power.} \\ &= \sqrt[4]{16} \cdot \sqrt[4]{3} && \text{Product rule} \\ &= 2\sqrt[4]{3} && \sqrt[4]{16} = 2 \end{aligned}$$

Is $2\sqrt[4]{3}$ *in simplest form?* Yes, because 3 does not have any factors that are perfect fourth powers.

ii) Use a factor tree to find the prime factorization of 48: $48 = 2^4 \cdot 3$.

$$\begin{aligned} \sqrt[4]{48} &= \sqrt[4]{2^4 \cdot 3} && 2^4 \cdot 3 \text{ is the prime factorization of 48.} \\ &= \sqrt[4]{2^4} \cdot \sqrt[4]{3} && \text{Product rule} \\ &= 2\sqrt[4]{3} && \sqrt[4]{2^4} = 2 \end{aligned}$$

Once again, both methods give us the same result.

[YOU TRY 2]

Simplify completely.

a) $\sqrt[3]{40}$ b) $\sqrt[4]{96}$

3 Use the Quotient Rule for Higher Roots

Definition Quotient Rule for Higher Roots

If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$, and n is a natural number then

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

We apply the quotient rule when working with nth roots the same way we apply it when working with square roots.

EXAMPLE 3

In-Class Example 3

Simplify completely.

a) $\sqrt[4]{\frac{48}{3}}$ b) $\frac{\sqrt[3]{56}}{\sqrt[3]{2}}$

Answer: a) 2 b) $\sqrt[3]{28}$

W Hint

Make a list of perfect cubes and numbers raised to fourth and fifth powers.

Simplify completely.

a) $\sqrt[3]{-\frac{81}{3}}$ b) $\frac{\sqrt[3]{96}}{\sqrt[3]{2}}$

Solution

a) We can think of $-\frac{81}{3}$ as $\frac{-81}{3}$ or $\frac{81}{-3}$. Let's think of it as $\frac{-81}{3}$.

Neither -81 nor 3 is a perfect cube, but if we simplify $\frac{-81}{3}$ we get -27, which *is* a perfect cube.

$$\sqrt[3]{-\frac{81}{3}} = \sqrt[3]{-27} = -3$$

b) Let's begin by applying the quotient rule to obtain a fraction under *one* radical, then simplify the fraction.

$$\begin{aligned}
\frac{\sqrt[3]{96}}{\sqrt[3]{2}} &= \sqrt[3]{\frac{96}{2}} && \text{Quotient rule} \\
&= \sqrt[3]{48} && \text{Simplify } \frac{96}{2}. \\
&= \sqrt[3]{8 \cdot 6} && \text{8 is a perfect cube.} \\
&= \sqrt[3]{8} \cdot \sqrt[3]{6} && \text{Product rule} \\
&= 2\sqrt[3]{6} && \sqrt[3]{8} = 2
\end{aligned}$$

Is $2\sqrt[3]{6}$ *in simplest form?* Yes, because 6 does not have any factors that are perfect cubes.

[YOU TRY 3]

Simplify completely.

a) $\sqrt[5]{-\frac{5}{160}}$ b) $\frac{\sqrt[3]{162}}{\sqrt[3]{3}}$

4 Simplify Radicals Containing Variables

In Section 9.2 we discussed the relationship between radical notation and fractional exponents. Recall that

Property $\sqrt[n]{a^m}$

If a is a nonnegative number and m and n are integers such that $n > 1$, then

$$\sqrt[n]{a^m} = a^{m/n}.$$

That is, the index of the radical becomes the denominator of the fractional exponent, and the power in the radicand becomes the numerator of the fractional exponent.

This is the principle we use to simplify radicals with indices greater than 2.

EXAMPLE 4

In-Class Example 4

Simplify completely.

a) $\sqrt[4]{x^{12}}$ b) $\sqrt[3]{8m^9n^6}$

c) $\sqrt[6]{\dfrac{f^{12}}{g^{24}}}$

Answer: a) x^3 b) $2m^3n^2$

c) $\dfrac{f^2}{g^4}$

Simplify completely.

a) $\sqrt[3]{y^{15}}$ b) $\sqrt[4]{16t^{24}u^8}$ c) $\sqrt[5]{\dfrac{c^{10}}{d^{30}}}$

Solution

a) $\sqrt[3]{y^{15}} = y^{15/3} = y^5$

b) $\sqrt[4]{16t^{24}u^8} = \sqrt[4]{16} \cdot \sqrt[4]{t^{24}} \cdot \sqrt[4]{u^8}$ Product rule

$= 2 \cdot t^{24/4} \cdot u^{8/4}$ Write with rational exponents.

$= 2t^6u^2$ Simplify exponents.

c) $\sqrt[5]{\dfrac{c^{10}}{d^{30}}} = \dfrac{\sqrt[5]{c^{10}}}{\sqrt[5]{d^{30}}} = \dfrac{c^{10/5}}{d^{30/5}} = \dfrac{c^2}{d^6}$ Quotient rule

[YOU TRY 4]

Simplify completely.

a) $\sqrt[5]{p^{30}}$ b) $\sqrt[3]{a^3b^{21}}$ c) $\sqrt[4]{\dfrac{m^{12}}{16n^{20}}}$

To simplify a radical expression if the power in the radicand does not divide evenly by the index, we use the same methods we used in Section 9.3 for simplifying similar expressions with square roots. We can use the product rule or we can use the idea of quotient and remainder in a division problem.

EXAMPLE 5

In-Class Example 5

Simplify $\sqrt[4]{d^{31}}$ completely in two ways.

Answer: $d^7\sqrt[4]{d^3}$

Simplify $\sqrt[4]{x^{23}}$ completely in two ways: i) use the product rule and ii) divide the exponent by the index and use the quotient and remainder.

Solution

i) Using the product rule:

To simplify $\sqrt[4]{x^{23}}$, write x^{23} as the product of two factors so that the exponent of one of the factors is the *largest* number less than 23 that is divisible by 4 (the index).

$\sqrt[4]{x^{23}} = \sqrt[4]{x^{20} \cdot x^3}$ 20 is the largest number less than 23 that is divisible by 4.

$= \sqrt[4]{x^{20}} \cdot \sqrt[4]{x^3}$ Product rule

$= x^{20/4} \cdot \sqrt[4]{x^3}$ Use a fractional exponent to simplify.

$= x^5\sqrt[4]{x^3}$ $20 \div 4 = 5$

ii) Using the quotient and remainder:

To simplify $\sqrt[4]{x^{23}}$, divide

$$\begin{array}{r} 5 \leftarrow \text{Quotient} \\ 4\overline{)\,23} \\ -20 \\ \hline 3 \leftarrow \text{Remainder} \end{array}$$

Recall from our work with square roots in Section 9.3 that

i) the exponent on the variable *outside* of the radical will be the *quotient* of the division problem,

and

ii) the exponent on the variable *inside* of the radical will be the *remainder* of the division problem.

$$\sqrt[4]{x^{23}} = x^5\sqrt[4]{x^3}$$

Is $x^5\sqrt[4]{x^3}$ in simplest form? Yes, because the exponent inside of the radical is less than the index, and they contain no common factors other than 1.

W Hint

Be sure to do this example both ways. Which method do you prefer and why?

YOU TRY 5 Simplify $\sqrt[5]{r^{32}}$ completely using both methods shown in Example 5.

We can apply the product and quotient rules together with the methods in Example 5 to simplify certain radical expressions.

EXAMPLE 6

In-Class Example 6

Simplify $\sqrt[4]{32p^5q^8}$ completely.

Answer: $2pq^2\sqrt[4]{2p}$

Completely simplify $\sqrt[3]{56a^{16}b^8}$.

Solution

$$\sqrt[3]{56a^{16}b^8} = \sqrt[3]{56} \cdot \sqrt[3]{a^{16}} \cdot \sqrt[3]{b^8} \qquad \text{Product rule}$$

$$= \sqrt[3]{8} \cdot \sqrt[3]{7} \cdot a^5\sqrt[3]{a^1} \cdot b^2\sqrt[3]{b^2}$$

Product rule; $16 \div 3$ gives a quotient of 5 and a remainder of 1.; $8 \div 3$ gives a quotient of 2 and a remainder of 2.

$$= 2\sqrt[3]{7} \cdot a^5\sqrt[3]{a} \cdot b^2\sqrt[3]{b^2} \qquad \text{Simplify } \sqrt[3]{8}.$$

$$= 2a^5b^2 \cdot \sqrt[3]{7} \cdot \sqrt[3]{a} \cdot \sqrt[3]{b^2} \qquad \text{Use the commutative property to rewrite the expression.}$$

$$= 2a^5b^2\sqrt[3]{7ab^2} \qquad \text{Product rule}$$

YOU TRY 6 Simplify completely.

a) $\sqrt[4]{48x^{15}y^{22}}$ b) $\sqrt[3]{\dfrac{r^{19}}{27s^{12}}}$

5 Multiply and Divide Radicals with Different Indices

The product and quotient rules for radicals apply only when the radicals have the *same* indices. To multiply or divide radicals with *different* indices, we first change the radical expressions to rational exponent form.

EXAMPLE 7

In-Class Example 7

Perform the indicated operation, and write the answer in simplest radical form.

a) $\sqrt[4]{m^5} \cdot \sqrt{m}$ b) $\dfrac{\sqrt[4]{b^3}}{\sqrt[3]{b}}$

Answer: a) $m\sqrt[4]{m^3}$ b) $\sqrt[12]{b^5}$

W Hint

Write your own procedure in your notes.

Multiply the expressions, and write the answer in simplest radical form.

$$\sqrt[3]{x^2} \cdot \sqrt{x}$$

Solution

The indices of $\sqrt[3]{x^2}$ and $\sqrt{x}$ are different, so we *cannot* use the product rule right now. Rewrite each radical as a fractional exponent, use the product rule for *exponents*, then convert the answer back to radical form.

$$\sqrt[3]{x^2} \cdot \sqrt{x} = x^{2/3} \cdot x^{1/2} \qquad \text{Change radicals to fractional exponents.}$$

$$= x^{4/6} \cdot x^{3/6} \qquad \text{Get a common denominator to add exponents.}$$

$$= x^{\frac{4}{6}+\frac{3}{6}} = x^{7/6} \qquad \text{Add exponents.}$$

$$= \sqrt[6]{x^7} = x\sqrt[6]{x} \qquad \text{Rewrite in radical form, and simplify.}$$

YOU TRY 7 Perform the indicated operation, and write the answer in simplest radical form.

a) $\sqrt[4]{y} \cdot \sqrt[6]{y}$ b) $\dfrac{\sqrt[3]{c^2}}{\sqrt{c}}$

ANSWERS TO [YOU TRY] EXERCISES

1) a) $\sqrt[4]{30}$ b) $\sqrt[5]{8k^2}$ 2) a) $2\sqrt[3]{5}$ b) $2\sqrt[4]{6}$ 3) a) $-\frac{1}{2}$ b) $3\sqrt[3]{2}$ 4) a) p^6 b) ab^7 c) $\frac{m^3}{2n^5}$

5) $r^6\sqrt[5]{r^2}$ 6) a) $2x^3y^5\sqrt[4]{3x^3y^2}$ b) $\frac{r^6\sqrt[3]{r}}{3s^4}$ 7) a) $\sqrt[12]{y^5}$ b) $\sqrt[6]{c}$

Evaluate 9.4 Exercises

Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Mixed Exercises: Objectives 1–3

1) In your own words, explain the product rule for radicals. Answers may vary.

2) In your own words, explain the quotient rule for radicals. Answers may vary.

3) How do you know that a radical expression containing a cube root is completely simplified?

4) How do you know that a radical expression containing a fourth root is completely simplified?

Assume all variables represent positive real numbers.

Objective 1: Multiply Higher Roots

Multiply.

5) $\sqrt[5]{6}\cdot\sqrt[5]{2}$ $\sqrt[5]{12}$

6) $\sqrt[3]{5}\cdot\sqrt[3]{4}$ $\sqrt[3]{20}$

7) $\sqrt[5]{9}\cdot\sqrt[5]{m^2}$ $\sqrt[5]{9m^2}$

8) $\sqrt[4]{11}\cdot\sqrt[4]{h^3}$ $\sqrt[4]{11h^3}$

9) $\sqrt[3]{a^2}\cdot\sqrt[3]{b}$ $\sqrt[3]{a^2b}$

10) $\sqrt[5]{t^2}\cdot\sqrt[5]{u^4}$ $\sqrt[5]{t^2u^4}$

Mixed Exercises: Objectives 2 and 3

Simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

11) $\sqrt[3]{56} = \sqrt[3]{8\cdot 7}$ Factor.
$= \sqrt[3]{8}\cdot\sqrt[3]{7}$ Product rule
$= 2\sqrt[3]{7}$ Simplify.

12) $\sqrt[4]{80} = \sqrt[4]{16\cdot 5}$ Factor.
$= \sqrt[4]{16}\cdot\sqrt[4]{5}$ Product rule
$= 2\sqrt[4]{5}$ Simplify.

13) $\sqrt[3]{24}$ $2\sqrt[3]{3}$

14) $\sqrt[3]{48}$ $2\sqrt[3]{6}$

15) $\sqrt[4]{64}$ $2\sqrt[4]{4}$

16) $\sqrt[4]{32}$ $2\sqrt[4]{2}$

17) $\sqrt[3]{54}$ $3\sqrt[3]{2}$

18) $\sqrt[3]{88}$ $2\sqrt[3]{11}$

19) $\sqrt[3]{2000}$ $10\sqrt[3]{2}$

20) $\sqrt[3]{108}$ $3\sqrt[3]{4}$

21) $\sqrt[5]{64}$ $2\sqrt[5]{2}$

22) $\sqrt[4]{162}$ $3\sqrt[4]{2}$

23) $\sqrt[3]{\frac{1}{125}}$ $\frac{1}{5}$

24) $\sqrt[4]{\frac{1}{16}}$ $\frac{1}{2}$

25) $\sqrt[3]{-\frac{54}{2}}$ -3

26) $\sqrt[4]{\frac{48}{3}}$ 2

27) $\sqrt[3]{\frac{48}{\sqrt[3]{2}}}$ $2\sqrt[3]{3}$

28) $\frac{\sqrt[3]{500}}{\sqrt[3]{2}}$ $5\sqrt[3]{2}$

29) $\frac{\sqrt[4]{240}}{\sqrt[4]{3}}$ $2\sqrt[4]{5}$

30) $\frac{\sqrt[3]{8000}}{\sqrt[3]{4}}$ $10\sqrt[3]{2}$

Objective 4: Simplify Radicals Containing Variables

Simplify completely.

31) $\sqrt[3]{d^6}$ d^2

32) $\sqrt[3]{g^9}$ g^3

33) $\sqrt[4]{n^{20}}$ n^5

34) $\sqrt[4]{t^{36}}$ t^9

35) $\sqrt[5]{x^5y^{15}}$ xy^3

36) $\sqrt[6]{a^{12}b^6}$ a^2b

37) $\sqrt[3]{w^{14}}$ $w^4\sqrt[3]{w^2}$

38) $\sqrt[3]{b^{19}}$ $b^6\sqrt[3]{b}$

39) $\sqrt[4]{y^9}$ $y^2\sqrt[4]{y}$

40) $\sqrt[4]{m^7}$ $m\sqrt[4]{m^3}$

41) $\sqrt[3]{d^5}$ $d\sqrt[3]{d^2}$

42) $\sqrt[3]{c^{29}}$ $c^9\sqrt[3]{c^2}$

43) $\sqrt[3]{u^{10}v^{15}}$ $u^3v^5\sqrt[3]{u}$

44) $\sqrt[3]{x^9y^{16}}$ $x^3y^5\sqrt[3]{y}$

45) $\sqrt[3]{b^{16}c^5}$ $b^5c\sqrt[3]{bc^2}$

46) $\sqrt[4]{r^{15}s^9}$ $r^3s^2\sqrt[4]{r^3s}$

47) $\sqrt[4]{m^3n^{18}}$ $n^4\sqrt[4]{m^3n^2}$

48) $\sqrt[3]{a^{11}b}$ $a^3\sqrt[3]{a^2b}$

49) $\sqrt[3]{24x^{10}y^{12}}$ $2x^3y^4\sqrt[3]{3x}$

50) $\sqrt[3]{54y^{10}z^{24}}$ $3y^3z^8\sqrt[3]{2y}$

51) $\sqrt[3]{72t^{17}u^7}$ $2t^5u^2\sqrt[3]{9t^2u}$

52) $\sqrt[3]{250w^4x^{16}}$ $5wx^5\sqrt[3]{2wx}$

53) $\sqrt[4]{\frac{m^8}{81}}$ $\frac{m^2}{3}$

54) $\sqrt[4]{\frac{16}{x^{12}}}$ $\frac{2}{x^3}$

55) $\sqrt[5]{\frac{32a^{23}}{b^{15}}}$ $\frac{2a^4\sqrt[5]{a^3}}{b^3}$

56) $\sqrt[3]{\frac{h^{17}}{125k^{21}}}$ $\frac{h^5\sqrt[3]{h^2}}{5k^7}$

57) $\sqrt[4]{\frac{t^9}{81s^{24}}}$ $\frac{t^2\sqrt[4]{t}}{3s^6}$

58) $\sqrt[5]{\frac{32c^9}{d^{20}}}$ $\frac{2c\sqrt[5]{c^4}}{d^4}$

59) $\sqrt[3]{\frac{u^{28}}{v^3}}$ $\frac{u^9\sqrt[3]{u}}{v}$

60) $\sqrt[4]{\frac{m^{13}}{n^8}}$ $\frac{m^3\sqrt[4]{m}}{n^2}$

Perform the indicated operation, and simplify.

61) $\sqrt[3]{6} \cdot \sqrt[3]{4}$ $2\sqrt[3]{3}$

62) $\sqrt[3]{4} \cdot \sqrt[3]{10}$ $2\sqrt[3]{5}$

63) $\sqrt[3]{9} \cdot \sqrt[3]{12}$ $3\sqrt[3]{4}$

64) $\sqrt[3]{9} \cdot \sqrt[3]{6}$ $3\sqrt[3]{2}$

65) $\sqrt[3]{20} \cdot \sqrt[3]{4}$ $2\sqrt[3]{10}$

66) $\sqrt[3]{28} \cdot \sqrt[3]{2}$ $2\sqrt[3]{7}$

67) $\sqrt[3]{m^4} \cdot \sqrt[3]{m^5}$ m^3

68) $\sqrt[3]{t^5} \cdot \sqrt[3]{t}$ t^2

69) $\sqrt[4]{k^7} \cdot \sqrt[4]{k^9}$ k^4

70) $\sqrt[4]{a^9} \cdot \sqrt[4]{a^{11}}$ a^5

71) $\sqrt[3]{r^7} \cdot \sqrt[3]{r^4}$ $r^3\sqrt[3]{r^2}$

72) $\sqrt[3]{y^2} \cdot \sqrt[3]{y^{17}}$ $y^6\sqrt[3]{y}$

73) $\sqrt[5]{p^{14}} \cdot \sqrt[5]{p^9}$ $p^4\sqrt[5]{p^3}$

74) $\sqrt[5]{c^{17}} \cdot \sqrt[5]{c^9}$ $c^5\sqrt[5]{c}$

75) $\sqrt[3]{9z^{11}} \cdot \sqrt[3]{3z^8}$ $3z^6\sqrt[3]{z}$

76) $\sqrt[3]{2h^4} \cdot \sqrt[3]{4h^{16}}$ $2h^6\sqrt[6]{h^2}$

77) $\sqrt[3]{\dfrac{h^{14}}{h^2}}$ h^4

78) $\sqrt[3]{\dfrac{a^{20}}{a^{14}}}$ a^2

79) $\sqrt[3]{\dfrac{c^{11}}{c^4}}$ $c^2\sqrt[3]{c}$

80) $\sqrt[3]{\dfrac{z^{16}}{z^5}}$ $z^3\sqrt[3]{z^2}$

81) $\sqrt[4]{\dfrac{162d^{21}}{2d^2}}$ $3d^4\sqrt[4]{d^3}$

82) $\sqrt[4]{\dfrac{48t^{11}}{3t^6}}$ $2t\sqrt[4]{t}$

Objective 5: Multiply and Divide Radicals with Different Indices

The following radical expressions do not have the same indices. Perform the indicated operation, and write the answer in simplest radical form.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

83) $\sqrt{a} \cdot \sqrt[4]{a^3} = a^{1/2} \cdot a^{3/4}$	Change radicals to fractional exponents.
$= a^{2/4} \cdot a^{3/4}$	Rewrite exponents with a common denominator.
$= a^{5/4}$	Add exponents.
$= \sqrt[4]{a^5}$	Rewrite in radical form.
$= a\sqrt[4]{a}$	Simplify.

84) $\sqrt[5]{r^4} \cdot \sqrt[3]{r^2} = r^{4/5} \cdot r^{2/3}$	Change radicals to fractional exponents.
$= r^{12/15} \cdot r^{10/15}$	Rewrite exponents with a common denominator.
$= r^{22/15}$	Add exponents.
$= \sqrt[15]{r^{22}}$	Rewrite in radical form.
$= r\sqrt[15]{r^7}$	Simplify.

85) $\sqrt{p} \cdot \sqrt[3]{p}$ $\sqrt[6]{p^5}$

86) $\sqrt[3]{y^2} \cdot \sqrt[4]{y}$ $\sqrt[12]{y^{11}}$

87) $\sqrt[4]{n^3} \cdot \sqrt{n}$ $n\sqrt[4]{n}$

88) $\sqrt[5]{k^4} \cdot \sqrt{k}$ $k\sqrt[10]{k^3}$

89) $\sqrt[5]{c^3} \cdot \sqrt[3]{c^2}$ $c\sqrt[15]{c^4}$

90) $\sqrt[3]{a^2} \cdot \sqrt[5]{a^2}$ $a\sqrt[15]{a}$

91) $\dfrac{\sqrt{w}}{\sqrt[4]{w}}$ $\sqrt[4]{w}$

92) $\dfrac{\sqrt[4]{m^3}}{\sqrt{m}}$ $\sqrt[4]{m}$

93) $\dfrac{\sqrt[4]{h^3}}{\sqrt[3]{h^2}}$ $\sqrt[12]{h}$

94) $\dfrac{\sqrt[5]{t^4}}{\sqrt[3]{t^2}}$ $\sqrt[15]{t^2}$

95) A block of candle wax in the shape of a cube has a volume of 64 in^3. The length of a side of the block, s, is given by $s = \sqrt[3]{V}$, where V is the volume of the block of candle wax. How long is each side of the block? 4 in.

96) The radius $r(V)$ of a sphere is a function of its volume V and can be described by the function $r(V) = \sqrt[3]{\dfrac{3V}{4\pi}}$. If a spherical water tank has a volume of $\dfrac{256\pi}{3}$ ft^3, what is the radius of the tank? 4 ft

Rethink

R1) In Objective 2, you learned two methods for simplifying the nth root of a number: one uses the prime factorization of the radicand and the other involves writing the radicand as the product of two factors, one of which is a perfect nth power. Which method do you prefer and why?

R2) Explain the connection between the rules for simplifying square root expressions and the rules for simplifying higher root expressions.

9.5 Adding, Subtracting, and Multiplying Radicals

Prepare | **Organize**

What are your objectives for Section 9.5?	How can you accomplish each objective?
1 Add and Subtract Radical Expressions	• Write the definition of *like radicals* in your notes. • Write the property for **Adding and Subtracting Radicals** in your notes. • Complete the given examples on your own. • Complete You Trys 1 and 2.
2 Simplify Before Adding and Subtracting	• Write the procedure for **Adding and Subtracting Radicals** in your own words. • Complete the given example on your own. • Complete You Try 3.
3 Multiply a Binomial Containing Radical Expressions by a Monomial	• Write a procedure that outlines how to multiply in this objective. • Complete the given example on your own. • Complete You Try 4.
4 Multiply Radical Expressions Using FOIL	• Compare using FOIL in Chapter 6 to using it for binomials containing radicals, and note what is similar and what is different. • Complete the given example on your own. • Complete You Try 5.
5 Square a Binomial Containing Radical Expressions	• Review the formulas for squaring a binomial in Chapter 6, if necessary. • Complete the given example on your own. • Complete You Try 6.
6 Multiply Radical Expressions of the Form $(a + b)(a - b)$	• Use the same formula derived in Chapter 6. • Complete the given example on your own. • Complete You Try 7.

Work **Read the explanations, follow the examples, take notes, and complete the You Trys.**

Just as we can add and subtract like terms such as $4x + 6x = 10x$ we can add and subtract *like radicals* such as $4\sqrt{3} + 6\sqrt{3}$.

Definition

Like radicals have the same index and the same radicand.

Some examples of like radicals are

$$4\sqrt{3} \text{ and } 6\sqrt{3}, \quad -\sqrt[3]{5} \text{ and } 8\sqrt[3]{5}, \quad \sqrt{x} \text{ and } 7\sqrt{x}, \quad 2\sqrt[3]{a^2b} \text{ and } \sqrt[3]{a^2b}$$

In this section, assume all variables represent nonnegative real numbers.

1 Add and Subtract Radical Expressions

Hint
Where have you seen similar definitions and procedures before?

Property Adding and Subtracting Radicals

In order to add or subtract radicals, they must be *like* radicals.

We add and subtract like radicals in the same way we add and subtract like terms—add or subtract the "coefficients" of the radicals and multiply that result by the radical. We are using the distributive property when we combine like terms in this way.

EXAMPLE 1

In-Class Example 1

Perform the operations, and simplify.
a) $9x + 2x$ b) $9\sqrt{5} + 2\sqrt{5}$
c) $\sqrt[3]{2} - 4\sqrt[3]{2}$
d) $8\sqrt{3} + 2\sqrt{5}$

Answer: a) $11x$ b) $11\sqrt{5}$
c) $-3\sqrt[3]{2}$ d) $8\sqrt{3} + 2\sqrt{5}$

Hint
When adding or subtracting "like radicals," the radical does not change.

Perform the operations, and simplify.

a) $4x + 6x$ b) $4\sqrt{3} + 6\sqrt{3}$ c) $\sqrt[4]{5} - 9\sqrt[4]{5}$ d) $7\sqrt{2} + 4\sqrt{3}$

Solution

a) First notice that $4x$ and $6x$ are like terms. Therefore, they can be added.

$$\begin{aligned} 4x + 6x &= (4 + 6)x && \text{Distributive property} \\ &= 10x && \text{Simplify.} \end{aligned}$$

Or, we can say that by just adding the coefficients, $4x + 6x = 10x$.

b) Before attempting to add $4\sqrt{3}$ and $6\sqrt{3}$ we must be certain that they are like radicals. Since they *are* like, they can be added.

$$\begin{aligned} 4\sqrt{3} + 6\sqrt{3} &= (4 + 6)\sqrt{3} && \text{Distributive property} \\ &= 10\sqrt{3} && \text{Simplify.} \end{aligned}$$

Or, we can say that by just adding the coefficients of $\sqrt{3}$, we get $4\sqrt{3} + 6\sqrt{3} = 10\sqrt{3}$.

c) $\sqrt[4]{5} - 9\sqrt[4]{5} = 1\sqrt[4]{5} - 9\sqrt[4]{5} = (1 - 9)\sqrt[4]{5} = -8\sqrt[4]{5}$

d) The radicands in $7\sqrt{2} + 4\sqrt{3}$ are different, so these expressions cannot be combined.

[YOU TRY 1]

Perform the operations, and simplify.

a) $9c + 7c$ b) $9\sqrt{10} + 7\sqrt{10}$ c) $\sqrt[3]{4} - 6\sqrt[3]{4}$ d) $5\sqrt{6} - 2\sqrt{3}$

EXAMPLE 2

In-Class Example 2

Perform the operations, and simplify.
a) $\sqrt{2} - 4 + 16\sqrt{2} + 19$
b) $9\sqrt{y} + 2\sqrt[3]{y} - 3\sqrt{y} + 2\sqrt[3]{y}$

Answer: a) $15 + 17\sqrt{2}$
b) $6\sqrt{y} + 4\sqrt[3]{y}$

Perform the operations, and simplify. $6\sqrt{x} + 11\sqrt[3]{x} + 2\sqrt{x} - 6\sqrt[3]{x}$

Solution

Begin by noticing that there are *two* different types of radicals: $\sqrt{x}$ and $\sqrt[3]{x}$. Write the like radicals together.

$$\begin{aligned} 6\sqrt{x} + 11\sqrt[3]{x} + 2\sqrt{x} - 6\sqrt[3]{x} &= 6\sqrt{x} + 2\sqrt{x} + 11\sqrt[3]{x} - 6\sqrt[3]{x} && \text{Commutative property} \\ &= (6 + 2)\sqrt{x} + (11 - 6)\sqrt[3]{x} && \text{Distributive property} \\ &= 8\sqrt{x} + 5\sqrt[3]{x} \end{aligned}$$

Is $8\sqrt{x} + 5\sqrt[3]{x}$ in simplest form? *Yes.* The radicals are not like (they have different indices) so they cannot be combined further. Also, each radical, $\sqrt{x}$ and $\sqrt[3]{x}$, is in simplest form.

[YOU TRY 2]

Perform the operations, and simplify. $8\sqrt[3]{2n} - 3\sqrt{2n} + 5\sqrt{2n} + 5\sqrt[3]{2n}$

2 Simplify Before Adding and Subtracting

Sometimes it looks like two radicals cannot be added or subtracted. But if the radicals can be *simplified* and they turn out to be *like* radicals, then we can add or subtract them.

Procedure Adding and Subtracting Radicals

1) Write each radical expression in simplest form.
2) Combine like radicals.

EXAMPLE 3

Perform the operations, and simplify.

a) $8\sqrt{2}+3\sqrt{50}-\sqrt{45}$ b) $-7\sqrt[3]{40}+\sqrt[3]{5}$

c) $10\sqrt{8t}-9\sqrt{2t}$ d) $\sqrt[3]{xy^6}+\sqrt[3]{x^7}$

In-Class Example 3

Perform the operations, and simplify.

a) $3\sqrt{20}+2\sqrt{12}+6\sqrt{5}$
b) $\sqrt[4]{6}+4\sqrt[4]{96}$
c) $4\sqrt{50n}-6\sqrt{2n}$
d) $\sqrt[3]{x^3y}+\sqrt[3]{y}$

Answer: a) $12\sqrt{5}+4\sqrt{3}$
b) $9\sqrt[4]{6}$
c) $14\sqrt{2n}$
d) $(x+1)+\sqrt[3]{y}$

Solution

a) The radicals $8\sqrt{2}, 3\sqrt{50}$, and $\sqrt{45}$ are not like. The first radical is in simplest form, but $3\sqrt{50}$ and $\sqrt{45}$ should be simplified to determine if any of the radicals can be combined.

$$\begin{aligned}8\sqrt{2}+3\sqrt{50}-\sqrt{45} &= 8\sqrt{2}+3\sqrt{25\cdot 2}-\sqrt{9\cdot 5} && \text{Factor.}\\ &= 8\sqrt{2}+3\sqrt{25}\cdot\sqrt{2}-\sqrt{9}\cdot\sqrt{5} && \text{Product rule}\\ &= 8\sqrt{2}+3\cdot 5\cdot\sqrt{2}-3\sqrt{5} && \text{Simplify radicals.}\\ &= 8\sqrt{2}+15\sqrt{2}-3\sqrt{5} && \text{Multiply.}\\ &= 23\sqrt{2}-3\sqrt{5} && \text{Add like radicals.}\end{aligned}$$

Hint

Write neatly, and write out each step very carefully to avoid making mistakes. After you complete one step, stop, look at what you have done, and ask yourself, *"What can I do next?"*

b)
$$\begin{aligned}-7\sqrt[3]{40}+\sqrt[3]{5} &= -7\sqrt[3]{8\cdot 5}+\sqrt[3]{5} && \text{8 is a perfect cube.}\\ &= -7\sqrt[3]{8}\cdot\sqrt[3]{5}+\sqrt[3]{5} && \text{Product rule}\\ &= -7\cdot 2\cdot\sqrt[3]{5}+\sqrt[3]{5} && \sqrt[3]{8}=2\\ &= -14\sqrt[3]{5}+\sqrt[3]{5} && \text{Multiply.}\\ &= -13\sqrt[3]{5} && \text{Add like radicals.}\end{aligned}$$

c) The radical $\sqrt{2t}$ is simplified, but $\sqrt{8t}$ is not. We must simplify $\sqrt{8t}$:

$$\sqrt{8t}=\sqrt{8}\cdot\sqrt{t}=\sqrt{4}\cdot\sqrt{2}\cdot\sqrt{t}=2\sqrt{2}\cdot\sqrt{t}=2\sqrt{2t}$$

Substitute $2\sqrt{2t}$ for $\sqrt{8t}$ in the original expression.

$$\begin{aligned}10\sqrt{8t}-9\sqrt{2t} &= 10(2\sqrt{2t})-9\sqrt{2t} && \text{Substitute } 2\sqrt{2t} \text{ for } \sqrt{8t}.\\ &= 20\sqrt{2t}-9\sqrt{2t} && \text{Multiply.}\\ &= 11\sqrt{2t} && \text{Subtract.}\end{aligned}$$

d) Each radical in the expression $\sqrt[3]{xy^6}+\sqrt[3]{x^7}$ must be simplified.

$$\sqrt[3]{xy^6}=\sqrt[3]{x}\cdot\sqrt[3]{y^6}=\sqrt[3]{x}\cdot y^2=y^2\sqrt[3]{x}$$

$$\sqrt[3]{x^7}=x^2\sqrt[3]{x^1}$$ $7 \div 3$ gives a quotient of 2 and a remainder of 1.

$$\begin{aligned}\sqrt[3]{xy^6}+\sqrt[3]{x^7} &= y^2\sqrt[3]{x}+x^2\sqrt[3]{x} && \text{Substitute the simplified radicals in the original expression.}\\ &= (y^2+x^2)\sqrt[3]{x} && \text{Factor out } \sqrt[3]{x} \text{ from exch term.}\end{aligned}$$

In this problem we cannot *add* $y^2\sqrt[3]{x}+x^2\sqrt[3]{x}$ like we added radicals in previous examples, but we *can* factor out $\sqrt[3]{x}$.

$(y^2+x^2)\sqrt[3]{x}$ is the completely simplified form of the sum.

YOU TRY 3

Perform the operations, and simplify.

a) $7\sqrt{3}-\sqrt{12}$ b) $2\sqrt{63}-11\sqrt{28}+2\sqrt{21}$ c) $\sqrt[3]{54}+5\sqrt[3]{16}$

d) $2\sqrt{6k}+4\sqrt{54k}$ e) $\sqrt[4]{mn^{11}}+\sqrt[4]{81mn^{3}}$

In the rest of this section, we will learn how to simplify expressions that combine multiplication, addition, and subtraction of radicals.

3 Multiply a Binomial Containing Radical Expressions by a Monomial

EXAMPLE 4

In-Class Example 4

Multiply and simplify.
a) $3(\sqrt{7}+\sqrt{63})$
b) $\sqrt{3}(\sqrt{6}+\sqrt{5})$
c) $\sqrt{m}(3\sqrt{m}+\sqrt{18n})$

Answer: a) $12\sqrt{7}$
b) $3\sqrt{2}+\sqrt{15}$
c) $3m+3\sqrt{2mn}$

Multiply and simplify.

a) $4(\sqrt{5}-\sqrt{20})$ b) $\sqrt{2}(\sqrt{10}+\sqrt{15})$ c) $\sqrt{x}(\sqrt{x}+\sqrt{32y})$

Solution

a) Because $\sqrt{20}$ can be simplified, we will do that first.

$$\sqrt{20}=\sqrt{4\cdot 5}=\sqrt{4}\cdot\sqrt{5}=2\sqrt{5}$$

Substitute $2\sqrt{5}$ for $\sqrt{20}$ in the original expression.

$$\begin{aligned} 4(\sqrt{5}-\sqrt{20}) &= 4(\sqrt{5}-2\sqrt{5}) && \text{Substitute } 2\sqrt{5} \text{ for } \sqrt{20}.\\ &= 4(-\sqrt{5}) && \text{Subtract.}\\ &= -4\sqrt{5} && \text{Multiply.}\end{aligned}$$

Hint

Use a pen with multiple colors to perform the steps in different colors as it is done in the examples.

b) Neither $\sqrt{10}$ nor $\sqrt{15}$ can be simplified. Begin by applying the distributive property.

$$\begin{aligned} \sqrt{2}(\sqrt{10}+\sqrt{15}) &= \sqrt{2}\cdot\sqrt{10}+\sqrt{2}\cdot\sqrt{15} && \text{Distribute.}\\ &= \sqrt{20}+\sqrt{30} && \text{Product rule}\end{aligned}$$

Is $\sqrt{20}+\sqrt{30}$ in simplest form? *No.* $\sqrt{20}$ can be simplified.

$$=\sqrt{4\cdot 5}+\sqrt{30}=\sqrt{4}\cdot\sqrt{5}+\sqrt{30}=2\sqrt{5}+\sqrt{30}$$

c) Since $\sqrt{32y}$ can be simplified, we will do that first.

$$\sqrt{32y}=\sqrt{32}\cdot\sqrt{y}=\sqrt{16\cdot 2}\cdot\sqrt{y}=\sqrt{16}\cdot\sqrt{2}\cdot\sqrt{y}=4\sqrt{2y}$$

Substitute $4\sqrt{2y}$ for $\sqrt{32y}$ in the original expression.

$$\begin{aligned} \sqrt{x}(\sqrt{x}+\sqrt{32y}) &= \sqrt{x}(\sqrt{x}+4\sqrt{2y}) && \text{Substitute } 4\sqrt{2y} \text{ for } \sqrt{32y}.\\ &= \sqrt{x}\cdot\sqrt{x}+\sqrt{x}\cdot 4\sqrt{2y} && \text{Distribute.}\\ &= x+4\sqrt{2xy} && \text{Multiply.}\end{aligned}$$

YOU TRY 4

Multiply and simplify.

a) $6(\sqrt{75}+2\sqrt{3})$ b) $\sqrt{3}(\sqrt{3}+\sqrt{21})$ c) $\sqrt{c}(\sqrt{c^{3}}-\sqrt{100d})$

4 Multiply Radical Expressions Using FOIL

In Chapter 6, we first multiplied binomials using **FOIL** (**F**irst **O**uter **I**nner **L**ast).

$$\begin{aligned}(2x+3)(x+4) &= \underset{F}{2x\cdot x} + \underset{O}{2x\cdot 4} + \underset{I}{3\cdot x} + \underset{L}{3\cdot 4}\\ &= 2x^2 + 8x + 3x + 12\\ &= 2x^2 + 11x + 12\end{aligned}$$

We can multiply binomials containing radicals the same way.

EXAMPLE 5

In-Class Example 5

Multiply and simplify.
a) $(3+\sqrt{3})(6+\sqrt{3})$
b) $(4\sqrt{2}+\sqrt{5})(2\sqrt{2}-2\sqrt{5})$
c) $(\sqrt{5}+\sqrt{2t})(\sqrt{5}+3\sqrt{2t})$

Answer: a) $21+9\sqrt{3}$
b) $6-6\sqrt{10}$
c) $5+4\sqrt{10t}+6t$

W Hint

Multiplication of binomials containing radicals uses the same procedure as multiplying binomials in Chapter 6!

Multiply and simplify.

a) $(2+\sqrt{5})(4+\sqrt{5})$ b) $(2\sqrt{3}+\sqrt{2})(\sqrt{3}-5\sqrt{2})$

c) $(\sqrt{r}+\sqrt{3s})(\sqrt{r}+8\sqrt{3s})$

Solution

a) Since we must multiply two binomials, we will use FOIL.

$$\begin{aligned}(2+\sqrt{5})(4+\sqrt{5}) &= \underset{F}{2\cdot 4} + \underset{O}{2\cdot\sqrt{5}} + \underset{I}{4\cdot\sqrt{5}} + \underset{L}{\sqrt{5}\cdot\sqrt{5}}\\ &= 8 + 2\sqrt{5} + 4\sqrt{5} + 5 && \text{Multiply.}\\ &= 13+6\sqrt{5} && \text{Combine like terms.}\end{aligned}$$

b) $(2\sqrt{3}+\sqrt{2})(\sqrt{3}-5\sqrt{2})$

$$\begin{aligned}&= \underset{F}{2\sqrt{3}\cdot\sqrt{3}} + \underset{O}{2\sqrt{3}\cdot(-5\sqrt{2})} + \underset{I}{\sqrt{2}\cdot\sqrt{3}} + \underset{L}{\sqrt{2}\cdot(-5\sqrt{2})}\\ &= 2\cdot 3 + (-10\sqrt{6}) + \sqrt{6} + (-5\cdot 2) && \text{Multiply.}\\ &= 6 - 10\sqrt{6} + \sqrt{6} - 10 && \text{Multiply.}\\ &= -4-9\sqrt{6} && \text{Combine like termes.}\end{aligned}$$

c) $(\sqrt{r}+\sqrt{3s})(\sqrt{r}+8\sqrt{3s})$

$$\begin{aligned}&= \underset{F}{\sqrt{r}\cdot\sqrt{r}} + \underset{O}{\sqrt{r}\cdot 8\sqrt{3s}} + \underset{I}{\sqrt{3s}\cdot\sqrt{r}} + \underset{L}{\sqrt{3s}\cdot 8\sqrt{3s}}\\ &= r + 8\sqrt{3rs} + \sqrt{3rs} + 8\cdot 3s && \text{Multiply.}\\ &= r + 8\sqrt{3rs} + \sqrt{3rs} + 24s && \text{Multiply.}\\ &= r+9\sqrt{3rs}+24s && \text{Combine like terms.}\end{aligned}$$

[YOU TRY 5]

Multiply and simplify.

a) $(6-\sqrt{7})(5+\sqrt{7})$ b) $(\sqrt{2}+4\sqrt{5})(3\sqrt{2}+\sqrt{5})$

c) $(\sqrt{6p}-\sqrt{2q})(\sqrt{6p}-3\sqrt{2q})$

5 Square a Binomial Containing Radical Expressions

Recall, again, from Chapter 6, that we can use FOIL to square a binomial or we can use these special formulas:

$$(a+b)^2 = a^2+2ab+b^2 \qquad (a-b)^2 = a^2-2ab+b^2$$

For example,

$$(k+7)^2 = (k)^2 + 2(k)(7) + (7)^2 = k^2 + 14k + 49 \quad \text{and} \quad (2p-5)^2 = (2p)^2 - 2(2p)(5) + (5)^2 = 4p^2 - 20p + 25$$

To square a binomial containing radicals, we can either use FOIL or we can use the formulas above. Understanding how to use the formulas to square a binomial will make it easier to solve radical equations in Section 9.7.

EXAMPLE 6

In-Class Example 6

Multiply and simplify.
a) $(\sqrt{5}+2)^2$
b) $(2\sqrt{t}-7)^2$

Answer: a) $9+4\sqrt{5}$
b) $4t-28\sqrt{t}+49$

Multiply and simplify.

a) $(\sqrt{10}+3)^2$ b) $(2\sqrt{x}-6)^2$

Solution

a) Use $(a+b)^2 = a^2 + 2ab + b^2$.

$$\begin{aligned}(\sqrt{10}+3)^2 &= (\sqrt{10})^2 + 2(\sqrt{10})(3) + (3)^2 && \text{Substitute } \sqrt{10} \text{ for } a \text{ and 3 for } b.\\ &= 10 + 6\sqrt{10} + 9 && \text{Multiply.}\\ &= 19 + 6\sqrt{10} && \text{Combine like terms.}\end{aligned}$$

b) Use $(a-b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned}(2\sqrt{x}-6)^2 &= (2\sqrt{x})^2 - 2(2\sqrt{x})(6) + (6)^2 && \text{Substitute } 2\sqrt{x} \text{ for } a \text{ and 6 for } b.\\ &= (4 \cdot x) - (4\sqrt{x})(6) + 36 && \text{Multiply.}\\ &= 4x - 24\sqrt{x} + 36 && \text{Multiply.}\end{aligned}$$

Hint
Write out *every* step on your paper; you will be less likely to make mistakes.

YOU TRY 6

Multiply and simplify.

a) $(\sqrt{6}+5)^2$ b) $(3\sqrt{2}-4)^2$ c) $(\sqrt{w}+\sqrt{11})^2$

6 Multiply Radical Expressions of the Form $(a + b)(a - b)$

We will review one last rule from Chapter 6 on multiplying binomials. We will use this in Section 9.6 when we divide radicals.

$$(a+b)(a-b) = a^2 - b^2$$

For example, $(t+8)(t-8) = (t)^2 - (8)^2 = t^2 - 64$.

The same rule applies when we multiply binomials containing radicals.

EXAMPLE 7

In-Class Example 7

Multiply and simplify.
a) $(3+\sqrt{7})(3-\sqrt{7})$
b) $(\sqrt{5}-\sqrt{2})(\sqrt{5}+\sqrt{2})$
c) $(\sqrt{r}+\sqrt{5})(\sqrt{r}-\sqrt{5})$

Answer: a) 2 b) 3
c) $r-5$

Multiply and simplify $(2\sqrt{x}+\sqrt{y})(2\sqrt{x}-\sqrt{y})$.

Solution

Use $(a+b)(a-b) = a^2 - b^2$.

$$\begin{aligned}(2\sqrt{x}+\sqrt{y})(2\sqrt{x}-\sqrt{y}) &= (2\sqrt{x})^2 - (\sqrt{y})^2 && \text{Substitute } 2\sqrt{x} \text{ for } a \text{ and } \sqrt{y} \text{ for } b.\\ &= 4(x) - y && \text{Square each term.}\\ &= 4x - y && \text{Simplify.}\end{aligned}$$

Note
When we multiply expressions of the form $(a + b)(a - b)$ containing square roots, the radicals are eliminated. *This will always be true.*

YOU TRY 7

Multiply and simplify.

a) $(4 + \sqrt{10})(4 - \sqrt{10})$ b) $(\sqrt{5h} - 2\sqrt{k})(\sqrt{5h} + 2\sqrt{k})$

ANSWERS TO YOU TRY EXERCISES

1) a) $16c$ b) $16\sqrt{10}$ c) $-5\sqrt[3]{4}$ d) $5\sqrt{6} - 2\sqrt{3}$ 2) $13\sqrt[3]{2n} + 2\sqrt{2n}$ 3) a) $5\sqrt{3}$ b) $-16\sqrt{7} + 2\sqrt{21}$ c) $13\sqrt[3]{2}$ d) $14\sqrt{6k}$ e) $(n^2 + 3)\sqrt[4]{mn^3}$ 4) a) $42\sqrt{3}$ b) $3 + 3\sqrt{7}$ c) $c^2 - 10\sqrt{cd}$ 5) a) $23 + \sqrt{7}$ b) $26 + 13\sqrt{10}$ c) $6p - 8\sqrt{3pq} + 6q$ 6) a) $31 + 10\sqrt{6}$ b) $34 - 24\sqrt{2}$ c) $w + 2\sqrt{11w} + 11$ 7) a) 6 b) $5h - 4k$

E Evaluate 9.5 Exercises

Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Assume all variables represent nonnegative real numbers.

Objective 1: Add and Subtract Radical Expressions

1) How do you know if two radicals are *like* radicals?
They have the same index and the same radicand.

2) Are $5\sqrt{3}$ and $7\sqrt[3]{3}$ like radicals? Why or why not? No. The indices are different.

Perform the operations, and simplify.

3) $5\sqrt{2} + 9\sqrt{2}$ $14\sqrt{2}$

4) $11\sqrt{7} + 7\sqrt{7}$ $18\sqrt{7}$

5) $7\sqrt[3]{4} + 8\sqrt[3]{4}$ $15\sqrt[3]{4}$

6) $10\sqrt[3]{5} - 2\sqrt[3]{5}$ $8\sqrt[3]{5}$

7) $6 - \sqrt{13} + 5 - 2\sqrt{13}$ $11 - 3\sqrt{13}$

8) $-8 + 3\sqrt{6} - 4\sqrt{6} + 9$ $1 - \sqrt{6}$

9) $15\sqrt[3]{z^2} - 20\sqrt[3]{z^2}$ $-5\sqrt[3]{z^2}$

10) $7\sqrt[3]{p} - 4\sqrt[3]{p}$ $3\sqrt[3]{p}$

11) $2\sqrt[3]{n^2} + 9\sqrt[5]{n^2} - 11\sqrt[3]{n^2} + \sqrt[5]{n^2}$ $-9\sqrt[3]{n^2} + 10\sqrt[5]{n^2}$

12) $5\sqrt[4]{s} - 3\sqrt[3]{s} + 2\sqrt[3]{s} + 4\sqrt[4]{s}$ $9\sqrt[4]{s} - \sqrt[3]{s}$

13) $\sqrt{5c} - 8\sqrt{6c} + \sqrt{5c} + 6\sqrt{6c}$ $2\sqrt{5c} - 2\sqrt{6c}$

14) $10\sqrt{2m} + 6\sqrt{3m} - \sqrt{2m} + 8\sqrt{3m}$ $9\sqrt{2m} + 14\sqrt{3m}$

Objective 2: Simplify Before Adding and Subtracting

15) What are the steps for adding or subtracting radicals?

16) Is $6\sqrt{2} + \sqrt{10}$ in simplified form? Explain.
Yes. The radicals are simplified, and they are unlike.

Perform the operations, and simplify.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

17) $\sqrt{48} + \sqrt{3}$
$= \sqrt{16 \cdot 3} + \sqrt{3}$ Factor.
$= \sqrt{16} \cdot \sqrt{3} + \sqrt{3}$ Product rule
$= 4\sqrt{3} + \sqrt{3}$ Simplify.
$= 5\sqrt{3}$ Add like radicals.

18) $\sqrt{44} - 8\sqrt{11}$
$= \sqrt{4 \cdot 11} - 8\sqrt{11}$ Factor.
$= \sqrt{4} \cdot \sqrt{11} - 8\sqrt{11}$ Product rule
$= 2\sqrt{11} - 8\sqrt{11}$ Simplify.
$= -6\sqrt{11}$ Subtract like radicals.

19) $6\sqrt{3} - \sqrt{12}$ $4\sqrt{3}$

20) $\sqrt{45} + 4\sqrt{5}$ $7\sqrt{5}$

21) $\sqrt{32} - 3\sqrt{8}$ $-2\sqrt{2}$

22) $3\sqrt{24} + \sqrt{96}$ $10\sqrt{6}$

23) $\sqrt{12} + \sqrt{75} - \sqrt{3}$ $6\sqrt{3}$

24) $\sqrt{96} + \sqrt{24} - 5\sqrt{54}$ $-9\sqrt{6}$

25) $8\sqrt[3]{9} + \sqrt[3]{72}$ $10\sqrt[3]{9}$

26) $5\sqrt[3]{88} + 2\sqrt[3]{11}$ $12\sqrt[3]{11}$

27) $\sqrt[3]{6} - \sqrt[3]{48}$ $-\sqrt[3]{6}$

28) $11\sqrt[3]{16} + 7\sqrt[3]{2}$ $29\sqrt[3]{2}$

29) $6q\sqrt{q} + 7\sqrt{q^3}$ $13q\sqrt{q}$

30) $11\sqrt{m^3} + 8m\sqrt{m}$ $19m\sqrt{m}$

31) $4d^2\sqrt{d} - 24\sqrt{d^5}$ $-20d^2\sqrt{d}$

32) $16k^4\sqrt{k} - 13\sqrt{k^9}$ $3k^4\sqrt{k}$

33) $9t^3\sqrt[3]{t} - 5\sqrt[3]{t^{10}}$ $4t^3\sqrt[3]{t}$

34) $8r^4\sqrt[3]{r} - 16\sqrt[3]{r^{13}}$ $-8r^4\sqrt[3]{r}$

35) $5a\sqrt[4]{a^7} + \sqrt[4]{a^{11}}$ $6a^2\sqrt[4]{a^3}$

36) $-3\sqrt[4]{c^{11}} + 6c^2\sqrt[4]{c^3}$ $3c^2\sqrt[4]{c^3}$

37) $2\sqrt{8p} - 6\sqrt{2p}$ $-2\sqrt{2p}$

38) $4\sqrt{63t} + 6\sqrt{7t}$ $18\sqrt{7t}$

39) $7\sqrt[3]{81a^5} + 4a\sqrt[3]{3a^2}$ $25a\sqrt[3]{3a^2}$

40) $3\sqrt[3]{40x} - 12\sqrt[3]{5x}$ $-6\sqrt[3]{5x}$

41) $\sqrt{xy^3} + 3y\sqrt{xy}$ $4y\sqrt{xy}$

42) $5a\sqrt{ab} + 2\sqrt{a^3b}$ $7a\sqrt{ab}$

43) $6c^2\sqrt{8d^3} - 9d\sqrt{2c^4d}$ $3c^2d\sqrt{2d}$

44) $11v\sqrt{5u^3} - 2u\sqrt{45uv^2}$ $5uv\sqrt{5u}$

45) $8p^2q\sqrt[3]{11pq^2} + 3p^2\sqrt[3]{88pq^5}$ $14p^2q\sqrt[3]{11pq^2}$

46) $18a^5\sqrt[3]{7a^2b} + 2a^3\sqrt[3]{7a^8b}$ $20a^5\sqrt[3]{7a^2b}$

47) $15cd\sqrt[4]{9cd} - \sqrt[4]{9c^5d^5}$ $14cd\sqrt[4]{9cd}$

48) $7yz^2\sqrt[4]{11y^4z} + 3z\sqrt[4]{11y^8z^5}$ $10y^2z^2\sqrt[4]{11z}$

49) $\sqrt[3]{a^9b} - \sqrt[3]{b^7}$ $\sqrt[3]{b}(a^3 - b^2)$

50) $\sqrt[3]{c^8} + \sqrt[3]{c^2d^3}$ $\sqrt[3]{c^2}(c^2 + d)$

Objective 3: Multiply a Binomial Containing Radical Expressions by a Monomial

Multiply and simplify.

51) $3(x + 5)$ $3x + 15$

52) $8(k + 3)$ $8k + 24$

53) $7(\sqrt{6} + 2)$ $7\sqrt{6} + 14$

54) $5(4 - \sqrt{7})$ $20 - 5\sqrt{7}$

55) $\sqrt{10}(\sqrt{3} - 1)$ $\sqrt{30} - \sqrt{10}$

56) $\sqrt{2}(9 + \sqrt{11})$ $9\sqrt{2} + \sqrt{22}$

57) $-6(\sqrt{32} + \sqrt{2})$ $-30\sqrt{2}$

58) $10(\sqrt{12} - \sqrt{3})$ $10\sqrt{3}$

59) $4(\sqrt{45} - \sqrt{20})$ $4\sqrt{5}$

60) $-3(\sqrt{18} + \sqrt{50})$ $-24\sqrt{2}$

61) $\sqrt{5}(\sqrt{24} - \sqrt{54})$ $-\sqrt{30}$

62) $\sqrt{2}(\sqrt{20} + \sqrt{45})$ $5\sqrt{10}$

63) $\sqrt[4]{3}(5 - \sqrt[4]{27})$ $5\sqrt[4]{3} - 3$

64) $\sqrt[3]{4}(2\sqrt[3]{5} + 7\sqrt[3]{4})$ $2\sqrt[3]{20} + 14\sqrt[3]{2}$

65) $\sqrt{t}(\sqrt{t} - \sqrt{81u})$ $t - 9\sqrt{tu}$

66) $\sqrt{s}(\sqrt{12r} + \sqrt{7s})$ $2\sqrt{3rs} + s\sqrt{7}$

67) $\sqrt{2xy}(\sqrt{2y} - y\sqrt{x})$ $2y\sqrt{x} - xy\sqrt{2y}$

68) $\sqrt{ab}(\sqrt{5a} + \sqrt{27b})$ $a\sqrt{5b} + 3b\sqrt{3a}$

69) $\sqrt[3]{c^2}(\sqrt[3]{c^2} + \sqrt[3]{125cd})$ $c\sqrt[3]{c} + 5c\sqrt[3]{d}$

70) $\sqrt[5]{mn^2}(\sqrt[5]{2m^2n} - n\sqrt[5]{mn^2})$ $\sqrt[5]{2m^3n^4} - n^2\sqrt[5]{m^2}$

Mixed Exercises: Objectives 4–6

71) How are the problems *Multiply* $(x + 8)(x + 3)$ and *Multiply* $(3 + \sqrt{2})(1 + \sqrt{2})$ similar? What method can be used to multiply each of them?

72) How are the problems *Multiply* $(y - 5)^2$ and *Multiply* $(\sqrt{7} - 2)^2$ similar? What method can be used to multiply each of them?

73) What formula can be used to multiply $(5 + \sqrt{6})(5 - \sqrt{6})$? $(a + b)(a - b) = a^2 - b^2$

74) What happens to the radical terms whenever we multiply $(a + b)(a - b)$ where the binomials contain square roots? The radicals are eliminated.

Objective 4: Multiply Radical Expressions Using FOIL

Multiply and simplify.

75) $(p + 7)(p + 6)$ $p^2 + 13p + 42$

76) $(z - 8)(z + 2)$ $z^2 - 6z - 16$

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

77) $(6 + \sqrt{7})(2 + \sqrt{7})$

$= \underline{6 \cdot 2 + 6\sqrt{7} + 2\sqrt{7} + \sqrt{7} \cdot \sqrt{7}}$ Use FOIL.

$= 12 + 6\sqrt{7} + 2\sqrt{7} + 7$ Multiply.

$= \underline{19 + 8\sqrt{7}}$ Combine like terms.

78) $(3 + \sqrt{5})(1 + \sqrt{5})$

$= 3 \cdot 1 + 3\sqrt{5} + 1\sqrt{5} + \sqrt{5} \cdot \sqrt{8}$ Use FOIL.

$= \underline{3 + 3\sqrt{5} + \sqrt{5} + 5}$ Multiply.

$= \underline{8 + 4\sqrt{5}}$ Combine like terms.

79) $(\sqrt{2} + 8)(\sqrt{2} - 3)$ $-22 + 5\sqrt{2}$

80) $(\sqrt{6} - 7)(\sqrt{6} + 2)$ $-8 - 5\sqrt{6}$

81) $(4 + \sqrt{x})(10 + \sqrt{x})$ $x + 14\sqrt{x} + 40$

82) $(\sqrt{y} + 9)(\sqrt{y} + 8)$ $y + 17\sqrt{y} + 72$

83) $(5\sqrt{2} - \sqrt{3})(2\sqrt{3} - \sqrt{2})$ $-16 + 11\sqrt{6}$

84) $(\sqrt{5} - 4\sqrt{3})(2\sqrt{5} - \sqrt{3})$ $22 - 9\sqrt{15}$

85) $(5 + 2\sqrt{3})(\sqrt{7} + \sqrt{2})$ $5\sqrt{7} + 5\sqrt{2} + 2\sqrt{21} + 2\sqrt{6}$

86) $(\sqrt{5} + 4)(\sqrt{3} - 6\sqrt{2})$ $\sqrt{15} - 6\sqrt{10} + 4\sqrt{3} - 24\sqrt{2}$

87) $(\sqrt{7n} + 1)(\sqrt{3n} - 8)$ $n\sqrt{21} - 8\sqrt{7n} + \sqrt{3n} - 8$

88) $(\sqrt{2c} - 5)(\sqrt{11c} - 4)$ $c\sqrt{22} - 4\sqrt{2c} - 5\sqrt{11c} + 20$

89) $(\sqrt[3]{25} - 3)(\sqrt[3]{5} - \sqrt[3]{6})$ $5 - \sqrt[3]{150} - 3\sqrt[3]{5} + 3\sqrt[3]{6}$

90) $(\sqrt[4]{8} - \sqrt[4]{3})(\sqrt[4]{6} + \sqrt[4]{2})$ $2\sqrt[4]{3} + 2 - \sqrt[4]{18} - \sqrt[4]{6}$

91) $(\sqrt{6p} - 2\sqrt{q})(8\sqrt{q} + 5\sqrt{6p})$ $-2\sqrt{6pq} + 30p - 16q$

92) $(4\sqrt{3r} + \sqrt{s})(3\sqrt{s} - 2\sqrt{3r})$ $10\sqrt{3rs} + 3s - 24r$

Objective 5: Square a Binomial Containing Radical Expressions

93) $(\sqrt{3} + 1)^2$ $4 + 2\sqrt{3}$

94) $(2 + \sqrt{5})^2$ $9 + 4\sqrt{5}$

95) $(\sqrt{k} - 6)^2$ $k - 12\sqrt{k} + 36$

96) $(8 + \sqrt{t})^2$ $t + 16\sqrt{t} + 64$

97) $(1 + 4\sqrt{3v})^2$ $48v + 8\sqrt{3v} + 1$

98) $(5\sqrt{2z} - 3)^2$ $50z - 30\sqrt{2z} + 9$

99) $(\sqrt{11} - \sqrt{5})^2$ $16 - 2\sqrt{55}$

100) $(\sqrt{3} + \sqrt{13})^2$ $16 + 2\sqrt{39}$

101) $(\sqrt{h} + \sqrt{7})^2$ $h + 2\sqrt{7h} + 7$

102) $(\sqrt{m} + \sqrt{3})^2$ $m + 2\sqrt{3m} + 3$

103) $(\sqrt{x} - \sqrt{y})^2$ $x - 2\sqrt{xy} + y$

104) $(\sqrt{b} - \sqrt{a})^2$ $a - 2\sqrt{ab} + b$

Objective 6: Multiply Radical Expressions of the Form $(a + b)(a - b)$

105) $(c + 9)(c - 9)$ $c^2 - 81$

106) $(g - 7)(g + 7)$ $g^2 - 49$

107) $(6 - \sqrt{5})(6 + \sqrt{5})$ 31

108) $(4 - \sqrt{7})(4 + \sqrt{7})$ 9

109) $(\sqrt{p}+7)(\sqrt{p}-7)$ $p-49$

110) $(\sqrt{w}+10)(\sqrt{w}-10)$ $w-100$

111) $(4\sqrt{3}+\sqrt{2})(4\sqrt{3}-\sqrt{2})$ 46

112) $(2\sqrt{2}-2\sqrt{7})(2\sqrt{2}+2\sqrt{7})$ -20

113) $(\sqrt[3]{2}-3)(\sqrt[3]{2}+3)$ $\sqrt[3]{4}-9$

114) $(1+\sqrt[3]{6})(1-\sqrt[3]{6})$ $1-\sqrt[3]{36}$

115) $(\sqrt{c}+\sqrt{d})(\sqrt{c}-\sqrt{d})$ $c-d$

116) $(\sqrt{2y}+\sqrt{z})(\sqrt{2y}-\sqrt{z})$ $2y-z$

117) $(8\sqrt{f}-\sqrt{g})(8\sqrt{f}+\sqrt{g})$ $64f-g$

118) $(\sqrt{a}+3\sqrt{4b})(\sqrt{a}-3\sqrt{4b})$ $a-36b$

Extension

Multiply and simplify.

119) $(1+2\sqrt[3]{5})(1-2\sqrt[3]{5}+4\sqrt[3]{25})$ 41

120) $(3+\sqrt[3]{2})(9-3\sqrt[3]{2}+\sqrt[3]{4})$ 29

Let $f(x)=x^2$. Find each function value.

121) $f(\sqrt{7}+2)$ $11+4\sqrt{7}$

122) $f(5-\sqrt{6})$ $31-10\sqrt{6}$

123) $f(1-2\sqrt{3})$ $13-4\sqrt{3}$

124) $f(3\sqrt{2}+4)$ $34+24\sqrt{2}$

R Rethink

R1) Explain the difference between mastering the concepts from Chapter 6 to completing these exercises. How did it compare?

R2) Explain how you could substitute "x" for the radical $\sqrt{3}$ and "y" for the radical $\sqrt{2}$ in Exercise 111 to help you multiply.

R3) Which exercises did you understand well, and which were more difficult for you? Identify where you had difficulty doing a problem.

9.6 Dividing Radicals

P Prepare | **O Organize**

What are your objectives for Section 9.6?	How can you accomplish each objective?
1 Rationalize a Denominator: One Square Root	• Understand what *rationalizing a denominator* means. • Complete the given examples on your own, and develop a procedure for rationalizing a denominator that contains one square root. • Complete You Trys 1–3.
2 Rationalize a Denominator: One Higher Root	• Follow the explanation to understand the logic behind rationalizing a higher root. • Complete the given examples on your own, and develop a procedure for rationalizing a denominator that contains one higher root. • Complete You Trys 4 and 5.
3 Rationalize a Denominator Containing Two Terms	• Understand the definition of a *conjugate,* and write a few examples in your notes. • Recall the formula for $(a+b)(a-b)$, and know how it can help you multiply conjugates. • Learn the procedure for **Rationalizing a Denominator That Contains Two Terms.** • Complete the given examples on your own. • Complete You Trys 6 and 7.

(continued)

What are your objectives for Section 9.6?	How can you accomplish each objective?
4 Rationalize a Numerator	• Follow the same procedures developed for rationalizing the denominator, but focus instead on the numerator. • Complete the given example on your own. • Complete You Try 8.
5 Divide Out Common Factors from the Numerator and Denominator	• Know how to apply the properties of real numbers, especially factoring and distributing, to simplify. • Complete the given example on your own. • Complete You Try 9.

W Work

Read the explanations, follow the examples, take notes, and complete the You Trys.

It is generally agreed that a radical expression is *not* in simplest form if its denominator contains a radical. For example, $\frac{1}{\sqrt{3}}$ is not simplified, but an equivalent form, $\frac{\sqrt{3}}{3}$, is simplified.

Later we will show that $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$. The process of eliminating radicals from the denominator of an expression is called **rationalizing the denominator.** We will look at two types of rationalizing problems.

1) Rationalizing a denominator containing one term
2) Rationalizing a denominator containing two terms

To rationalize a denominator, we will use the fact that multiplying the numerator and denominator of a fraction by the same quantity results in an equivalent fraction:

$$\frac{2}{3} \cdot \frac{4}{4} = \frac{8}{12} \qquad \frac{2}{3} \text{ and } \frac{8}{12} \text{ are equivalent because } \frac{4}{4} = 1$$

We use the same idea to rationalize the denominator of a radical expression.

1 Rationalize a Denominator: One Square Root

The goal of rationalizing a denominator is to eliminate the radical from the denominator. With regard to square roots, recall that $\sqrt{a} \cdot \sqrt{a} = \sqrt{a^2} = a$ for $a \geq 0$. For example,

$$\sqrt{19} \cdot \sqrt{19} = \sqrt{(19)^2} = 19 \text{ and } \sqrt{t} \cdot \sqrt{t} = \sqrt{t^2} = t \ (t \geq 0)$$

We will use this property to rationalize the denominators of the following expressions.

EXAMPLE 1

Rationalize the denominator of each expression.

a) $\frac{1}{\sqrt{3}}$ b) $\frac{36}{\sqrt{18}}$ c) $\frac{5\sqrt{3}}{\sqrt{2}}$

In-Class Example 1

Rationalize the denominator of each expression.

a) $\frac{2}{\sqrt{5}}$ b) $\frac{4}{\sqrt{20}}$ c) $\frac{7\sqrt{11}}{\sqrt{6}}$

Answer: a) $\frac{2\sqrt{5}}{5}$ b) $\frac{2\sqrt{5}}{5}$ c) $\frac{7\sqrt{66}}{6}$

Solution

a) To eliminate the square root from the denominator of $\frac{1}{\sqrt{3}}$, ask yourself, "By what do I multiply $\sqrt{3}$ to get a *perfect square* under the square root?" The answer is $\sqrt{3}$ because $\sqrt{3}\cdot\sqrt{3} = \sqrt{3^2} = \sqrt{9} = 3$. Multiply by $\sqrt{3}$ in the numerator *and* denominator. (We are actually multiplying by 1.)

$$\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \underbrace{\frac{\sqrt{3}}{\sqrt{3}}}_{\text{Rationalize the denominator.}} = \frac{\sqrt{3}}{\sqrt{3^2}} = \frac{\sqrt{3}}{\sqrt{9}} = \frac{\sqrt{3}}{3}$$

BE CAREFUL

$\frac{\sqrt{3}}{3}$ is in simplest form. We cannot "simplify" terms inside and outside of the radical.

$$\frac{\sqrt{3}}{3} = \frac{\sqrt{\cancel{3}^{1}}}{\cancel{3}_{1}} = \sqrt{1} = 1 \qquad \textbf{Incorrect!}$$

Hint

Be sure you understand *why* you multiply by a particular radical to rationalize the denominator!

b) First, simplify the denominator of $\frac{36}{\sqrt{18}}$.

$$\frac{36}{\sqrt{18}} = \underbrace{\frac{36}{3\sqrt{2}}}_{\text{Simplify } \sqrt{18}.} = \underbrace{\frac{12}{\sqrt{2}}}_{\text{Simplify.}} = \frac{12}{\sqrt{2}} \cdot \underbrace{\frac{\sqrt{2}}{\sqrt{2}}}_{\text{Rationalize the denominator.}} = \frac{12\sqrt{2}}{2} = 6\sqrt{2}$$

c) To rationalize $\frac{5\sqrt{3}}{\sqrt{2}}$, multiply the numerator and denominator by $\sqrt{2}$.

$$\frac{5\sqrt{3}}{\sqrt{2}} = \frac{5\sqrt{3}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{6}}{2}$$

[YOU TRY 1]

Rationalize the denominator of each expression.

a) $\frac{1}{\sqrt{7}}$ b) $\frac{15}{\sqrt{27}}$ c) $\frac{9\sqrt{6}}{\sqrt{5}}$

Sometimes we will apply the quotient or product rule before rationalizing.

EXAMPLE 2

Simplify completely.

a) $\sqrt{\frac{3}{24}}$ b) $\sqrt{\frac{5}{14}} \cdot \sqrt{\frac{7}{3}}$

In-Class Example 2

Simplify completely.

a) $\sqrt{\frac{6}{20}}$ b) $\sqrt{\frac{3}{11}} \cdot \sqrt{\frac{22}{9}}$

Answer: a) $\frac{\sqrt{30}}{10}$ b) $\frac{\sqrt{6}}{3}$

Solution

a) Begin by simplifying the fraction $\frac{3}{24}$ under the radical.

$$\sqrt{\frac{3}{24}} = \sqrt{\frac{1}{8}} \qquad \text{Simplify.}$$

$$= \frac{\sqrt{1}}{\sqrt{8}} = \frac{1}{\sqrt{4}\cdot\sqrt{2}} = \frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2\cdot 2} = \frac{\sqrt{2}}{4}$$

b) Begin by using the product rule to multiply the radicands.

$$\sqrt{\frac{5}{14}} \cdot \sqrt{\frac{7}{3}} = \sqrt{\frac{5}{14} \cdot \frac{7}{3}} \qquad \text{Product rule}$$

Multiply the fractions under the radical.

$$= \sqrt{\frac{5}{\underset{2}{\cancel{14}}} \cdot \frac{\overset{1}{\cancel{7}}}{3}} = \sqrt{\frac{5}{6}} \qquad \text{Multiply.}$$

$$= \frac{\sqrt{5}}{\sqrt{6}} = \frac{\sqrt{5}}{\sqrt{6}} \cdot \frac{\sqrt{6}}{\sqrt{6}} = \frac{\sqrt{30}}{6}$$

YOU TRY 2

Simplify completely.

a) $\sqrt{\frac{10}{35}}$ b) $\sqrt{\frac{21}{10}} \cdot \sqrt{\frac{2}{7}}$

We work with radical expressions containing variables the same way. **In the rest of this section, we will assume that all variables represent positive real numbers.**

EXAMPLE 3

Simplify completely.

a) $\frac{2}{\sqrt{x}}$ b) $\sqrt{\frac{12m^3}{7n}}$ c) $\sqrt{\frac{6cd^2}{cd^3}}$

In-Class Example 3

Simplify completely.

a) $\frac{7}{\sqrt{m}}$ b) $\sqrt{\frac{27p^4}{q^3}}$

c) $\sqrt{\frac{10xy}{x^4y^2}}$

Answer: a) $\frac{7\sqrt{m}}{m}$

b) $\frac{3p^2\sqrt{3q}}{q^2}$ c) $\frac{\sqrt{10xy}}{x^2y}$

W Hint

Ask yourself the questions found in parts a) and b). This will help you understand the next objective too.

Solution

a) Ask yourself, "By what do I multiply $\sqrt{x}$ to get a *perfect square* under the square root?" The perfect square we want to get is $\sqrt{x^2}$.

$$\sqrt{x} \cdot \sqrt{?} = \sqrt{x^2} = x$$

$$\sqrt{x} \cdot \sqrt{x} = \sqrt{x^2} = x$$

$$\frac{2}{\sqrt{x}} = \frac{2}{\sqrt{x}} \cdot \frac{\sqrt{x}}{\sqrt{x}} = \frac{2\sqrt{x}}{\sqrt{x^2}} = \frac{2\sqrt{x}}{x}$$

Rationalize the denominator.

b) Before rationalizing, apply the quotient rule and simplify the numerator.

$$\sqrt{\frac{12m^3}{7n}} = \frac{\sqrt{12m^3}}{\sqrt{7n}} = \frac{2m\sqrt{3m}}{\sqrt{7n}}$$

Rationalize the denominator. "By what do I multiply $\sqrt{7n}$ to get a *perfect square* under the square root?" The perfect square we want to get is $\sqrt{7^2n^2}$ or $\sqrt{49n^2}$.

$$\sqrt{7n} \cdot \sqrt{?} = \sqrt{7^2n^2} = 7n$$

$$\sqrt{7n} \cdot \sqrt{7n} = \sqrt{7^2n^2} = 7n$$

$$\sqrt{\frac{12m^3}{7n}} = \frac{2m\sqrt{3m}}{\sqrt{7n}}$$

$$= \frac{2m\sqrt{3m}}{\sqrt{7n}} \cdot \frac{\sqrt{7n}}{\sqrt{7n}} = \frac{2m\sqrt{21mn}}{7n}$$

Rationalize the denominator.

c) $\sqrt{\dfrac{6cd^2}{cd^3}} = \sqrt{\dfrac{6}{d}}$ Simplify the radicand using the quotient rule for exponents.

$$= \frac{\sqrt{6}}{\sqrt{d}} = \frac{\sqrt{6}}{\sqrt{d}} \cdot \frac{\sqrt{d}}{\sqrt{d}} = \frac{\sqrt{6d}}{d}$$

YOU TRY 3

Simplify completely.

a) $\dfrac{5}{\sqrt{p}}$ b) $\dfrac{\sqrt{18k^5}}{\sqrt{10m}}$ c) $\sqrt{\dfrac{20r^3s}{s^2}}$

2 Rationalize a Denominator: One Higher Root

Many students assume that to rationalize *all* denominators we simply multiply the numerator and denominator of the expression by the denominator as in $\dfrac{4}{\sqrt{3}} = \dfrac{4}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{4\sqrt{3}}{3}$. *We will see, however, why this reasoning is incorrect.*

To rationalize an expression like $\dfrac{4}{\sqrt{3}}$ we asked ourselves, "By what do I multiply $\sqrt{3}$ to get a *perfect square* under the *square root*?"

To rationalize an expression like $\dfrac{5}{\sqrt[3]{2}}$ we must ask ourselves, "By what do I multiply $\sqrt[3]{2}$ to get a *perfect cube* under the *cube root*?" The perfect cube we want is 2^3 (since we began with 2) so that $\sqrt[3]{2} \cdot \sqrt[3]{2^2} = \sqrt[3]{2^3} = 2$.

We will practice some fill-in-the-blank problems to eliminate radicals before we move on to rationalizing.

EXAMPLE 4

In-Class Example 4

Fill in the blank.
a) $\sqrt[3]{6} \cdot \sqrt[3]{?} = \sqrt[3]{6^3} = 6$
b) $\sqrt[3]{49} \cdot \sqrt[3]{?} = \sqrt[3]{7^3} = 7$
c) $\sqrt[3]{?} \cdot \sqrt[3]{m} = \sqrt[3]{m^3} = m$
d) $\sqrt[4]{4} \cdot \sqrt[4]{?} = \sqrt[4]{2^4} = 2$
e) $\sqrt[5]{4} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$

Answer: a) 6^2 b) 7 c) m^2 d) 2^2 e) 2^3

Hint
Get in the habit of asking yourself these questions!

Fill in the blank.

a) $\sqrt[3]{5} \cdot \sqrt[3]{?} = \sqrt[3]{5^3} = 5$
b) $\sqrt[3]{3} \cdot \sqrt[3]{?} = \sqrt[3]{3^3} = 3$
c) $\sqrt[3]{x^2} \cdot \sqrt[3]{?} = \sqrt[3]{x^3} = x$
d) $\sqrt[5]{8} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$
e) $\sqrt[4]{27} \cdot \sqrt[4]{?} = \sqrt[4]{3^4} = 3$

Solution

a) Ask yourself, "By what do I multiply $\sqrt[3]{5}$ to get $\sqrt[3]{5^3}$?" The answer is $\sqrt[3]{5^2}$.

$$\sqrt[3]{5} \cdot \sqrt[3]{?} = \sqrt[3]{5^3} = 5$$
$$\sqrt[3]{5} \cdot \sqrt[3]{5^2} = \sqrt[3]{5^3} = 5$$

b) "By what do I multiply $\sqrt[3]{3}$ to get $\sqrt[3]{3^3}$?" $\sqrt[3]{3^2}$

$$\sqrt[3]{3} \cdot \sqrt[3]{?} = \sqrt[3]{3^3} = 3$$
$$\sqrt[3]{3} \cdot \sqrt[3]{3^2} = \sqrt[3]{3^3} = 3$$

c) "By what do I multiply $\sqrt[3]{x^2}$ to get $\sqrt[3]{x^3}$?" $\sqrt[3]{x}$

$$\sqrt[3]{x^2} \cdot \sqrt[3]{?} = \sqrt[3]{x^3} = x$$
$$\sqrt[3]{x^2} \cdot \sqrt[3]{x} = \sqrt[3]{x^3} = x$$

d) In this example, $\sqrt[5]{8}\cdot\sqrt[5]{?}=\sqrt[5]{2^5}=2$, why are we trying to obtain $\sqrt[5]{2^5}$ instead of $\sqrt[5]{8^5}$? Because in the first radical, $\sqrt[5]{8}$, 8 *is a power of* 2. Before attempting to fill in the blank, rewrite 8 as 2^3.

$$\sqrt[5]{8}\cdot\sqrt[5]{?}=\sqrt[5]{2^5}=2$$
$$\sqrt[5]{2^3}\cdot\sqrt[5]{?}=\sqrt[5]{2^5}=2$$
$$\sqrt[5]{2^3}\cdot\sqrt[5]{2^2}=\sqrt[5]{2^5}=2$$

W Hint
Don't move to the next example unless you have fully grasped this concept first!

e)
$$\sqrt[4]{27}\cdot\sqrt[4]{?}=\sqrt[4]{3^4}=3$$
$$\sqrt[4]{3^3}\cdot\sqrt[4]{?}=\sqrt[4]{3^4}=3 \qquad \text{Since 27 is a power of 3, rewrite } \sqrt[4]{27} \text{ as } \sqrt[4]{3^3}.$$
$$\sqrt[4]{3^3}\cdot\sqrt[4]{3}=\sqrt[4]{3^4}=3$$

[YOU TRY 4]

Fill in the blank.

a) $\sqrt[3]{2}\cdot\sqrt[3]{?}=\sqrt[3]{2^3}=2$ b) $\sqrt[5]{t^2}\cdot\sqrt[5]{?}=\sqrt[5]{t^5}=t$

c) $\sqrt[4]{125}\cdot\sqrt[4]{?}=\sqrt[4]{5^4}=5$

We will use the technique presented in Example 4 to rationalize denominators with indices higher than 2.

EXAMPLE 5

In-Class Example 5

Rationalize the denominator.

a) $\frac{8}{\sqrt[3]{3}}$ b) $\sqrt[4]{\frac{2}{9}}$ c) $\frac{9}{\sqrt[5]{t}}$

Answer: a) $\frac{8\sqrt[3]{9}}{3}$ b) $\frac{\sqrt[4]{18}}{3}$ c) $\frac{9\sqrt[5]{t^4}}{t}$

Rationalize the denominator.

a) $\frac{7}{\sqrt[3]{3}}$ b) $\sqrt[5]{\frac{3}{4}}$ c) $\frac{7}{\sqrt[4]{n}}$

Solution

a) *First* identify what we want the denominator to be *after* multiplying. **We want to obtain $\sqrt[3]{3^3}$ since $\sqrt[3]{3^3}=3$.**

$$\frac{7}{\sqrt[3]{3}}\cdot\frac{\ }{\uparrow}=\frac{\ }{\sqrt[3]{3^3}} \longleftarrow \text{This is what we want to get.}$$

What is needed here?

Ask yourself, "By what do I multiply $\sqrt[3]{3}$ to get $\sqrt[3]{3^3}$?" $\sqrt[3]{3^2}$

W Hint
Be sure to ask yourself this question so that you multiply by the correct radical!

$$\frac{7}{\sqrt[3]{3}}\cdot\frac{\sqrt[3]{3^2}}{\sqrt[3]{3^2}}=\frac{7\sqrt[3]{3^2}}{\sqrt[3]{3^3}} \qquad \text{Multiply.}$$
$$=\frac{7\sqrt[3]{9}}{3} \qquad \text{Simplify.}$$

b) Use the quotient rule for radicals to rewrite $\sqrt[5]{\frac{3}{4}}$ as $\frac{\sqrt[5]{3}}{\sqrt[5]{4}}$. Then, write 4 as 2^2 to get

$$\frac{\sqrt[5]{3}}{\sqrt[5]{4}}=\frac{\sqrt[5]{3}}{\sqrt[5]{2^2}}$$

What denominator do we want to get *after* multiplying? **We want to obtain $\sqrt[5]{2^5}$ since $\sqrt[5]{2^5}=2$.**

$$\frac{\sqrt[5]{3}}{\sqrt[5]{2^2}}\cdot\frac{\ }{\uparrow}=\frac{\ }{\sqrt[5]{2^5}} \longleftarrow \text{This is what we want to get.}$$

What is needed here?

"By what do I multiply $\sqrt[5]{2^2}$ to get $\sqrt[5]{2^5}$?" $\sqrt[5]{2^3}$

$$\frac{\sqrt[5]{3}}{\sqrt[5]{2^2}} \cdot \frac{\sqrt[5]{2^3}}{\sqrt[5]{2^3}} = \frac{\sqrt[5]{3} \cdot \sqrt[5]{2^3}}{\sqrt[5]{2^5}} \qquad \text{Multiply.}$$

$$= \frac{\sqrt[5]{3} \cdot \sqrt[5]{8}}{2} = \frac{\sqrt[5]{24}}{2} \qquad \text{Multiply.}$$

W Hint

In your own words, summarize how to rationalize denominators in Objective 2.

c) What denominator do we want to get *after* multiplying? **We want to obtain $\sqrt[4]{n^4}$ since $\sqrt[4]{n^4} = n$.**

$$\frac{7}{\sqrt[4]{n}} \cdot \frac{}{} = \frac{}{\sqrt[4]{n^4}} \longleftarrow \text{This is what we want to get.}$$

↑

What is needed here?

Ask yourself, "By what do I multiply $\sqrt[4]{n}$ to get $\sqrt[4]{n^4}$?" $\sqrt[4]{n^3}$

$$\frac{7}{\sqrt[4]{n}} \cdot \frac{\sqrt[4]{n^3}}{\sqrt[4]{n^3}} = \frac{7\sqrt[4]{n^3}}{\sqrt[4]{n^4}} \qquad \text{Multiply.}$$

$$= \frac{7\sqrt[4]{n^3}}{n} \qquad \text{Simplify.}$$

[YOU TRY 5]

Rationalize the denominator.

a) $\dfrac{4}{\sqrt[3]{7}}$ b) $\sqrt[4]{\dfrac{2}{27}}$ c) $\sqrt[5]{\dfrac{8}{w^3}}$

3 Rationalize a Denominator Containing Two Terms

To rationalize the denominator of an expression like $\dfrac{1}{5+\sqrt{3}}$, we multiply the numerator and the denominator of the expression by the *conjugate* of $5+\sqrt{3}$.

Definition

The **conjugate** of a binomial is the binomial obtained by changing the sign between the two terms.

Expression	Conjugate
$\sqrt{7} - 2\sqrt{5}$	$\sqrt{7} + 2\sqrt{5}$
$\sqrt{a} + \sqrt{b}$	$\sqrt{a} - \sqrt{b}$

In Section 9.5 we applied the formula $(a+b)(a-b) = a^2 - b^2$ to multiply binomials containing square roots. Recall that the terms containing the square roots were eliminated.

EXAMPLE 6 Multiply $8 - \sqrt{6}$ by its conjugate.

In-Class Example 6

Multiply $4 + \sqrt{3}$ by its conjugate.

Answer: 13

Solution

The conjugate of $8 - \sqrt{6}$ is $8 + \sqrt{6}$. We will first multiply using FOIL to show *why* the radical drops out, then we will multiply using the formula

$$(a + b)(a - b) = a^2 - b^2$$

i) Use FOIL to multiply.

$$\begin{aligned}(8 - \sqrt{6})(8 + \sqrt{6}) &= \underset{F}{8 \cdot 8} + \underset{O}{8 \cdot \sqrt{6}} - \underset{I}{8 \cdot \sqrt{6}} - \underset{L}{\sqrt{6} \cdot \sqrt{6}} \\ &= 64 - 6 \\ &= 58\end{aligned}$$

ii) Use $(a + b)(a - b) = a^2 - b^2$.

$$\begin{aligned}(8 - \sqrt{6})(8 + \sqrt{6}) &= (8)^2 - (\sqrt{6})^2 && \text{Substitute 8 for } a \text{ and } \sqrt{6} \text{ for } b. \\ &= 64 - 6 = 58\end{aligned}$$

Each method gives the same result.

[YOU TRY 6] Multiply $2 + \sqrt{11}$ by its conjugate.

Procedure Rationalize a Denominator Containing Two Terms

If the denominator of an expression contains two terms, including one or two square roots, then to rationalize the denominator we multiply the numerator and denominator of the expression by the conjugate of the denominator.

EXAMPLE 7 Rationalize the denominator, and simplify completely.

a) $\dfrac{3}{5 + \sqrt{3}}$ b) $\dfrac{\sqrt{a} + b}{\sqrt{b} - a}$

In-Class Example 7

Rationalize the denominator, and simplify completely.

a) $\dfrac{3}{6 + \sqrt{2}}$ b) $\dfrac{\sqrt{c} - d}{\sqrt{d} + c}$

Answer: a) $\dfrac{18 - 3\sqrt{2}}{34}$

b) $\dfrac{\sqrt{cd} - c\sqrt{c} + d\sqrt{d} + cd}{d - c^2}$

Solution

a) The denominator of $\dfrac{3}{5 + \sqrt{3}}$ has two terms, so we multiply the numerator and denominator by $5 - \sqrt{3}$, the conjugate of the denominator.

$$\begin{aligned}&\frac{3}{5 + \sqrt{3}} \cdot \frac{5 - \sqrt{3}}{5 - \sqrt{3}} && \text{Multiply by the conjugate.} \\ &= \frac{3(5 - \sqrt{3})}{(5)^2 - (\sqrt{3})^2} && (a + b)(a - b) = a^2 - b^2 \\ &= \frac{15 - 3\sqrt{3}}{25 - 3} && \text{Simplify.} \\ &= \frac{15 - 3\sqrt{3}}{22} && \text{Subtract.}\end{aligned}$$

b) $\dfrac{\sqrt{a}+b}{\sqrt{b}-a} \cdot \dfrac{\sqrt{b}+a}{\sqrt{b}+a}$ Multiply by the conjugate.

In the numerator we must multiply $(\sqrt{a}+b)(\sqrt{b}+a)$. We will use FOIL.

$$\frac{\sqrt{a}+b}{\sqrt{b}-a} \cdot \frac{\sqrt{b}+a}{\sqrt{b}+\mathrm{a}} = \frac{\sqrt{ab}+a\sqrt{a}+b\sqrt{b}+ab}{(\sqrt{b})^2-(a)^2} \quad \begin{array}{l}\leftarrow\text{Use FOIL in the numerator.}\\ \leftarrow(a+b)(a-b)=a^2-b^2\end{array}$$

$$= \frac{\sqrt{ab}+a\sqrt{a}+b\sqrt{b}+ab}{b-a^2} \quad \text{Square the terms.}$$

[YOU TRY 7]

Rationalize the denominator, and simplify completely.

a) $\dfrac{1}{\sqrt{7}-2}$ b) $\dfrac{c-\sqrt{d}}{c+\sqrt{d}}$

4 Rationalize a Numerator

In higher-level math courses, sometimes it is necessary to rationalize the *numerator* of a radical expression so that the numerator does not contain a radical.

EXAMPLE 8

Rationalize the numerator, and simplify completely.

a) $\dfrac{\sqrt{7}}{\sqrt{2}}$ b) $\dfrac{8-\sqrt{5}}{3}$

In-Class Example 8

Rationalize the numerator, and simplify completely.

a) $\dfrac{\sqrt{11}}{\sqrt{7}}$ b) $\dfrac{4-\sqrt{6}}{3}$

Answer: a) $\dfrac{11}{\sqrt{77}}$

b) $\dfrac{10}{12+3\sqrt{6}}$

Solution

a) Rationalizing the numerator of $\dfrac{\sqrt{7}}{\sqrt{2}}$ means eliminating the square root from the *numerator*. Multiply the numerator and denominator by $\sqrt{7}$.

$$\frac{\sqrt{7}}{\sqrt{2}} = \frac{\sqrt{7}}{\sqrt{2}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{7}{\sqrt{14}}$$

Hint

Follow the same process you did for rationalizing the denominator, but focus on the numerator.

b) To rationalize the numerator we must multiply the numerator and denominator by $8+\sqrt{5}$, the conjugate of the numerator.

$$\frac{8-\sqrt{5}}{3} \cdot \frac{8+\sqrt{5}}{8+\sqrt{5}} \quad \text{Multiply by the conjugate.}$$

$$= \frac{8^2-(\sqrt{5})^2}{3(8+\sqrt{5})} \quad \begin{array}{l}\leftarrow (a+b)(a-b)=a^2-b^2\\ \leftarrow \text{Multiply.}\end{array}$$

$$= \frac{64-5}{24+3\sqrt{5}} = \frac{59}{24+3\sqrt{5}}$$

[YOU TRY 8]

Rationalize the numerator, and simplify completely.

a) $\dfrac{\sqrt{3}}{\sqrt{5}}$ b) $\dfrac{6+\sqrt{7}}{4}$

5 Divide Out Common Factors from the Numerator and Denominator

Sometimes it is necessary to simplify a radical expression by dividing out common factors from the numerator and denominator. This is a skill we will need in Chapter 10 to solve quadratic equations, so we will look at an example here.

EXAMPLE 9

In-Class Example 9

Simplify completely: $\frac{9\sqrt{3}-18}{9}$.

Answer: $\sqrt{3}-2$

Simplify completely: $\frac{4\sqrt{5}+12}{4}$.

Solution

BE CAREFUL

It is tempting to do one of the following:

$$\frac{\cancel{4}\sqrt{5}+12}{\cancel{4}} = \sqrt{5}+12 \qquad \textbf{Incorrect!}$$

or

$$\frac{4\sqrt{5}+\overset{3}{\cancel{12}}}{\cancel{4}} = 4\sqrt{5}+3 \qquad \textbf{Incorrect!}$$

Each is incorrect because $4\sqrt{5}$ is a *term* in a sum and 12 is a *term* in a sum.

Hint

We will use this skill in Chapter 10, so be sure you understand it.

The correct way to simplify $\frac{4\sqrt{5}+12}{4}$ is to begin by factoring out a 4 in the numerator and *then* divide the numerator and denominator by any common factors.

$$\frac{4\sqrt{5}+12}{4} = \frac{4(\sqrt{5}+3)}{4} \qquad \text{Factor out 4 from the numerator.}$$

$$= \frac{\overset{1}{\cancel{4}}(\sqrt{5}+3)}{\underset{1}{\cancel{4}}} \qquad \text{Divide by 4.}$$

$$= \sqrt{5}+3 \qquad \text{Simplify.}$$

We can divide the numerator and denominator by 4 in $\frac{4(\sqrt{5}+3)}{4}$ because the 4 in the numerator is part of a *product*, not a sum or difference.

YOU TRY 9

Simplify completely.

a) $\frac{5\sqrt{7}+40}{5}$ b) $\frac{20-6\sqrt{2}}{4}$

ANSWERS TO YOU TRY EXERCISES

1) a) $\frac{\sqrt{7}}{7}$ b) $\frac{5\sqrt{3}}{3}$ c) $\frac{9\sqrt{30}}{5}$ 2) a) $\frac{\sqrt{14}}{7}$ b) $\frac{\sqrt{15}}{5}$ 3) a) $\frac{5\sqrt{p}}{p}$ b) $\frac{3k^2\sqrt{5km}}{5m}$ c) $\frac{2r\sqrt{5rs}}{s}$ 4) a) 2^2 or 4 b) t^3 c) 5 5) a) $\frac{4\sqrt[3]{49}}{7}$ b) $\frac{\sqrt[4]{6}}{3}$ c) $\frac{\sqrt[5]{8w^2}}{w}$ 6) -7 7) a) $\frac{\sqrt{7}+2}{3}$ b) $\frac{c^2-2c\sqrt{d}+d}{c^2-d}$ 8) a) $\frac{3}{\sqrt{15}}$ b) $\frac{29}{24-4\sqrt{7}}$ 9) a) $\sqrt{7}+8$ b) $\frac{10-3\sqrt{2}}{2}$

*Additional answers can be found in the Answers to Exercises appendix.

Assume all variables represent positive real numbers.

Objective 1: Rationalize a Denominator: One Square Root

1) What does it mean to rationalize the denominator of a radical expression?
Eliminate the radical from the denominator.

2) In your own words, explain how to rationalize the denominator of an expression containing one term in the denominator. Answers may vary.

Rationalize the denominator of each expression.

3) $\dfrac{1}{\sqrt{5}}$ $\quad \dfrac{\sqrt{5}}{5}$

4) $\dfrac{1}{\sqrt{6}}$ $\quad \dfrac{\sqrt{6}}{6}$

5) $\dfrac{9}{\sqrt{6}}$ $\quad \dfrac{3\sqrt{6}}{2}$

6) $\dfrac{25}{\sqrt{10}}$ $\quad \dfrac{5\sqrt{10}}{2}$

7) $-\dfrac{20}{\sqrt{8}}$ $\quad -5\sqrt{2}$

8) $-\dfrac{18}{\sqrt{45}}$ $\quad -\dfrac{6\sqrt{5}}{5}$

9) $\dfrac{\sqrt{3}}{\sqrt{28}}$ $\quad \dfrac{\sqrt{21}}{14}$

10) $\dfrac{\sqrt{8}}{\sqrt{27}}$ $\quad \dfrac{2\sqrt{6}}{9}$

11) $\sqrt{\dfrac{20}{60}}$ $\quad \dfrac{\sqrt{3}}{3}$

12) $\sqrt{\dfrac{12}{80}}$ $\quad \dfrac{\sqrt{15}}{10}$

13) $\dfrac{\sqrt{56}}{\sqrt{48}}$ $\quad \dfrac{\sqrt{42}}{6}$

14) $\dfrac{\sqrt{66}}{\sqrt{12}}$ $\quad \dfrac{\sqrt{22}}{2}$

Multiply and simplify.

15) $\sqrt{\dfrac{10}{7}} \cdot \sqrt{\dfrac{7}{3}}$ $\quad \dfrac{\sqrt{30}}{3}$

16) $\sqrt{\dfrac{11}{5}} \cdot \sqrt{\dfrac{5}{2}}$ $\quad \dfrac{\sqrt{22}}{2}$

17) $\sqrt{\dfrac{6}{5}} \cdot \sqrt{\dfrac{1}{8}}$ $\quad \dfrac{\sqrt{15}}{10}$

18) $\sqrt{\dfrac{11}{10}} \cdot \sqrt{\dfrac{8}{11}}$ $\quad \dfrac{2\sqrt{5}}{5}$

Simplify completely.

19) $\dfrac{8}{\sqrt{y}}$ $\quad \dfrac{8\sqrt{y}}{y}$

20) $\dfrac{4}{\sqrt{w}}$ $\quad \dfrac{4\sqrt{w}}{w}$

21) $\dfrac{\sqrt{5}}{\sqrt{t}}$ $\quad \dfrac{\sqrt{5t}}{t}$

22) $\dfrac{\sqrt{2}}{\sqrt{m}}$ $\quad \dfrac{\sqrt{2m}}{m}$

23) $\sqrt{\dfrac{64v^7}{5w}}$ $\quad \dfrac{8v^3\sqrt{5vw}}{5w}$

24) $\sqrt{\dfrac{81c^5}{2d}}$ $\quad \dfrac{9c^2\sqrt{2cd}}{2d}$

25) $\sqrt{\dfrac{a^3b^3}{3ab^4}}$ $\quad \dfrac{a\sqrt{3b}}{3b}$

26) $\sqrt{\dfrac{m^2n^5}{7m^3n}}$ $\quad \dfrac{n^2\sqrt{7m}}{7m}$

27) $-\dfrac{\sqrt{75}}{\sqrt{b^3}}$ $\quad -\dfrac{5\sqrt{3b}}{b^2}$

28) $-\dfrac{\sqrt{24}}{\sqrt{v^3}}$ $\quad -\dfrac{2\sqrt{6v}}{v^2}$

29) $\dfrac{\sqrt{13}}{\sqrt{j^5}}$ $\quad \dfrac{\sqrt{13j}}{j^3}$

30) $\dfrac{\sqrt{22}}{\sqrt{w^7}}$ $\quad \dfrac{\sqrt{22w}}{w^4}$

Objective 2: Rationalize a Denominator: One Higher Root

Fill in the blank.

31) $\sqrt[3]{2} \cdot \sqrt[3]{?} = \sqrt[3]{2^3} = 2$ $\quad 2^2$ or 4

32) $\sqrt[3]{5} \cdot \sqrt[3]{?} = \sqrt[3]{5^3} = 5$ $\quad 5^2$ or 25

33) $\sqrt[3]{9} \cdot \sqrt[3]{?} = \sqrt[3]{3^3} = 3$ $\quad 3$

34) $\sqrt[3]{4} \cdot \sqrt[3]{?} = \sqrt[3]{2^3} = 2$ $\quad 2$

35) $\sqrt[3]{c} \cdot \sqrt[3]{?} = \sqrt[3]{c^3} = c$ $\quad c^2$

36) $\sqrt[3]{p} \cdot \sqrt[3]{?} = \sqrt[3]{p^3} = p$ $\quad p^2$

37) $\sqrt[5]{4} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$ $\quad 2^3$ or 8

38) $\sqrt[5]{16} \cdot \sqrt[5]{?} = \sqrt[5]{2^5} = 2$ $\quad 2$

39) $\sqrt[4]{m^3} \cdot \sqrt[4]{?} = \sqrt[4]{m^4} = m$ $\quad m$

40) $\sqrt[4]{k} \cdot \sqrt[4]{?} = \sqrt[4]{k^4} = k$ $\quad k^3$

41) Inez simplifies $\dfrac{8}{\sqrt[3]{3}}$ like this: $\dfrac{8}{\sqrt[3]{3}} \cdot \dfrac{\sqrt[3]{3}}{\sqrt[3]{3}} = \dfrac{8\sqrt[3]{3}}{3}$. Is she right or wrong? If her answer is wrong, explain her mistake and do the problem correctly.
Her answer is wrong because $\sqrt[3]{3} \cdot \sqrt[3]{3} = \sqrt[3]{9} \neq 3$. The right way to simplify $\dfrac{8}{\sqrt[3]{3}}$ is $\dfrac{8}{\sqrt[3]{3}} \cdot \dfrac{\sqrt[3]{9}}{\sqrt[3]{9}} = \dfrac{8\sqrt[3]{9}}{\sqrt[3]{27}} = \dfrac{8\sqrt[3]{9}}{3}$.

42) Stavros simplifies $\dfrac{5}{\sqrt[8]{x^3}}$ like this: $\dfrac{5}{\sqrt[8]{x^3}} \cdot \dfrac{\sqrt[8]{x^5}}{\sqrt[8]{x^5}} = \dfrac{5\sqrt[8]{x^5}}{x}$. Is he right or wrong? If his answer is wrong, explain his mistake and do the problem correctly.
His answer is right: $\sqrt[8]{x^3} \cdot \sqrt[8]{x^5} = \sqrt[8]{x^8} = x$.

Rationalize the denominator of each expression.

43) $\dfrac{4}{\sqrt[3]{3}}$ $\quad \dfrac{4\sqrt[3]{9}}{3}$

44) $\dfrac{26}{\sqrt[3]{5}}$ $\quad \dfrac{26\sqrt[3]{25}}{5}$

45) $\dfrac{12}{\sqrt[3]{2}}$ $\quad 6\sqrt[3]{4}$

46) $\dfrac{21}{\sqrt[3]{3}}$ $\quad 7\sqrt[3]{9}$

47) $\dfrac{9}{\sqrt[3]{25}}$ $\quad \dfrac{9\sqrt[3]{5}}{5}$

48) $\dfrac{6}{\sqrt[3]{4}}$ $\quad 3\sqrt[3]{2}$

49) $\sqrt[4]{\dfrac{5}{9}}$ $\quad \dfrac{\sqrt[4]{45}}{3}$

50) $\sqrt[4]{\dfrac{2}{25}}$ $\quad \dfrac{\sqrt[4]{50}}{5}$

51) $\sqrt[5]{\dfrac{3}{8}}$ $\quad \dfrac{\sqrt[5]{12}}{2}$

52) $\sqrt[5]{\dfrac{7}{4}}$ $\quad \dfrac{\sqrt[5]{56}}{2}$

53) $\dfrac{10}{\sqrt[3]{z}}$ $\quad \dfrac{10\sqrt[3]{z^2}}{z}$

54) $\dfrac{6}{\sqrt[3]{u}}$ $\quad \dfrac{6\sqrt[3]{u^2}}{u}$

55) $\sqrt[3]{\dfrac{3}{n^2}}$ $\quad \dfrac{\sqrt[3]{3n}}{n}$

56) $\sqrt[3]{\dfrac{5}{x^2}}$ $\quad \dfrac{\sqrt[3]{5x}}{x}$

57) $\dfrac{\sqrt[3]{7}}{\sqrt[3]{2k^2}}$ $\quad \dfrac{\sqrt[3]{28k}}{2k}$

58) $\dfrac{\sqrt[3]{2}}{\sqrt[3]{25t}}$ $\quad \dfrac{\sqrt[3]{10t^2}}{5t}$

59) $\dfrac{9}{\sqrt[5]{a^3}}$ $\quad \dfrac{9\sqrt[5]{a^2}}{a}$

60) $\dfrac{8}{\sqrt[5]{h^2}}$ $\quad \dfrac{8\sqrt[5]{h^3}}{h}$

61) $\sqrt[4]{\dfrac{5}{2m}}$ $\quad \dfrac{\sqrt[4]{40m^3}}{2m}$

62) $\sqrt[4]{\dfrac{2}{3t^2}}$ $\quad \dfrac{\sqrt[4]{54t^2}}{3t}$

Objective 3: Rationalize a Denominator Containing Two Terms

63) How do you find the conjugate of an expression with two radical terms?
Change the sign between the two terms.

64) When you multiply a binomial containing a square root by its conjugate, what happens to the radical?
The radical is eliminated.

Find the conjugate of each expression. Then, multiply the expression by its conjugate.

65) $(5+\sqrt{2})$ $(5-\sqrt{2})$; 23

66) $(\sqrt{5}-4)$ $(\sqrt{5}+4)$; -11

67) $(\sqrt{2}+\sqrt{6})$ $(\sqrt{2}-\sqrt{6})$; -4

68) $(\sqrt{3}-\sqrt{10})$ $(\sqrt{3}+\sqrt{10})$; -7

69) $(\sqrt{t}-8)$ $(\sqrt{t}+8)$; $t-64$

70) $(\sqrt{p}+5)$ $(\sqrt{p}-5)$; $p-25$

Rationalize the denominator, and simplify completely.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

71) $\dfrac{6}{4-\sqrt{5}}=\dfrac{6}{4-\sqrt{5}}\cdot\dfrac{4+\sqrt{5}}{4+\sqrt{5}}$ Multiply by the conjugate.

$=\dfrac{6(4+\sqrt{5})}{(4)^2-(\sqrt{5})^2}$ $(a+b)(a-b)=a^2-b^2$

$=\dfrac{24+6\sqrt{5}}{16-5}$ Multiply terms in numerator; square terms in denominator.

$=\dfrac{24+6\sqrt{5}}{11}$ Simplify.

72) $\dfrac{\sqrt{6}}{\sqrt{7}+\sqrt{2}}=\dfrac{\sqrt{6}}{\sqrt{7}+\sqrt{2}}\cdot\dfrac{\sqrt{7}-\sqrt{2}}{\sqrt{7}-\sqrt{2}}$ Multiply by the conjugate.

$=\dfrac{\sqrt{6}(\sqrt{7}-\sqrt{2})}{(\sqrt{7})^2-(\sqrt{2})^2}$ $(a+b)(a-b)=a^2-b^2$

$=\dfrac{\sqrt{42}-\sqrt{12}}{7-2}$ Multiply terms in numerator; square terms in denominator.

$=\dfrac{\sqrt{42}-2\sqrt{3}}{5}$ Simplify.

73) $\dfrac{3}{2+\sqrt{3}}$ $6-3\sqrt{3}$

74) $\dfrac{8}{6-\sqrt{5}}$ $\dfrac{48+8\sqrt{5}}{31}$

75) $\dfrac{10}{9-\sqrt{2}}$ $\dfrac{90+10\sqrt{2}}{79}$

76) $\dfrac{5}{4+\sqrt{6}}$ $\dfrac{4-\sqrt{6}}{2}$

77) $\dfrac{\sqrt{8}}{\sqrt{3}+\sqrt{2}}$ $2\sqrt{6}-4$

78) $\dfrac{\sqrt{32}}{\sqrt{5}-\sqrt{7}}$ $-2\sqrt{10}-2\sqrt{14}$

79) $\dfrac{\sqrt{3}-\sqrt{5}}{\sqrt{10}-\sqrt{3}}$

80) $\dfrac{\sqrt{3}+\sqrt{6}}{\sqrt{2}+\sqrt{5}}$

81) $\dfrac{\sqrt{m}}{\sqrt{m}+\sqrt{n}}$ $\dfrac{m-\sqrt{mn}}{m-n}$

82) $\dfrac{\sqrt{u}}{\sqrt{u}-\sqrt{v}}$ $\dfrac{u+\sqrt{uv}}{u-v}$

83) $\dfrac{b-25}{\sqrt{b}-5}$ $\sqrt{b}+5$

84) $\dfrac{d-9}{\sqrt{d}+3}$ $\sqrt{d}-3$

85) $\dfrac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}$

86) $\dfrac{\sqrt{f}-\sqrt{g}}{\sqrt{f}+\sqrt{g}}$ $\dfrac{f-2\sqrt{fg}+g}{f-g}$

Objective 4: Rationalize a Numerator

Rationalize the numerator of each expression, and simplify.

87) $\dfrac{\sqrt{5}}{3}$ $\dfrac{5}{3\sqrt{5}}$

88) $\dfrac{\sqrt{2}}{9}$ $\dfrac{2}{9\sqrt{2}}$

89) $\dfrac{\sqrt{x}}{\sqrt{7}}$ $\dfrac{x}{\sqrt{7x}}$

90) $\dfrac{\sqrt{8a}}{\sqrt{b}}$ $\dfrac{4a}{\sqrt{2ab}}$

91) $\dfrac{2+\sqrt{3}}{6}$ $\dfrac{1}{12-6\sqrt{3}}$

92) $\dfrac{1+\sqrt{7}}{3}$ $\dfrac{2}{\sqrt{7}-1}$

93) $\dfrac{\sqrt{x}-2}{x-4}$ $\dfrac{1}{\sqrt{x}+2}$

94) $\dfrac{3-\sqrt{n}}{n-9}$ $-\dfrac{1}{3+\sqrt{n}}$

95) $\dfrac{4-\sqrt{c+11}}{c-5}$

96) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

97) Does rationalizing the denominator of an expression change the value of the original expression? Explain your answer.

98) Does rationalizing the numerator of an expression change the value of the original expression? Explain your answer.

Objective 5: Divide Out Common Factors from the Numerator and Denominator

Simplify completely.

99) $\dfrac{5+10\sqrt{3}}{5}$ $1+2\sqrt{3}$

100) $\dfrac{18-6\sqrt{7}}{6}$ $3-\sqrt{7}$

101) $\dfrac{30-18\sqrt{5}}{4}$ $\dfrac{15-9\sqrt{5}}{2}$

102) $\dfrac{36+20\sqrt{2}}{12}$ $\dfrac{9+5\sqrt{2}}{3}$

103) $\dfrac{\sqrt{45}+6}{9}$ $\dfrac{\sqrt{5}+2}{3}$

104) $\dfrac{\sqrt{48}+28}{4}$ $\sqrt{3}+7$

105) $\dfrac{-10-\sqrt{50}}{5}$ $-2-\sqrt{2}$

106) $\dfrac{-35+\sqrt{200}}{15}$ $\dfrac{-7+2\sqrt{2}}{3}$

107) The function $r(A) = \sqrt{\frac{A}{\pi}}$ describes the radius of a circle, $r(A)$, in terms of its area, A.

a) If the area of a circle is measured in square inches, find $r(8\pi)$ and explain what it means in the context of the problem.

b) If the area of a circle is measured in square inches, find $r(7)$ and rationalize the denominator. Explain the meaning of $r(7)$ in the context of the problem.

c) Obtain an equivalent form of the function by rationalizing the denominator. $r(A) = \frac{\sqrt{A\pi}}{\pi}$

108) The function $r(V) = \sqrt[3]{\frac{3V}{4\pi}}$ describes the radius of a sphere, $r(V)$, in terms of its volume, V.

a) If the volume of a sphere is measured in cubic centimeters, find $r(36\pi)$ and explain what it means in the context of the problem.

b) If the volume of a sphere is measured in cubic centimeters, find $r(11)$ and rationalize the denominator. Explain the meaning of $r(11)$ in the context of the problem.

c) Obtain an equivalent form of the function by rationalizing the denominator. $r(V) = \frac{\sqrt[3]{6\pi^2 V}}{2\pi}$

R Rethink

R1) Why is it important to always read the directions carefully?

R2) Look at Exercises 3 and 43. Explain how rationalizing their denominators is similar as well as how that process is different.

R3) Did you circle any problems you struggled with? Ask about them in class!

Putting It All Together

P Prepare | **O Organize**

What are your objectives?	How can you accomplish each objective?
1 Review the concepts of Sections 9.1–9.6.	• Complete the given examples on your own. • Complete You Try 1.

W Work **Read the explanations, follow the examples, take notes, and complete the You Trys.**

1 Review the Concepts Presented in 9.1–9.6

In Section 9.1, we learned how to find roots of numbers. For example, $\sqrt[3]{-64} = -4$ because $(-4)^3 = -64$. We also learned about square root and cube root functions.

In Section 9.2, we learned about the relationship between rational exponents and radicals. Recall that if m and n are positive integers and $\frac{m}{n}$ is in lowest terms, then $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$ provided that $a^{1/n}$ is a real number. **For the rest of this section we will assume that all variables represent positive real numbers.**

EXAMPLE 1

Simplify completely. The answer should contain only positive exponents.

a) $(32)^{4/5}$ b) $\left(\frac{a^7b^{9/8}}{25a^9b^{3/4}}\right)^{-3/2}$

In-Class Example 1

Simplify completely. The answer should contain only positive exponents.

a) $(125)^{2/3}$ b) $\left(\frac{4c^9d^{11/6}}{c^{-1}d^2}\right)^{-5/2}$

Answer: a) 25 b) $\left(\frac{d^{5/12}}{32c^{25}}\right)$

Solution

a) The denominator of the fractional exponent is the index of the radical, and the numerator is the power to which we raise the radical expression.

$$\begin{aligned} 32^{4/5} &= (\sqrt[5]{32})^4 && \text{Write in radical form.} \\ &= (2)^4 && \sqrt[5]{32} = 2 \\ &= 16 \end{aligned}$$

b) $$\left(\frac{a^7b^{9/8}}{25a^9b^{3/4}}\right)^{-3/2} = \left(\frac{25a^9b^{3/4}}{a^7b^{9/8}}\right)^{3/2}$$ Eliminate the negative from the outermost exponent by taking the reciprocal of the base.

Simplify the expression inside the parentheses by subtracting the exponents.

$$= (25a^{9-7}b^{3/4-9/8})^{3/2} = (25a^2b^{6/8-9/8})^{3/2} = (25a^2b^{-3/8})^{3/2}$$

Apply the power rule, and simplify.

$$= (25)^{3/2}(a^2)^{3/2}(b^{-3/8})^{3/2} = (\sqrt{25})^3a^3b^{-9/16} = 5^3a^3b^{-9/16} = \frac{125a^3}{b^{9/16}}$$

In Sections 9.3–9.6 we learned how to simplify, multiply, divide, add, and subtract radicals. Let's look at these operations together so that we will learn to recognize the techniques needed to perform these operations.

EXAMPLE 2

Perform the operations, and simplify.

a) $\sqrt{3} + 10\sqrt{6} - 4\sqrt{3}$ b) $\sqrt{3}(10\sqrt{6} - 4\sqrt{3})$

In-Class Example 2

Perform the operations, and simplify.

a) $\sqrt{5} + 7\sqrt{3} - 11\sqrt{5}$

b) $\sqrt{6}(4\sqrt{3} + 2\sqrt{6})$

Answer: a) $7\sqrt{3} - 10\sqrt{5}$
b) $12\sqrt{2} + 12$

W Hint

Look carefully at expressions like these so that you can distinguish between a product and a sum or difference.

Solution

a) This is the *sum and difference* of radicals. Remember that we can only add and subtract radicals that are like radicals.

$$\begin{aligned} \sqrt{3} + 10\sqrt{6} - 4\sqrt{3} &= \sqrt{3} - 4\sqrt{3} + 10\sqrt{6} && \text{Write like radicals together.} \\ &= -3\sqrt{3} + 10\sqrt{6} && \text{Subtract.} \end{aligned}$$

b) This is the *product* of radical expressions. We must multiply the binomial $10\sqrt{6} - 4\sqrt{3}$ by $\sqrt{3}$ using the distributive property.

$$\begin{aligned} \sqrt{3}(10\sqrt{6} - 4\sqrt{3}) &= \sqrt{3} \cdot 10\sqrt{6} - \sqrt{3} \cdot 4\sqrt{3} && \text{Distribute.} \\ &= 10\sqrt{18} - 4 \cdot 3 && \text{Product rule; } \sqrt{3} \cdot \sqrt{3} = 3. \\ &= 10\sqrt{18} - 12 && \text{Multiply.} \end{aligned}$$

Ask yourself, "Is $10\sqrt{18} - 12$ in simplest form?" *No.* $\sqrt{18}$ can be simplified.

$$\begin{aligned} &= 10\sqrt{9 \cdot 2} - 12 && \text{9 is a perfect square.} \\ &= 10\sqrt{9} \cdot \sqrt{2} - 12 && \text{Product rule} \\ &= 10 \cdot 3\sqrt{2} - 12 && \sqrt{9} = 3 \\ &= 30\sqrt{2} - 12 && \text{Multiply.} \end{aligned}$$

The expression is now in simplest form.

Next we will look at multiplication problems involving binomials. Remember that the rules we used to multiply binomials like $(x+4)(x-9)$ are the same rules we use to multiply binomials containing radicals.

EXAMPLE 3

In-Class Example 3

Multiply and simplify.
a) $(1+\sqrt{3})(8-\sqrt{7})$
b) $(\sqrt{k}+\sqrt{11})(\sqrt{k}-\sqrt{11})$
c) $(4\sqrt{3}-5)^2$

Answer:
a) $8-\sqrt{7}+8\sqrt{3}-\sqrt{21}$
b) $k-11$ c) $23-40\sqrt{3}$

Multiply and simplify.

a) $(8+\sqrt{2})(9-\sqrt{11})$ b) $(\sqrt{n}+\sqrt{7})(\sqrt{n}-\sqrt{7})$

c) $(2\sqrt{5}-3)^2$

Solution

a) Since we must multiply two binomials, we will use FOIL.

$$\begin{aligned}(8+\sqrt{2})(9-\sqrt{11}) &= \overset{F}{8\cdot 9} - \overset{O}{8\cdot\sqrt{11}} + \overset{I}{9\cdot\sqrt{2}} - \overset{L}{\sqrt{2}\cdot\sqrt{11}} && \text{Use FOIL.}\\ &= 72 - 8\sqrt{11} + 9\sqrt{2} - \sqrt{22} && \text{Multiply.}\end{aligned}$$

All radicals are simplified and none of them are like radicals, so this expression is in simplest form.

Hint

Notice the connection between what you learned in Chapter 6 and multiplying binomials containing radicals.

b) We can multiply $(\sqrt{n}+\sqrt{7})(\sqrt{n}-\sqrt{7})$ using FOIL or, if we notice that this product is in the form $(a+b)(a-b)$ we can apply the rule $(a+b)(a-b)=a^2-b^2$. Either method will give us the correct answer. We will use the second method.

$$(a+b)(a-b) = a^2 - b^2$$
$$(\sqrt{n}+\sqrt{7})(\sqrt{n}-\sqrt{7}) = (\sqrt{n})^2 - (\sqrt{7})^2 = n-7 \quad \text{Substitute } \sqrt{n} \text{ for } a \text{ and } \sqrt{7} \text{ for } b.$$

c) Once again we can either use FOIL to expand $(2\sqrt{5}-3)^2$, or we can use the special formula we learned for squaring a binomial.

We will use $(a-b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned}(2\sqrt{5}-3)^2 &= (2\sqrt{5})^2 - 2(2\sqrt{5})(3) + (3)^2 && \text{Substitute } 2\sqrt{5} \text{ for } a \text{ and } 3 \text{ for } b.\\ &= (4\cdot 5) - 4\sqrt{5}(3) + 9 && \text{Multiply.}\\ &= 20 - 12\sqrt{5} + 9 && \text{Multiply.}\\ &= 29 - 12\sqrt{5} && \text{Combine like terms.}\end{aligned}$$

Remember that an expression is not considered to be in simplest form if it contains a radical in its denominator. To rationalize the denominator of a radical expression, we must keep in mind the index on the radical and the number of terms in the denominator.

EXAMPLE 4

In-Class Example 4

Rationalize the denominator of each expression.
a) $\dfrac{14}{\sqrt{2a}}$ b) $\dfrac{14}{\sqrt[3]{2a}}$
c) $\dfrac{\sqrt{14}}{\sqrt{2}+3}$

Answer: a) $\dfrac{7\sqrt{2a}}{a}$
b) $\dfrac{7\sqrt[3]{4a^2}}{a}$ c) $\dfrac{3\sqrt{14}-2\sqrt{7}}{7}$

Rationalize the denominator of each expression.

a) $\dfrac{10}{\sqrt{2x}}$ b) $\dfrac{10}{\sqrt[3]{2x}}$ c) $\dfrac{\sqrt{10}}{\sqrt{2}-1}$

Solution

a) First, notice that the denominator of $\dfrac{10}{\sqrt{2x}}$ contains only one term and it is a *square* root. Ask yourself, "By what do I multiply $\sqrt{2x}$ to get a perfect *square* under the radical?" The answer is $\sqrt{2x}$ since $\sqrt{2x}\cdot\sqrt{2x} = \sqrt{4x^2} = 2x$. Multiply the numerator and denominator by $\sqrt{2x}$, and simplify.

$$\frac{10}{\sqrt{2x}} = \frac{10}{\sqrt{2x}}\cdot\frac{\sqrt{2x}}{\sqrt{2x}} \quad \text{Rationalize the denominator.}$$
$$= \frac{10\sqrt{2x}}{\sqrt{4x^2}} = \frac{10\sqrt{2x}}{2x} = \frac{5\sqrt{2x}}{x}$$

W Hint

In your notes, summarize the similarities and differences in rationalizing the denominators in each expression in this example.

b) The denominator of $\dfrac{10}{\sqrt[3]{2x}}$ contains only one term, but it is a *cube* root. Ask yourself, "By what do I multiply $\sqrt[3]{2x}$ to get a radicand that is a perfect *cube*?" The answer is $\sqrt[3]{4x^2}$ since $\sqrt[3]{2x} \cdot \sqrt[3]{4x^2} = \sqrt[3]{8x^3} = 2x$. Multiply the numerator and denominator by $\sqrt[3]{4x^2}$, and simplify.

$$\frac{10}{\sqrt[3]{2x}} = \frac{10}{\sqrt[3]{2x}} \cdot \frac{\sqrt[3]{4x^2}}{\sqrt[3]{4x^2}} \qquad \text{Rationalize the denominator.}$$

$$= \frac{10\sqrt[3]{4x^2}}{\sqrt[3]{8x^3}} = \frac{10\sqrt[3]{4x^2}}{2x} = \frac{5\sqrt[3]{4x^2}}{x}$$

c) The denominator of $\dfrac{\sqrt{10}}{\sqrt{2}-1}$ contains two terms, so how do we rationalize the denominator of this expression? We multiply the numerator and denominator by the *conjugate* of the denominator.

$$\frac{\sqrt{10}}{\sqrt{2}-1} = \frac{\sqrt{10}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} \qquad \text{Multiply by the conjugate.}$$

$$= \frac{\sqrt{10}(\sqrt{2}+1)}{(\sqrt{2})^2-(1)^2} \qquad \text{Multiply. } (a+b)(a-b)=a^2-b^2$$

$$= \frac{\sqrt{20}+\sqrt{10}}{1} \qquad \text{Distribute. Simplify.}$$

$$= 2\sqrt{5}+\sqrt{10} \qquad \sqrt{20}=2\sqrt{5}\text{; simplify.}$$

[YOU TRY 1]

a) Perform the operations, and simplify.

i) $(\sqrt{w}+8)^2$ ii) $(3-\sqrt{5a})(4+\sqrt{5a})$ iii) $\sqrt{2}(9\sqrt{10}-\sqrt{2})$

iv) $\sqrt{2}+9\sqrt{10}-5\sqrt{2}$ v) $(2\sqrt{3}+y)(2\sqrt{3}-y)$

b) Find each root.

i) $-\sqrt{\dfrac{121}{16}}$ ii) $\sqrt[3]{-1000}$ iii) $\sqrt{0.09}$ iv) $\sqrt{-49}$

W Hint

Read all directions first.

c) Simplify completely. The answer should contain only positive exponents.

i) $(-64)^{2/3}$ ii) $\left(\dfrac{81x^3y^{1/2}}{x^{-5}y^6}\right)^{-3/4}$

d) Rationalize the denominator of each expression.

i) $\dfrac{24}{\sqrt[3]{9h}}$ ii) $\dfrac{7+\sqrt{6}}{4+\sqrt{6}}$ iii) $\dfrac{56}{\sqrt{7}}$

ANSWERS TO [YOU TRY] EXERCISES

1) a) i) $w+16\sqrt{w}+64$ ii) $12-\sqrt{5a}-5a$ iii) $18\sqrt{5}-2$ iv) $-4\sqrt{2}+9\sqrt{10}$ v) $12-y^2$

b) i) $-\dfrac{11}{4}$ ii) -10 iii) 0.3 iv) not a real number

c) i) 16 ii) $\dfrac{y^{33/8}}{27x^6}$ d) i) $\dfrac{8\sqrt[3]{3h^2}}{h}$ ii) $\dfrac{22-3\sqrt{6}}{10}$ iii) $8\sqrt{7}$

Putting It All Together Exercises

Evaluate Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Objective 1: Review the Concepts Presented in 9.1–9.6

Fill in the blank with *always, sometimes*, or *never* to make the statement true.

1) The odd root of a real number is ___always___ a real number.

2) The even root of a negative number is ___never___ a real number.

Assume all variables represent positive real numbers.

Find each root, if possible.

3) $\sqrt[4]{81}$ 3

4) $\sqrt[3]{-1000}$ -10

5) $-\sqrt[6]{64}$ -2

6) $\sqrt{121}$ 11

7) $\sqrt{-169}$ not a real number

8) $\sqrt{\dfrac{144}{49}}$ $\dfrac{12}{7}$

Simplify completely. The answer should contain only positive exponents.

9) $(144)^{1/2}$ 12

10) $(-32)^{4/5}$ 16

11) $-1000^{2/3}$ -100

12) $\left(-\dfrac{16}{81}\right)^{3/4}$ not a real number

13) $125^{-1/3}$ $\dfrac{1}{5}$

14) $\left(\dfrac{100}{9}\right)^{-3/2}$ $\dfrac{27}{1000}$

15) $k^{-3/5} \cdot k^{3/10}$ $\dfrac{1}{k^{3/10}}$

16) $(t^{3/8})^{16}$ t^6

17) $\left(\dfrac{27a^{-8}}{b^9}\right)^{2/3}$ $\dfrac{9}{a^{16/3}b^6}$

18) $\left(\dfrac{18x^{-9}y^{4/3}}{2x^3y}\right)^{-5/2}$ $\dfrac{x^{30}}{243y^{5/6}}$

Simplify completely.

19) $\sqrt{24}$ $2\sqrt{6}$

20) $\sqrt[4]{32}$ $2\sqrt[4]{2}$

21) $\sqrt[3]{72}$ $2\sqrt[3]{9}$

22) $\sqrt[3]{\dfrac{500}{2}}$ $5\sqrt[3]{2}$

23) $\sqrt[4]{243}$ $3\sqrt[4]{3}$

24) $\sqrt{45c^{11}}$ $3c^5\sqrt{5c}$

25) $\sqrt[3]{96m^7n^{15}}$ $2m^2n^5\sqrt[3]{12m}$

26) $\sqrt[5]{\dfrac{64x^{19}}{y^{20}}}$ $\dfrac{2x^3\sqrt[5]{2x^4}}{y^4}$

Perform the operations, and simplify.

27) $\sqrt[3]{12} \cdot \sqrt[3]{2}$ $2\sqrt[3]{3}$

28) $\sqrt[4]{\dfrac{96k^{11}}{2k^3}}$ $2k^2\sqrt[4]{3}$

29) $(6+\sqrt{7})(2+\sqrt{7})$ $19+8\sqrt{7}$

30) $4c^2\sqrt[3]{108c} - 15\sqrt[3]{32c^7}$ $-18c^2\sqrt[3]{4c}$

31) $(2\sqrt{3}+10)^2$ $112+40\sqrt{3}$

32) $(\sqrt{2}+3)(\sqrt{2}-3)$ -7

33) $\dfrac{18}{\sqrt{6}}$ $3\sqrt{6}$

34) $\dfrac{5}{\sqrt{3}-\sqrt{2}}$ $5\sqrt{3}+5\sqrt{2}$

35) $3\sqrt{75m^3n} + m\sqrt{12mn}$ $17m\sqrt{3mn}$

36) $\sqrt{6p^7q^3} \cdot \sqrt{15pq^2}$ $3p^4q^2\sqrt{10q}$

37) $\dfrac{\sqrt{60t^8u^3}}{\sqrt{5t^2u}}$ $2t^3u\sqrt{3}$

38) $\dfrac{9}{\sqrt[3]{2}}$ $\dfrac{9\sqrt[3]{4}}{2}$

39) $(8-\sqrt{5w})^2$ $5w-16\sqrt{5w}+64$

40) $(2\sqrt{x}-\sqrt{y})(5\sqrt{x}-6\sqrt{y})$ $10x+6y-17\sqrt{xy}$

41) $\dfrac{\sqrt{2}}{4+\sqrt{a}}$ $\dfrac{4\sqrt{2}-\sqrt{2a}}{16-a}$

42) $\sqrt[3]{r^2} \cdot \sqrt{r}$ $r\sqrt[6]{r}$

43) $\sqrt[3]{\dfrac{b^2}{9c}}$ $\dfrac{\sqrt[3]{3b^2c^2}}{3c}$

44) $\dfrac{\sqrt[4]{32}}{\sqrt[4]{w^{11}}}$ $\dfrac{2\sqrt[4]{2w}}{w^3}$

For each function, find the domain and graph the function.

45) $f(x) = \sqrt{x-2}$

46) $g(x) = \sqrt[3]{x}$

47) $h(x) = \sqrt[3]{x} + 1$

48) $k(x) = \sqrt{x+1}$

49) $g(x) = \sqrt{-x}$

50) $f(x) = \sqrt{x} - 2$

Rethink

R1) Which types of problems did you struggle with the most? What made them difficult?

R2) Were you able to easily distinguish between the methods to use for the different problems in this homework? How often did you have to look back at your notes or the book to do the problems? If you had to take a quiz or test on this material now, would you be ready?

9.7 Solving Radical Equations

Prepare | **Organize**

What are your objectives for Section 9.7?	How can you accomplish each objective?
1 Understand the Steps for Solving a Radical Equation	• Write the procedure for **Solving Radical Equations Containing Square Roots** in your own words, and be aware of *extraneous solutions.*
2 Solve an Equation Containing One Square Root	• Use the steps for **Solving Radical Equations Containing Square Roots.** • Be aware that you will, often, have to square a binomial. • Complete the given examples on your own. • Complete You Trys 1 and 2.
3 Solve an Equation Containing Two Square Roots	• Use the steps for **Solving Radical Equations Containing Square Roots.** • Recognize when you need to square both sides of the equation twice. • Complete the given examples on your own. • Complete You Trys 3–5.
4 Solve an Equation Containing a Cube Root	• Follow the explanation to create your own procedure for solving radical equations containing a cube root. • Complete the given example on your own. • Complete You Try 6.

Work

Read the explanations, follow the examples, take notes, and complete the You Trys.

In this section, we will learn how to solve *radical equations.*

An equation containing a variable in the radicand is a **radical equation.** Some examples of radical equations are

$$\sqrt{p} = 7, \qquad \sqrt[3]{n} = 2, \qquad \sqrt{2x+1} + 1 = x, \qquad \sqrt{5w+6} - \sqrt{4w+1} = 1$$

1 Understand the Steps for Solving a Radical Equation

Let's review what happens when we square a square root expression: If $x \ge 0$, then $(\sqrt{x})^2 = x$. That is, to eliminate the radical from $\sqrt{x}$, we *square* the expression. Therefore to solve equations like those above containing *square roots,* we *square* both sides of the equation to obtain new equations. The solutions of the new equations contain all of the solutions of the original equation and may also contain *extraneous solutions.*

An **extraneous solution** is a value that satisfies one of the new equations but does not satisfy the original equation. Extraneous solutions occur frequently when solving radical equations, so we *must* check all possible solutions in the original equation and discard any that are extraneous.

Procedure Solving Radical Equations Containing Square Roots

Step 1: Get a radical on a side by itself.

Step 2: Square both sides of the equation to eliminate a radical.

Step 3: Combine like terms on each side of the equation.

Step 4: If the equation still contains a radical, repeat Steps 1–3.

Step 5: Solve the equation.

Step 6: Check the proposed solutions *in the original equation,* and discard extraneous solutions.

2 Solve an Equation Containing One Square Root

EXAMPLE 1

In-Class Example 1

Solve.
a) $\sqrt{b-8}=2$
b) $\sqrt{w+3}+7=3$

Answer: a) {12} b) $\varnothing$

W Hint

Write out the steps as you are reading them. Be sure to write neatly and in a very orderly way.

Solve.

a) $\sqrt{c-2}=3$ b) $\sqrt{t+5}+6=0$

Solution

a) ***Step 1:*** The radical *is* on a side by itself: $\sqrt{c-2}=3$

Step 2: *Square* both sides to eliminate the *square root.*

$$(\sqrt{c-2})^2 = 3^2 \qquad \text{Square both sides.}$$
$$c-2=9$$

Steps 3 and 4 do not apply because there are no like terms to combine and no radicals remain.

Step 5: Solve the equation.

$$c = 11 \qquad \text{Add 2 to each side.}$$

Step 6: Check $c=11$ in the *original* equation.

$$\sqrt{c-2}=3$$
$$\sqrt{11-2} \stackrel{?}{=} 3$$
$$\sqrt{9}=3 \quad \checkmark$$

The solution set is {11}.

b) The first step is to get the radical on a side by itself.

$$\sqrt{t+5}+6=0$$
$$\sqrt{t+5}=-6 \qquad \text{Subtract 6 from each side.}$$
$$(\sqrt{t+5})^2=(-6)^2 \qquad \text{Square both sides to eliminate the radical.}$$
$$t+5=36 \qquad \text{The square root has been eliminated.}$$
$$t=31 \qquad \text{Solve the equation.}$$

Check $t=31$ in the *original* equation.

$$\sqrt{t+5}+6=0$$
$$\sqrt{31+5}+6 \stackrel{?}{=} 0$$
$$6+6 \stackrel{?}{=} 0 \qquad \text{FALSE}$$

Because $t=31$ gives us a false statement, it is an *extraneous solution.* The equation has no real solution. The solution set is $\varnothing$.

YOU TRY 1

Solve.

a) $\sqrt{a+4} = 7$ b) $\sqrt{m-7} + 12 = 9$

Sometimes, we have to square a binomial in order to solve a radical equation. Don't forget that when we square a binomial we can either use FOIL or one of the following formulas: $(a+b)^2 = a^2 + 2ab + b^2$ or $(a-b)^2 = a^2 - 2ab + b^2$.

EXAMPLE 2

In-Class Example 2

Solve $\sqrt{2y+4} - 2 = y$.

Answer: $\{-2, 0\}$

Hint

Review Section 6.4 if you need help squaring a binomial.

Solve $\sqrt{2x+1} + 1 = x$.

Solution

Start by getting the radical on a side by itself.

$$\sqrt{2x+1} = x - 1 \quad \text{Subtract 1 from each side.}$$
$$(\sqrt{2x+1})^2 = (x-1)^2 \quad \text{Square both sides to eliminate the radical.}$$
$$2x + 1 = x^2 - 2x + 1 \quad \text{Simplify; square the binomial.}$$
$$0 = x^2 - 4x \quad \text{Subtract } 2x\text{; subtract 1.}$$
$$0 = x(x-4) \quad \text{Factor.}$$

$$x = 0 \quad \text{or} \quad x - 4 = 0 \quad \text{Set each factor equal to zero.}$$
$$x = 0 \quad \text{or} \quad x = 4 \quad \text{Solve.}$$

Check $x = 0$ and $x = 4$ in the *original* equation.

$x = 0$: $\sqrt{2x+1} + 1 = x$

$$\sqrt{2(0)+1} + 1 \stackrel{?}{=} 0$$
$$\sqrt{1} + 1 \stackrel{?}{=} 0$$
$$2 \stackrel{?}{=} 1 \quad \text{FALSE}$$

$x = 4$: $\sqrt{2x+1} + 1 = x$

$$\sqrt{2(4)+1} + 1 \stackrel{?}{=} 4$$
$$\sqrt{9} + 1 \stackrel{?}{=} 4$$
$$3 + 1 = 4 \quad \text{TRUE}$$

$x = 4$ *is* a solution but $x = 0$ is ***not*** because $x = 0$ does not satisfy the original equation. The solution set is $\{4\}$.

YOU TRY 2

Solve.

a) $\sqrt{3p+10} - 4 = p$ b) $\sqrt{4h-3} - h = -2$

3 Solve an Equation Containing Two Square Roots

Next, we will take our first look at solving an equation containing two square roots.

EXAMPLE 3

In-Class Example 3

Solve $\sqrt{3b+4} - 2\sqrt{b-1} = 0$.

Answer: $\{8\}$

Solve $\sqrt{2a+4} - 3\sqrt{a-5} = 0$.

Solution

Begin by getting a radical on a side by itself.

$$\sqrt{2a+4} = 3\sqrt{a-5} \quad \text{Add } 3\sqrt{a-5} \text{ to each side.}$$
$$(\sqrt{2a+4})^2 = (3\sqrt{a-5})^2 \quad \text{Square both sides to eliminate the radicals.}$$
$$2a + 4 = 9(a-5) \quad 3^2 = 9$$
$$2a + 4 = 9a - 45 \quad \text{Distribute.}$$
$$-7a = -49$$
$$a = 7 \quad \text{Solve.}$$

Check $a = 7$ in the original equation.

$$\begin{aligned}\sqrt{2a+4} - 3\sqrt{a-5} &= 0\\ \sqrt{2(7)+4} - 3\sqrt{7-5} &= 0\\ \sqrt{14+4} - 3\sqrt{2} &\stackrel{?}{=} 0\\ \sqrt{18} - 3\sqrt{2} &\stackrel{?}{=} 0\\ 3\sqrt{2} - 3\sqrt{2} &= 0 \quad \checkmark\end{aligned}$$

The solution set is $\{7\}$.

YOU TRY 3

Solve $4\sqrt{r-3} - \sqrt{6r+2} = 0$.

Recall from Section 9.5 that we can square binomials containing radical expressions just like we squared $(x-1)^2$ in Example 2. We can use FOIL or the formulas

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \text{or} \qquad (a-b)^2 = a^2 - 2ab + b^2$$

EXAMPLE 4

In-Class Example 4

Square and simplify $(6 + \sqrt{r+1})^2$.

Answer: $37 + 12\sqrt{r+1} + r$

W Hint

Make sure you understand this example before going on to the next.

Square and simplify $(3 - \sqrt{m+2})^2$.

Solution

Use the formula $(a-b)^2 = a^2 - 2ab + b^2$.

$$\begin{aligned}(3 - \sqrt{m+2})^2 &= (3)^2 - 2(3)(\sqrt{m+2}) + (\sqrt{m+2})^2 && \text{Substitute 3 for } a \text{ and } \sqrt{m+2} \text{ for } b.\\ &= 9 - 6\sqrt{m+2} + (m+2)\\ &= m + 11 - 6\sqrt{m+2} && \text{Combine like terms.}\end{aligned}$$

YOU TRY 4

Square and simplify each expression.

a) $(\sqrt{z} - 4)^2$ b) $(5 + \sqrt{3d-1})^2$

To solve the next two equations, we will have to square both sides of the equation twice to eliminate the radicals. Be very careful when you are squaring the binomials that contain a radical.

EXAMPLE 5

In-Class Example 5

Solve each equation.

a) $\sqrt{c} - \sqrt{c-9} = 1$

b) $\sqrt{7k+8} - \sqrt{3k+4} = 2$

Answer: a) $\{25\}$ b) $\{4\}$

W Hint

Always use parentheses to organize your work when squaring both sides.

Solve each equation.

a) $\sqrt{x+5} + \sqrt{x} = 5$ b) $\sqrt{5w+6} - \sqrt{4w+1} = 1$

Solution

a) This equation contains two radicals *and* a constant. Get one of the radicals on a side by itself, then square both sides.

$$\begin{aligned}\sqrt{x+5} &= 5 - \sqrt{x} && \text{Subtract } \sqrt{x} \text{ from each side.}\\ (\sqrt{x+5})^2 &= (5 - \sqrt{x})^2 && \text{Square both sides.}\\ x + 5 &= (5)^2 - 2(5)(\sqrt{x}) + (\sqrt{x})^2 && \text{Use the formula } (a-b)^2 = a^2 - 2ab + b^2.\\ x + 5 &= 25 - 10\sqrt{x} + x && \text{Simplify.}\end{aligned}$$

The equation still contains a radical. Therefore, repeat Steps 1–3. Begin by getting the radical on a side by itself.

$$5 = 25 - 10\sqrt{x} \quad \text{Subtract } x \text{ from each side.}$$
$$-20 = -10\sqrt{x} \quad \text{Subtract 25 from each side.}$$
$$2 = \sqrt{x} \quad \text{Divide by } -10.$$
$$2^2 = (\sqrt{x})^2 \quad \text{Square both sides.}$$
$$4 = x \quad \text{Solve.}$$

The check is left to the student. The solution set is $\{4\}$.

W Hint

These problems can be very long, so write out each step *carefully* and *neatly* to minimize mistakes.

b) ***Step 1:*** Get a radical on a side by itself.

$$\sqrt{5w+6} - \sqrt{4w+1} = 1$$
$$\sqrt{5w+6} = 1 + \sqrt{4w+1} \quad \text{Add } \sqrt{4w+1} \text{ to each side.}$$

Step 2: Square both sides of the equation to eliminate a radical.

$$(\sqrt{5w+6})^2 = (1 + \sqrt{4w+1})^2 \quad \text{Square both sides.}$$
$$5w+6 = (1)^2 + 2(1)(\sqrt{4w+1}) + (\sqrt{4w+1})^2 \quad \text{Use the formula } (a+b)^2 = a^2 + 2ab + b^2.$$
$$5w+6 = 1 + 2\sqrt{4w+1} + 4w + 1$$

Step 3: Combine like terms on the right side.

$$5w+6 = 4w + 2 + 2\sqrt{4w+1} \quad \text{Combine like terms.}$$

Step 4: The equation still contains a radical, so repeat Steps 1–3.

Step 1: Get the radical on a side by itself.

$$5w+6 = 4w+2+2\sqrt{4w+1}$$
$$w+4 = 2\sqrt{4w+1} \quad \text{Subtract } 4w \text{ and subtract 2.}$$

We do not need to eliminate the 2 from in front of the radical before squaring both sides. The radical must not be a part of a *sum* or *difference* when we square.

Step 2: Square both sides of the equation to eliminate the radical.

$$(w+4)^2 = (2\sqrt{4w+1})^2 \quad \text{Square both sides.}$$
$$w^2 + 8w + 16 = 4(4w+1) \quad \text{Square the binomial; } 2^2 = 4.$$

Steps 3 and 4 no longer apply.

Step 5: Solve the equation.

$$w^2 + 8w + 16 = 16w + 4 \quad \text{Distribute.}$$
$$w^2 - 8w + 12 = 0 \quad \text{Subtract } 16w \text{ and subtract 4.}$$
$$(w-2)(w-6) = 0 \quad \text{Factor.}$$
$$w-2 = 0 \quad \text{or} \quad w-6 = 0 \quad \text{Set each factor equal to zero.}$$
$$w = 2 \quad \text{or} \quad w = 6 \quad \text{Solve.}$$

Step 6: The check is left to the student. Verify that $w = 2$ and $w = 6$ each satisfy the original equation. The solution set is $\{2, 6\}$.

[YOU TRY 5]

Solve each equation.

a) $\sqrt{2y+1} - \sqrt{y} = 1$ b) $\sqrt{3t+4} + \sqrt{t+2} = 2$

BE CAREFUL Watch out for two common mistakes that students make when solving an equation like the one in Example 5b.

1) Do not square both sides before getting a radical on a side by itself.

This is incorrect: $(\sqrt{5w+6} - \sqrt{4w+1})^2 = 1^2$

$$5w+6-(4w+1) = 1$$

2) The *second* time we perform Step 2, watch out for this common error.

This is incorrect: $(w+4)^2 = (2\sqrt{4w+1})^2$

$$w^2+16 = 2(4w+1)$$

On the left we must multiply using FOIL or the formula $(a+b)^2 = a^2+2ab+b^2$, and on the right we must remember to square the 2.

4 Solve an Equation Containing a Cube Root

We can solve many equations containing cube roots the same way we solve equations containing square roots except, to eliminate a *cube root,* we *cube* both sides of the equation.

EXAMPLE 6

Solve $\sqrt[3]{7a+1} - 2\sqrt[3]{a-1} = 0$.

In-Class Example 6

Solve $2\sqrt[3]{5k+2} - 3\sqrt[3]{k+3} = 0$.

Answer: $\{5\}$

Hint Write a procedure to help you solve equations containing a cube root.

Solution

Begin by getting a radical on a side by itself.

$\sqrt[3]{7a+1} = 2\sqrt[3]{a-1}$	Add $2\sqrt[3]{a-1}$ to each side.
$(\sqrt[3]{7a+1})^3 = (2\sqrt[3]{a-1})^3$	Cube both sides to eliminate the radicals.
$7a+1 = 8(a-1)$	Simplify; $2^3 = 8$.
$7a+1 = 8a-8$	Distribute.
$9 = a$	Subtract $7a$; add 8.

Check $a = 9$ in the original equation.

$$\sqrt[3]{7a+1} - 2\sqrt[3]{a-1} = 0$$
$$\sqrt[3]{7(9)+1} - 2\sqrt[3]{9-1} \stackrel{?}{=} 0$$
$$\sqrt[3]{64} - 2\sqrt[3]{8} \stackrel{?}{=} 0$$
$$4-2(2) \stackrel{?}{=} 0$$
$$4-4 = 0 \quad ✓$$

The solution set is $\{9\}$.

[YOU TRY 6] Solve $3\sqrt[3]{r-4} - \sqrt[3]{5r+2} = 0$.

Using Technology

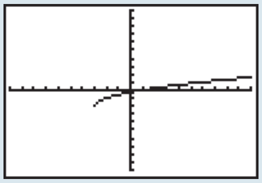
We can use a graphing calculator to solve a radical equation in one variable. First subtract every term on the right side of the equation from both sides of the equation, and enter the result in Y_1. Graph the equation in Y_1. The zeros or x-intercepts of the graph are the solutions to the equation.

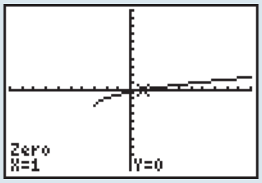
We will solve $\sqrt{x+3} = 2$ using a graphing calculator.

1) Enter $\sqrt{x+3} - 2$ in Y_1.

2) Press ZOOM 6 to graph the function in Y_1 as shown.

3) Press 2nd TRACE 2:zero, move the cursor to the left of the zero and press ENTER, move the cursor to the right of the zero and press ENTER, and move the cursor close to the zero and press ENTER to display the zero. The solution to the equation is $x = 1$.





Solve each equation using a graphing calculator.

1) $\sqrt{x-2} = 1$
2) $\sqrt{3x-2} = 5$
3) $\sqrt{3x-2} = \sqrt{x} + 2$
4) $\sqrt{4x-5} = \sqrt{x+4}$
5) $\sqrt{2x-7} = \sqrt{x} - 1$
6) $\sqrt{\sqrt{x}-1} = 1$

ANSWERS TO [YOU TRY] EXERCISES

1) a) $\{45\}$ b) $\varnothing$ 2) a) $\{-3, -2\}$ b) $\{7\}$ 3) $\{5\}$ 4) a) $z - 8\sqrt{z} + 16$ b) $3d + 24 + 10\sqrt{3d-1}$ 5) a) $\{0, 4\}$ b) $\{-1\}$ 6) $\{5\}$

ANSWERS TO TECHNOLOGY EXERCISES

1) $\{3\}$ 2) $\{9\}$ 3) $\{9\}$ 4) $\{3\}$ 5) $\{4\}$ 6) $\{4\}$

E Evaluate

9.7 Exercises

Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Objective 1: Understand the Steps for Solving a Radical Equation

1) Why is it necessary to check the proposed solutions to a radical equation in the original equation?
Sometimes there are extraneous solutions.

2) How do you know, without actually solving and checking the solution, that $\sqrt{y} = -3$ has no solution?
The principle square root of a number cannot equal a negative number.

Objective 2: Solve an Equation Containing One Square Root

Fill in the blank with *always, sometimes*, or *never* to make the statement true.

3) A negative number can ___sometimes___ be a solution to an equation containing a square root expression.

4) An equation containing a square root expression will ___sometimes___ have extraneous solutions.

Solve.

5) $\sqrt{q} = 7$ $\{49\}$
6) $\sqrt{z} = 10$ $\{100\}$
7) $\sqrt{w} - \frac{2}{3} = 0$ $\left\{\frac{4}{9}\right\}$
8) $\sqrt{r} - \frac{3}{5} = 0$ $\left\{\frac{9}{25}\right\}$
9) $\sqrt{a} + 5 = 3$ $\varnothing$
10) $\sqrt{k} + 8 = 2$ $\varnothing$
11) $\sqrt{b-11} - 3 = 0$ $\{20\}$
12) $\sqrt{d+3} - 5 = 0$ $\{22\}$
13) $\sqrt{4g-1} + 7 = 1$ $\varnothing$
14) $\sqrt{3v+4} + 10 = 6$ $\varnothing$
15) $\sqrt{3f+2} + 9 = 11$ $\left\{\frac{2}{3}\right\}$
16) $\sqrt{5u-4} + 12 = 17$ $\left\{\frac{29}{5}\right\}$
17) $m = \sqrt{m^2 - 3m + 6}$ $\{2\}$
18) $b = \sqrt{b^2 + 4b - 24}$ $\{6\}$
19) $\sqrt{9r^2 - 2r + 10} = 3r$ $\{5\}$
20) $\sqrt{4p^2 - 3p + 6} = 2p$ $\{2\}$

Square each binomial, and simplify.

21) $(n+5)^2$ $n^2+10n+25$

22) $(z-3)^2$ z^2-6z+9

23) $(c-6)^2$ $c^2-12c+36$

24) $(2k+1)^2$ $4k^2+4k+1$

Solve.

25) $c-7=\sqrt{2c+1}$ $\{12\}$

26) $p+6=\sqrt{12+p}$ $\{-3\}$

27) $6+\sqrt{c^2+3c-9}=c$ $\varnothing$

28) $-4+\sqrt{z^2+5z-8}=z$ $\varnothing$

29) $w-\sqrt{10w+6}=-3$ $\{1, 3\}$

30) $3-\sqrt{8t+9}=-t$ $\{0, 2\}$

31) $3v=8+\sqrt{3v+4}$ $\{4\}$

32) $4k=3+\sqrt{10k+5}$ $\{2\}$

33) $m+4=5\sqrt{m}$ $\{1, 16\}$

34) $b+5=6\sqrt{b}$ $\{1, 25\}$

35) $y+2\sqrt{6-y}=3$ $\{-3\}$

36) $r-3\sqrt{r+2}=2$ $\{14\}$

37) $\sqrt{r^2-8r-19}=r-9$ $\{10\}$

38) $\sqrt{x^2+x+4}=x+8$ $\{-4\}$

Objective 3: Solve an Equation Containing Two Square Roots

Solve.

39) $5\sqrt{1-5h}=4\sqrt{1-8h}$ $\{-3\}$

40) $3\sqrt{6a-2}-4\sqrt{3a+3}=0$ $\{11\}$

41) $3\sqrt{3x+6}-2\sqrt{9x-9}=0$ $\{10\}$

42) $5\sqrt{q+11}=2\sqrt{8q+25}$ $\{25\}$

43) $\sqrt{m}=3\sqrt{7}$ $\{63\}$

44) $4\sqrt{3}=\sqrt{p}$ $\{48\}$

45) $2\sqrt{3t+4}+\sqrt{t-6}=0$ $\varnothing$

46) $\sqrt{2w-1}+2\sqrt{w+4}=0$ $\varnothing$

Square each expression, and simplify.

47) $(\sqrt{x}+5)^2$ $x+10\sqrt{x}+25$

48) $(\sqrt{y}-8)^2$ $y-16\sqrt{y}+64$

49) $(9-\sqrt{a+4})^2$ $85-18\sqrt{a+4}+a$

50) $(4+\sqrt{p+5})^2$ $21+8\sqrt{p+5}+p$

51) $(2\sqrt{3n-1}+7)^2$ $12n+28\sqrt{3n-1}+45$

52) $(5-3\sqrt{2k-3})^2$ $18k-30\sqrt{2k-3}-2$

Solve.

53) $\sqrt{2y-1}=2+\sqrt{y-4}$ $\{5, 13\}$

54) $\sqrt{3n+4}=\sqrt{2n+1}+1$ $\{0, 4\}$

55) $1+\sqrt{3s-2}=\sqrt{2s+5}$ $\{2\}$

56) $\sqrt{4p+12}-1=\sqrt{6p-11}$ $\{6\}$

57) $\sqrt{5a+19}-\sqrt{a+12}=1$ $\left\{\frac{1}{4}\right\}$

58) $\sqrt{2u+3}-\sqrt{5u+1}=-1$ $\{3\}$

59) $\sqrt{3k+1}-\sqrt{k-1}=2$ $\{1, 5\}$

60) $\sqrt{4z-3}-\sqrt{5z+1}=-1$ $\{3, 7\}$

61) $\sqrt{3x+4}-5=\sqrt{3x-11}$ $\varnothing$

62) $\sqrt{4c-7}=\sqrt{4c+1}-4$ $\varnothing$

63) $\sqrt{3v+3}-\sqrt{v-2}=3$ $\{2, 11\}$

64) $\sqrt{2y+1}-\sqrt{y}=1$ $\{0, 4\}$

Objective 4: Solve an Equation Containing a Cube Root

65) How do you eliminate the radical from an equation like $\sqrt[3]{x}=2$? Raise both sides of the equation to the third power.

66) Give a reason why $\sqrt[3]{h}=-3$ has no extraneous solutions. When you solve the equation you get $h=-27$, and the cube root of -27 is -3.

Solve.

67) $\sqrt[3]{y}=5$ $\{125\}$

68) $\sqrt[3]{c}=3$ $\{27\}$

69) $\sqrt[3]{m}=-4$ $\{-64\}$

70) $\sqrt[3]{t}=-2$ $\{-8\}$

71) $\sqrt[3]{2x-5}+3=1$ $\left\{-\frac{3}{2}\right\}$

72) $\sqrt[3]{4a+1}+7=4$ $\{-7\}$

73) $\sqrt[3]{6j-2}=\sqrt[3]{j-7}$ $\{-1\}$

74) $\sqrt[3]{w+3}=\sqrt[3]{2w-11}$ $\{14\}$

75) $\sqrt[3]{3y-1}-\sqrt[3]{2y-3}=0$ $\{-2\}$

76) $\sqrt[3]{2-2b}+\sqrt[3]{b-5}=0$ $\{-3\}$

77) $\sqrt[3]{2n^2}=\sqrt[3]{7n+4}$ $\left\{-\frac{1}{2}, 4\right\}$

78) $\sqrt[3]{4c^2-5c+11}=\sqrt[3]{c^2+9}$ $\left\{\frac{2}{3}, 1\right\}$

Extension

Solve.

79) $p^{1/2}=6$ $\{36\}$

80) $\frac{2}{3}=t^{1/2}$ $\left\{\frac{4}{9}\right\}$

81) $7=(2z-3)^{1/2}$ $\{26\}$

82) $(3k+1)^{1/2}=4$ $\{5\}$

83) $(y+4)^{1/3}=3$ $\{23\}$

84) $-5=(a-2)^{1/3}$ $\{-123\}$

85) $\sqrt[4]{n+7}=2$ $\{9\}$

86) $\sqrt[4]{x-3}=-1$ $\varnothing$

87) $\sqrt{13+\sqrt{r}}=\sqrt{r+7}$ $\{9\}$

88) $\sqrt{m-1}=\sqrt{m-\sqrt{m-4}}$ $\{5\}$

89) $\sqrt{y+\sqrt{y+5}}=\sqrt{y+2}$ $\{-1\}$

90) $\sqrt{2d-\sqrt{d+6}}=\sqrt{d+6}$ $\{10\}$

Mixed Exercises: Objectives 2 and 4

Solve for the indicated variable.

91) $v=\sqrt{\frac{2E}{m}}$ for E $E=\frac{mv^2}{2}$

92) $V=\sqrt{\frac{300VP}{m}}$ for P $P=\frac{mV}{300}$

93) $c=\sqrt{a^2+b^2}$ for b^2 $b^2=c^2-a^2$

94) $r=\sqrt{\frac{A}{\pi}}$ for A $A=\pi r^2$

95) $T = \sqrt[4]{\dfrac{E}{\sigma}}$ for σ $\quad \sigma = \dfrac{E}{T^4}$

96) $r = \sqrt[3]{\dfrac{3V}{4\pi}}$ for V $\quad V = \dfrac{4}{3}\pi r^3$

97) The speed of sound is proportional to the square root of the air temperature in still air. The speed of sound is given by the formula.

$$V_S = 20\sqrt{T + 273}$$

where V_S is the speed of sound in meters/second and T is the temperature of the air in °Celsius.

a) What is the speed of sound when the temperature is -17°C (about 1°F)? 320 m/sec

b) What is the speed of sound when the temperature is 16°C (about 61°F)? 340 m/sec

c) What happens to the speed of sound as the temperature increases? The speed of sound increases.

d) Solve the equation for T. $T = \dfrac{V_s^2}{400} - 273$

98) If the area of a square is A and each side has length l, then the length of a side is given by

$$l = \sqrt{A}$$

A square rug has an area of 25 ft^2.

a) Find the dimensions of the rug. 5 ft × 5 ft

b) Solve the equation for A. $A = l^2$

99) Let V represent the volume of a cylinder, h represent its height, and r represent its radius. V, h, and r are related according to the formula

$$r = \sqrt{\frac{V}{\pi h}}$$

a) A cylindrical soup can has a volume of 28π in^3. It is 7 in. high. What is the radius of the can? 2 in.

b) Solve the equation for V. $V = \pi r^2 h$

100) For shallow water waves, the wave velocity is given by

$$c = \sqrt{gH}$$

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where g is the acceleration due to gravity (32 ft/sec^2) and H is the depth of the water (in feet).

a) Find the velocity of a wave in 8 ft of water. 16 ft/sec

b) Solve the equation for H. $H = \dfrac{c^2}{g}$

101) Refer to the formula given in Exercise 100.

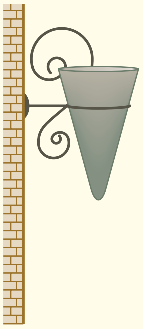
The catastrophic Indian Ocean tsunami that hit Banda Aceh, Sumatra, Indonesia, on December 26, 2004, was caused by an earthquake whose epicenter was off the coast of northern Sumatra. The tsunami originated in about 14,400 ft of water.

a) Find the velocity of the wave near the epicenter, in miles per hour. Round the answer to the nearest unit. (Hint: 1 mile = 5280 ft.) 463 mph

b) Banda Aceh, the area hardest hit by the tsunami, was about 60 mi from the tsunami's origin. Approximately how many minutes after the earthquake occurred did the tsunami hit Banda Aceh? (*Exploring Geology*, McGraw-Hill, 2008.) about 8 min.

102) The radius r of a cone with height h and volume V is given by $r = \sqrt{\dfrac{3V}{\pi h}}$.

A hanging glass vase in the shape of a cone is 8 in. tall, and the radius of the top of the cone is 2 in. How much water will the vase hold? Give an exact answer and an approximation to the tenths place. $\dfrac{32}{3}\pi$ in^3; 33.5 in^3



Use the following information for Exercises 103 and 104.

The distance a person can see to the horizon is approximated by the function $D(h) = 1.2\sqrt{h}$, where D is the number of miles a person can see to the horizon from a height of h ft.

103) Sig is the captain of an Alaskan crab fishing boat and can see 4.8 mi to the horizon when he is sailing his ship. Find his height above the sea. 16 ft

104) Phil is standing on the deck of a boat and can see 3.6 mi to the horizon. What is his height above the water? 9 ft

Use the following information for Exercises 105 and 106.

When the air temperature is 0°F, the wind chill temperature, W, in degrees Fahrenheit is a function of the velocity of the wind, V, in miles per hour and is given by the formula

$$W(V) = 35.74 - 35.75V^{4/25}$$

105) Calculate the wind speed when the wind chill temperature is -10°F. Round to the nearest whole number 5 mph

106) Find V so that $W(V) = -20$. Round to the nearest whole number. Explain your result in the context of the problem. (www.nws.noaa.gov)
When the wind chill temperature is -20°F, the speed of the wind is 16 mph.

R Rethink

R1) How has a good understanding of multiplying binomials helped you complete these exercises?

R2) Look at Exercise 57. Next to your work solving this equation, explain, in words, the steps you used to solve the equation.

9.8 Complex Numbers

P Prepare | **O Organize**

What are your objectives for Section 9.8?	How can you accomplish each objective?
1 Find the Square Root of a Negative Number	• Understand the definition of the *imaginary number, i.* • Learn the definition of a *complex number.* • Understand the property of **Real Numbers and Complex Numbers.** • Complete the given example on your own. • Complete You Try 1.
2 Multiply and Divide Square Roots Containing Negative Numbers	• Follow the explanation and example to create a procedure for this objective. • Complete the given example on your own. • Complete You Try 2.
3 Add and Subtract Complex Numbers	• Write the procedure for **Adding and Subtracting Complex Numbers** in your own words. • Complete the given example on your own. • Complete You Try 3.
4 Multiply Complex Numbers	• Compare the steps for multiplying complex numbers to the notes you took on multiplying polynomials. • Complete the given example on your own. • Complete You Try 4.
5 Multiply a Complex Number by Its Conjugate	• Understand the definition of a *complex conjugate.* • Follow the explanation, and write the summary of **Complex Conjugates** in your notes. • Complete the given example on your own. • Complete You Try 5.
6 Divide Complex Numbers	• Write the procedure for **Dividing Complex Numbers** in your own words. • Complete the given example on your own. • Complete You Try 6.
7 Simplify Powers of i	• Follow the explanation, and summarize how to simplify powers of i in your notes. • Complete the given example on you own. • Complete You Try 7.

Work Read the explanations, follow the examples, take notes, and complete the You Trys.

1 Find the Square Root of a Negative Number

We have seen throughout this chapter that the square root of a negative number does not exist in the real number system because there is no real number that, when squared, will result in a negative number. For example, $\sqrt{-4}$ is not a real number because there is no real number whose square is -4.

The square roots of negative numbers do exist, however, under another system of numbers called *complex numbers*. Before we define a complex number, we must define the number i. The number i is called an *imaginary number*.

Definition

The **imaginary number *i*** is defined as

$$i = \sqrt{-1}.$$

Therefore, squaring both sides gives us

$$i^2 = -1.$$

Note
$i = \sqrt{-1}$ and $i^2 = -1$ are two *very* important facts to remember. We will be using them often!

Definition

A **complex number** is a number of the form $a + bi$, where a and b are real numbers; a is called the **real part,** and b is called the **imaginary part.**

The following table lists some examples of complex numbers and their real and imaginary parts.

Complex Number	Real Part	Imaginary Part
$-5 + 2i$	-5	2
$\frac{1}{3} - 7i$	$\frac{1}{3}$	-7
$8i$	0	8
4	4	0

Note
The complex number $8i$ can be written in the form $a + bi$ as $0 + 8i$. Likewise, besides being a real number, 4 is a complex number because it can be written as $4 + 0i$.

Because all real numbers, a, can be written in the form $a + 0i$, all real numbers are also complex numbers.

Property Real Numbers and Complex Numbers

The set of real numbers is a subset of the set of complex numbers.

Since we defined i as $i = \sqrt{-1}$, we can now evaluate square roots of negative numbers.

EXAMPLE 1

In-Class Example 1

Simplify.
a) $\sqrt{-4}$ b) $\sqrt{-11}$ c) $\sqrt{-54}$

Answer: a) $2i$ b) $i\sqrt{11}$ c) $3i\sqrt{6}$

Hint
Create a procedure for simplifying square roots of negative numbers.

Simplify.

a) $\sqrt{-9}$ b) $\sqrt{-7}$ c) $\sqrt{-12}$

Solution

a) $\sqrt{-9} = \sqrt{-1 \cdot 9} = \sqrt{-1} \cdot \sqrt{9} = i \cdot 3 = 3i$

b) $\sqrt{-7} = \sqrt{-1 \cdot 7} = \sqrt{-1} \cdot \sqrt{7} = i\sqrt{7}$

c) $\sqrt{-12} = \sqrt{-1 \cdot 12} = \sqrt{-1} \cdot \sqrt{12} = i\sqrt{4}\sqrt{3} = i \cdot 2\sqrt{3} = 2i\sqrt{3}$

Note
In Example 1b) we wrote $i\sqrt{7}$ instead of $\sqrt{7}i$, and in Example 1c) we wrote $2i\sqrt{3}$ instead of $2\sqrt{3}i$. We do this to be clear that the i is not under the radical. It is good practice to write the i *before* the radical.

YOU TRY 1

Simplify.

a) $\sqrt{-36}$ b) $\sqrt{-13}$ c) $\sqrt{-20}$

2 Multiply and Divide Square Roots Containing Negative Numbers

When multiplying or dividing square roots with negative radicands, write each radical in terms of i first. Remember, also, that since $i = \sqrt{-1}$ it follows that $i^2 = -1$. We must keep this in mind when simplifying expressions.

Note
Whenever an i^2 appears in an expression, replace it with -1.

EXAMPLE 2

In-Class Example 2

Multiply and simplify.
$\sqrt{-12} \cdot \sqrt{-3}$

Answer: -6

Multiply and simplify. $\sqrt{-8} \cdot \sqrt{-2}$

Solution

$\sqrt{-8} \cdot \sqrt{-2} = i\sqrt{8} \cdot i\sqrt{2}$ Write each radical in terms of i *before* multiplying.

$= i^2\sqrt{16}$ Multiply.

$= (-1)(4)$ Replace i^2 with -1.

$= -4$

YOU TRY 2

Perform the operation, and simplify.

a) $\sqrt{-6} \cdot \sqrt{-3}$ b) $\dfrac{\sqrt{-72}}{\sqrt{-2}}$

3 Add and Subtract Complex Numbers

Just as we can add, subtract, multiply, and divide real numbers, we can perform all of these operations with complex numbers.

Procedure Adding and Subtracting Complex Numbers

1) To add complex numbers, add the real parts and add the imaginary parts.
2) To subtract complex numbers, apply the distributive property and combine the real parts and combine the imaginary parts.

EXAMPLE 3

In-Class Example 3

Add or subtract.
a) $(7 + 5i) + (3 + 6i)$
b) $(4 - 3i) - (10 + 2i)$

Answer: a) $10 + 11i$
b) $-6 - 5i$

Add or subtract.

a) $(8 + 3i) + (4 + 2i)$ b) $(7 + i) - (3 - 4i)$

Solution

a) $(8 + 3i) + (4 + 2i) = (8 + 4) + (3 + 2)i$ Add real parts; add imaginary parts.
$= 12 + 5i$

b) $(7 + i) - (3 - 4i) = 7 + i - 3 + 4i$ Distributive property
$= (7 - 3) + (1 + 4)i$ Add real parts; add imaginary parts.
$= 4 + 5i$

YOU TRY 3

Add or subtract.

a) $(-10 + 6i) + (1 + 8i)$ b) $(2 - 5i) - (-1 + 6i)$

4 Multiply Complex Numbers

We multiply complex numbers just like we would multiply polynomials. There may be an additional step, however. Remember to replace i^2 with -1.

EXAMPLE 4

In-Class Example 4

Multiply and simplify.
a) $7(4 - 11i)$
b) $(9 - 5i)(-2 + 3i)$
c) $(9 + i)(9 - i)$

Answer: a) $28 - 77i$
b) $-3 + 37i$ c) 82

Multiply and simplify.

a) $5(-2 + 3i)$ b) $(8 + 3i)(-1 + 4i)$ c) $(6 + 2i)(6 - 2i)$

Solution

a) $5(-2 + 3i) = -10 + 15i$ Distributive property

Hint

How does this compare to the techniques you learned in Section 6.4?

b) Look carefully at $(8 + 3i)(-1 + 4i)$. Each complex number has two terms, similar to, say, $(x + 3)(x + 4)$. How can we multiply these two binomials? We can use FOIL.

$$\begin{aligned} & \qquad\quad\ \ \text{F} \qquad\quad\ \text{O} \qquad\quad\ \text{I} \qquad\quad\ \text{L} \\ (8+3i)(-1+4i) &= (8)(-1) + (8)(4i) + (3i)(-1) + (3i)(4i) \\ &= \ -8 \quad + \ 32i \quad - \quad 3i \quad + \ 12i^2 \\ &= -8 + 29i + 12(-1) && \text{Replace } i^2 \text{ with } -1. \\ &= -8 + 29i - 12 \\ &= -20 + 29i \end{aligned}$$

c) Use FOIL to find the product $(6 + 2i)(6 - 2i)$.

$$\begin{aligned} & \qquad\quad\ \text{F} \qquad\quad\ \text{O} \qquad\quad\ \text{I} \qquad\quad\ \text{L} \\ 6+2i)(6-2i) &= (6)(6) + (6)(-2i) + (2i)(6) + (2i)(-2i) \\ &= \ 36 \quad - \ 12i \quad + \ 12i \quad - \quad 4i^2 \\ &= 36 - 4(-1) && \text{Replace } i^2 \text{ with } -1. \\ &= 36 + 4 \\ &= 40 \end{aligned}$$

[YOU TRY 4]

Multiply and simplify.

a) $-3(6 - 7i)$ b) $(5 + i)(4 + 8i)$ c) $(-2 - 9i)(-2 + 9i)$

5 Multiply a Complex Number by Its Conjugate

In Section 9.6, we learned about conjugates of radical expressions. For example, the conjugate of $3 + \sqrt{5}$ is $3 - \sqrt{5}$.

The complex numbers in Example 4c, $6 + 2i$ and $6 - 2i$, are **complex conjugates.**

Definition

The **conjugate** of $a + bi$ is $a - bi$.

We found that $(6 + 2i)(6 - 2i) = 40$ which is a real number. **The product of a complex number and its conjugate is *always* a real number,** as illustrated next.

$$\begin{aligned} & \qquad\quad\ \text{F} \qquad\quad\ \text{O} \qquad\quad\ \text{I} \qquad\quad\ \text{L} \\ (a+bi)(a-bi) &= (a)(a) + (a)(-bi) + (bi)(a) + (bi)(-bi) \\ &= \ a^2 \quad - \ abi \quad + \ abi \quad - \ b^2i^2 \\ &= a^2 - b^2(-1) && \text{Replace } i^2 \text{ with } -1. \\ &= a^2 + b^2 \end{aligned}$$

We can summarize these facts about complex numbers and their conjugates as follows:

Summary Complex Conjugates

1) The conjugate of $a + bi$ is $a - bi$.
2) The product of $a + bi$ and $a - bi$ is a real number.
3) We can find the product $(a + bi)(a - bi)$ by using FOIL or by using $(a + bi)(a - bi) = a^2 + b^2$.

EXAMPLE 5

In-Class Example 5

Multiply $-7 + 2i$ by its conjugate using the formula $(a + bi)(a - bi) = a^2 + b^2$.

Answer: 53

Hint
Always use parentheses!

Multiply $-3 + 4i$ by its conjugate using the formula $(a + bi)(a - bi) = a^2 + b^2$.

Solution

The conjugate of $-3 + 4i$ is $-3 - 4i$.

$$\begin{aligned}(-3 + 4i)(-3 - 4i) &= (-3)^2 + (4)^2 && a = -3,\ b = 4\\ &= 9 + 16\\ &= 25\end{aligned}$$

YOU TRY 5 Multiply $2 - 9i$ by its conjugate using the formula $(a + bi)(a - bi) = a^2 + b^2$.

6 Divide Complex Numbers

To rationalize the denominator of a radical expression like $\dfrac{2}{3 + \sqrt{5}}$, we multiply the numerator and denominator by $3 - \sqrt{5}$, the conjugate of the denominator. We divide complex numbers in the same way.

Procedure Dividing Complex Numbers

To divide complex numbers, multiply the numerator and denominator by the *conjugate of the denominator*. Write the quotient in the form $a + bi$.

EXAMPLE 6

In-Class Example 6

Divide. Write the result in the form $a + bi$.

a) $\dfrac{9}{6 + 3i}$ b) $\dfrac{-3 + i}{4 - 11i}$

Answer: a) $\dfrac{6}{5} - \dfrac{3}{5}i$
b) $-\dfrac{23}{137} - \dfrac{29}{137}i$

Hint
Be sure to use parentheses so that you distribute correctly.

Divide. Write the quotient in the form $a + bi$.

a) $\dfrac{3}{4 - 5i}$ b) $\dfrac{6 - 2i}{-7 + i}$

Solution

a)
$$\begin{aligned}\frac{3}{4 - 5i} &= \frac{3}{(4 - 5i)} \cdot \frac{(4 + 5i)}{(4 + 5i)} && \text{Multiply the numerator and denominator by the conjugate of the denominator.}\\ &= \frac{12 + 15i}{(4)^2 + (5)^2} && \text{Multiply numerators. } (a + bi)(a - bi) = a^2 + b^2\\ &= \frac{12 + 15i}{16 + 25}\\ &= \frac{12 + 15i}{41}\\ &= \frac{12}{41} + \frac{15}{41}i && \text{Write the quotient in the form } a + bi.\end{aligned}$$

Recall that we can find the product $(4 - 5i)(4 + 5i)$ using FOIL *or* by using the formula $(a + bi)(a - bi) = a^2 + b^2$.

b)
$$\begin{aligned}\frac{6 - 2i}{-7 + i} &= \frac{(6 - 2i)}{(-7 + i)} \cdot \frac{(-7 - i)}{(-7 - i)} && \text{Multiply the numerator and denominator by the conjugate of the denominator.}\\ &= \frac{-42 - 6i + 14i + 2i^2}{(-7)^2 + (1)^2} && \text{Multiply using FOIL. } (a + bi)(a - bi) = a^2 + b^2\\ &= \frac{-42 + 8i - 2}{49 + 1} = \frac{-44 + 8i}{50} = -\frac{44}{50} + \frac{8}{50}i = -\frac{22}{25} + \frac{4}{25}i\end{aligned}$$

YOU TRY 6

Divide. Write the result in the form $a + bi$.

a) $\dfrac{9}{2+i}$ b) $\dfrac{5+3i}{-6-4i}$

7 Simplify Powers of i

All powers of i larger than i^1 (or just i) can be simplified. We use the fact that $i^2 = -1$ to simplify powers of i.

Let's write i through i^4 in their simplest forms.

i is in simplest form.

$$i^2 = -1$$
$$i^3 = i^2 \cdot i = -1 \cdot i = -i$$
$$i^4 = (i^2)^2 = (-1)^2 = 1$$

Let's continue by simplifying i^5 and i^6.

$$\begin{aligned} i^5 &= i^4 \cdot i \\ &= (i^2)^2 \cdot i \\ &= (-1)^2 \cdot i \\ &= 1i \\ &= i \end{aligned} \qquad \begin{aligned} i^6 &= (i^2)^3 \\ &= (-1)^3 \\ &= -1 \end{aligned}$$

The pattern repeats so that all powers of i can be simplified to i, -1, $-i$, or 1.

EXAMPLE 7

In-Class Example 7

Simplify each power of i.
a) i^{12} b) i^{22} c) i^{15} d) i^{41}

Answer: a) 1 b) -1 c) $-i$ d) i

Simplify each power of i.

a) i^8 b) i^{14} c) i^{11} d) i^{37}

Solution

a) Use the power rule for exponents to simplify i^8. Since the exponent is even, we can rewrite it in terms of i^2.

$i^8 = (i^2)^4$	Power rule
$= (-1)^4$	$i^2 = -1$
$= 1$	Simplify.

b) As in Example 7a), the exponent is even. Rewrite i^{14} in terms of i^2.

$i^{14} = (i^2)^7$	Power rule
$= (-1)^7$	$i^2 = -1$
$= -1$	Simplify.

c) The exponent of i^{11} is odd, so first use the product rule to write i^{11} as a product of i and i^{11-1} or i^{10}.

$i^{11} = i^{10} \cdot i$	Product rule
$= (i^2)^5 \cdot i$	10 is even; write i^{10} in terms of i^2.
$= (-1)^5 \cdot i$	$i^2 = -1$
$= -1 \cdot i$	Simplify.
$= -i$	Multiply.

d) The exponent of i^{37} is odd. Use the product rule to write i^{37} as a product of i and i^{37-1} or i^{36}.

$i^{37} = i^{36} \cdot i$	Product rule
$= (i^2)^{18} \cdot i$	36 is even; write i^{36} in terms of i^2.
$= (-1)^{18}$	$i^2 = -1$
$= 1 \cdot i$	Simplify.
$= i$	Multiply.

YOU TRY 7

Simplify each power of i.

a) i^{18} b) i^{32} c) i^7 d) i^{25}

Using Technology

We can use a graphing calculator to perform operations on complex numbers or to evaluate square roots of negative numbers.

If the calculator is in the default REAL mode the result is an error message "ERR: NONREAL ANS," which indicates that $\sqrt{-4}$ is a complex number rather than a real number. Before evaluating $\sqrt{-4}$ on the home screen of your calculator, check the mode by pressing MODE and looking at row 7. Change the mode to complex numbers by selecting $a + bi$, as shown at the left below.

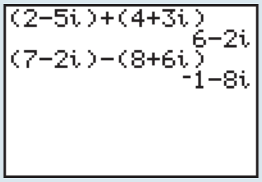
Now evaluating $\sqrt{-4}$ on the home screen results in the correct answer $2i$, as shown on the right below.

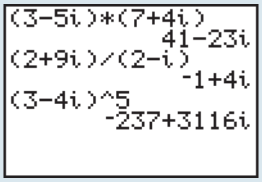
Operations can be performed on complex numbers with the calculator in either REAL or $a + bi$ mode. Simply use the arithmetic operators on the right column on your calculator. To enter the imaginary number i, press 2nd . . To add $2 - 5i$ and $4 + 3i$, enter $(2 - 5i) + (4 + 3i)$ on the home screen and press ENTER as shown on the left screen below. To subtract $8 + 6i$ from $7 - 2i$, enter $(7 - 2i) - (8 + 6i)$ on the home screen and press ENTER as shown.

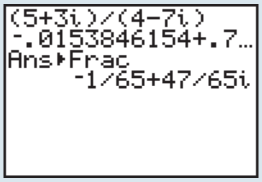
To multiply $3 - 5i$ and $7 + 4i$, enter $(3 - 5i) \cdot (7 + 4i)$ on the home screen and press ENTER as shown on the middle screen below. To divide $2 + 9i$ by $2 - i$, enter $(2 + 9i)/(2 - i)$ on the home screen and press ENTER as shown.

To raise $3 - 4i$ to the fifth power, enter $(3 - 4i)$^5 on the home screen and press ENTER as shown.

Consider the quotient $(5 + 3i)/(4 - 7i)$. The exact answer is $-\frac{1}{65} + \frac{47}{65}i$. The calculator automatically displays the decimal result. Press MATH 1 ENTER to convert the decimal result to the exact fractional result, as shown on the right screen below.







Perform the indicated operation using a graphing calculator.

1) Simplify $\sqrt{-36}$ 2) $(3 + 7i) + (5 - 8i)$ 3) $(10 - 3i) - (4 + 8i)$

4) $(3 + 2i)(6 - 3i)$ 5) $(4 + 3i) \div (1 - i)$ 6) $(5 - 3i)^3$

ANSWERS TO [YOU TRY] EXERCISES

1) a) $6i$ b) $i\sqrt{13}$ c) $2i\sqrt{5}$ 2) a) $-3\sqrt{2}$ b) 6 3) a) $-9 + 14i$ b) $3 - 11i$

4) a) $-18 + 21i$ b) $12 + 44i$ c) 85 5) 85 6) a) $\frac{18}{5} - \frac{9}{5}i$ b) $-\frac{21}{26} + \frac{1}{26}i$

7) a) -1 b) 1 c) $-i$ d) i

ANSWERS TO TECHNOLOGY EXERCISES

1) $6i$ 2) $8 - i$ 3) $6 - 11i$ 4) $24 + 3i$ 5) $\frac{1}{2} + \frac{7}{2}i$ 6) $-10 - 198i$

E Evaluate 9.8 Exercises

Do the exercises, and check your work.

*Additional answers can be found in the Answers to Exercises appendix.

Objective 1: Find the Square Root of a Negative Number

Determine if each statement is true or false.

1) Every complex number is a real number. False
2) Every real number is a complex number. True
3) Since $i = \sqrt{-1}$, it follows that $i^2 = -1$. True
4) In the complex number $-6 + 5i$, -6 is the real part and $5i$ is the imaginary part. False

Simplify.

5) $\sqrt{-81}$ $9i$
6) $\sqrt{-16}$ $4i$
7) $\sqrt{-25}$ $5i$
8) $\sqrt{-169}$ $13i$
9) $\sqrt{-6}$ $i\sqrt{6}$
10) $\sqrt{-30}$ $i\sqrt{30}$
11) $\sqrt{-27}$ $3i\sqrt{3}$
12) $\sqrt{-75}$ $5i\sqrt{3}$
13) $\sqrt{-60}$ $2i\sqrt{15}$
14) $\sqrt{-28}$ $2i\sqrt{7}$

Objective 2: Multiply and Divide Square Roots Containing Negative Numbers

Find the error in each of the following exercises, then find the correct answer.

15) $$\begin{aligned}\sqrt{-5}\cdot\sqrt{-10} &= \sqrt{-5\cdot(-10)}\\ &= \sqrt{50}\\ &= \sqrt{25}\cdot\sqrt{2}\\ &= 5\sqrt{2}\end{aligned}$$

16) $$\begin{aligned}(\sqrt{-7})^2 &= \sqrt{(-7)^2}\\ &= \sqrt{49}\\ &= 7\end{aligned}$$

Perform the indicated operation, and simplify.

17) $\sqrt{-1}\cdot\sqrt{-5}$ $-\sqrt{5}$
18) $\sqrt{-5}\cdot\sqrt{-15}$ $-5\sqrt{3}$
19) $\sqrt{-20}\cdot\sqrt{-5}$ -10
20) $\sqrt{-12}\cdot\sqrt{-3}$ -6
21) $\dfrac{\sqrt{-60}}{\sqrt{-15}}$ 2
22) $\dfrac{\sqrt{-2}}{\sqrt{-128}}$ $\frac{1}{8}$
23) $(\sqrt{-13})^2$ -13
24) $(\sqrt{-1})^2$ -1

Mixed Exercises: Objectives 3–6

25) Explain how to add complex numbers.
Add the real parts, and add the imaginary parts.
26) How is multiplying $(1 + 3i)(2 - 7i)$ similar to multiplying $(x + 3)(2x - 7)$?
Both are products of binomials, so we can multiply both using FOIL.
27) When i^2 appears in an expression, it should be replaced with what? -1
28) Explain how to divide complex numbers. Multiply the numerator and denominator by the conjugate of the denominator.

Objective 3: Add and Subtract Complex Numbers

Perform the indicated operations.

29) $(-4 + 9i) + (7 + 2i)$ $3 + 11i$
30) $(6 + i) + (8 - 5i)$ $14 - 4i$
31) $(13 - 8i) - (9 + i)$ $4 - 9i$
32) $(-12 + 3i) - (-7 - 6i)$ $-5 + 9i$
33) $\left(-\frac{3}{4} - \frac{1}{6}i\right) - \left(-\frac{1}{2} + \frac{2}{3}i\right)$ $-\frac{1}{4} - \frac{5}{6}i$
34) $\left(\frac{1}{2} + \frac{7}{9}i\right) - \left(\frac{7}{8} - \frac{1}{6}i\right)$ $-\frac{3}{8} + \frac{17}{18}i$
35) $16i - (3 + 10i) + (3 + i)$ $7i$
36) $(-6 - 5i) + (2 + 6i) - (-4 + i)$ 0

Objective 4: Multiply Complex Numbers

Multiply and simplify.

37) $3(8 - 5i)$ $24 - 15i$
38) $-6(8 - i)$ $-48 + 6i$
39) $\frac{2}{3}(-9 + 2i)$ $-6 + \frac{4}{3}i$
40) $\frac{1}{2}(18 + 7i)$ $9 + \frac{7}{2}i$
41) $-4i(6 + 11i)$ $44 - 24i$
42) $6i(5 + 6i)$ $-36 + 30i$
43) $(2 + 5i)(1 + 6i)$ $-28 + 17i$
44) $(2 + i)(10 + 5i)$ $15 + 20i$
45) $(-1 + 3i)(4 - 6i)$ $14 + 18i$
46) $(-4 - 9i)(3 - i)$ $-21 - 23i$
47) $(5 - 3i)(9 - 3i)$ $36 - 42i$
48) $(3 - 4i)(6 + 7i)$ $46 - 3i$
49) $\left(\frac{3}{4} + \frac{3}{4}i\right)\left(\frac{2}{5} + \frac{1}{5}i\right)$
50) $\left(\frac{1}{3} - \frac{4}{3}i\right)\left(\frac{3}{4} + \frac{2}{3}i\right)$

Objective 5: Multiply a Complex Number by Its Conjugate

Identify the conjugate of each complex number, then multiply the number and its conjugate.

51) $11 + 4i$ conjugate: $11 - 4i$; product: 137
52) $-1 - 2i$ conjugate: $-1 + 2i$; product: 5
53) $-3 - 7i$ conjugate: $-3 + 7i$; product: 58
54) $4 + 9i$ conjugate: $4 - 9i$; product: 97
55) $-6 + 4i$ conjugate: $-6 - 4i$; product: 52
56) $6 - 5i$ conjugate: $6 + 5i$; product: 61

57) How are conjugates of complex numbers like conjugates of expressions containing real numbers and radicals? Answers may vary.

58) Is the product of two complex numbers always a complex number? Explain your answer. No. For example, a complex number times its conjugate is a real number.

Objective 6: Divide Complex Numbers

Divide. Write the result in the form $a + bi$.

59) $\dfrac{4}{2-3i}$ $\quad \dfrac{8}{13}+\dfrac{12}{13}i$

60) $\dfrac{-10}{8-9i}$ $\quad -\dfrac{16}{29}-\dfrac{18}{29}i$

61) $\dfrac{8i}{4+i}$ $\quad \dfrac{8}{17}+\dfrac{32}{17}i$

62) $\dfrac{i}{6-5i}$ $\quad -\dfrac{5}{61}+\dfrac{6}{61}i$

63) $\dfrac{2i}{-3+7i}$ $\quad \dfrac{7}{29}-\dfrac{3}{29}i$

64) $\dfrac{9i}{-4+10i}$ $\quad \dfrac{45}{58}-\dfrac{9}{29}i$

65) $\dfrac{3-8i}{-6+7i}$ $\quad -\dfrac{74}{85}+\dfrac{27}{85}i$

66) $\dfrac{-5+2i}{4-i}$ $\quad -\dfrac{22}{17}+\dfrac{3}{17}i$

67) $\dfrac{2+3i}{5-6i}$ $\quad -\dfrac{8}{61}+\dfrac{27}{61}i$

68) $\dfrac{1+6i}{5+2i}$ $\quad \dfrac{17}{29}+\dfrac{28}{29}i$

69) $\dfrac{9}{i}$ $\quad -9i$

70) $\dfrac{16+3i}{-i}$ $\quad -3+16i$

Objective 7: Simplify Powers of i

Simplify each power of i.

Fill It In

Fill in the blanks with either the missing mathematical step or reason for the given step.

71) $i^{24} = \underline{(i^2)^{12}}$ — Rewrite i^{24} in terms of i^2 using the power rule.
$\quad = (-1)^{12}$ — $\underline{i^2 = -1}$
$\quad = \underline{1}$ — Simplify.

72) $i^{31} = i^{30} \cdot i$ — Product rule
$\quad = (i^2)^{15} \cdot i$ — Rewrite i^{30} in terms of i^2 using the power rule.
$\quad = \underline{(-1)^{15} \cdot i}$ — $i^2 = -1$
$\quad = \underline{-1 \cdot i}$ — Simplify.
$\quad = \underline{-i}$ — Multiply.

73) i^{24} $\quad 1$

74) i^{16} $\quad 1$

75) i^{28} $\quad 1$

76) i^{30} $\quad -1$

77) i^{9} $\quad i$

78) i^{19} $\quad -i$

79) i^{35} $\quad -i$

80) i^{29} $\quad i$

81) i^{23} $\quad -i$

82) i^{40} $\quad 1$

83) i^{42} $\quad -1$

84) i^{33} $\quad i$

85) $(2i)^5$ $\quad 32i$

86) $(2i)^6$ $\quad -64$

87) $(-i)^{14}$ $\quad -1$

88) $(-i)^{15}$ $\quad i$

Expand.

89) $(-2+5i)^3$ $\quad 142-65i$

90) $(3-4i)^3$ $\quad -117-44i$

Simplify each expression. Write the result in the form $a + bi$.

91) $1+\sqrt{-8}$ $\quad 1+2i\sqrt{2}$

92) $-7-\sqrt{-48}$ $\quad -7-4i\sqrt{-3}$

93) $8-\sqrt{-45}$ $\quad 8-3i\sqrt{5}$

94) $3+\sqrt{-20}$ $\quad 3+2i\sqrt{5}$

95) $\dfrac{-12+\sqrt{-32}}{4}$ $\quad -3+i\sqrt{2}$

96) $\dfrac{21-\sqrt{-18}}{3}$ $\quad 7-i\sqrt{2}$

Used in the field of electronics, the **impedance,** Z, is the total opposition to the current flow of an alternating current (AC) within an electronic component, circuit, or system. It is expressed as a complex number $\boldsymbol{Z = R + Xj}$, where the i used to represent an imaginary number in most areas of mathematics is replaced by j in electronics. $\boldsymbol{R}$ represents the resistance of a substance, and $\boldsymbol{X}$ represents the reactance.

The **total impedance,** $\boldsymbol{Z}$**,** of components connected in series is the *sum* of the individual impedances of each component.

Each exercise contains the impedance of individual circuits. Find the total impedance of a system formed by connecting the circuits in series by finding the sum of the individual impedances.

97) $Z_1 = 3+2j$
$\quad Z_2 = 7+4j$
$\quad Z = 10+6j$

98) $Z_1 = 5+3j$
$\quad Z_2 = 9+6j$
$\quad Z = 14+9j$

99) $Z_1 = 5-2j$
$\quad Z_2 = 11+6j$
$\quad Z = 16+4j$

100) $Z_1 = 4-1.5j$
$\quad Z_2 = 3+0.5j$
$\quad Z = 7-1.0j$

Rethink

R1) Explain, in your own words, the meaning of the imaginary number i. How does this help us find the square root of a negative number?

R2) Which learning objectives from previous chapters helped you master the concepts of this exercise set?

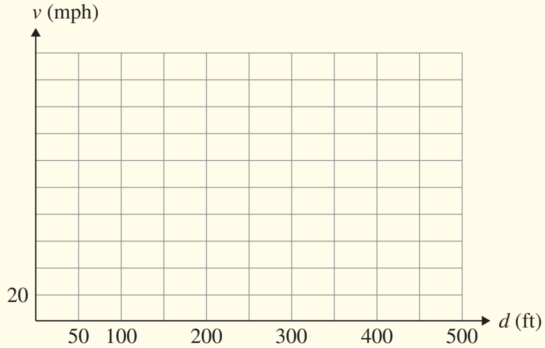
Group Activity – Radical Equations

The science of forensics is used in accident investigations. The speed of a car before brakes are applied can be approximated from the length of its skid marks by using the following equation:

$$v = 5.47\sqrt{\mu d} \quad \text{where} \quad \begin{aligned} v &= \text{speed of the car (in mph)} \\ d &= \text{length of skid marks (in feet)} \\ \mu &= \text{coefficient of friction determined by road conditions} \end{aligned}$$

Graph the equation $v = 5.47\sqrt{\mu d}$ by filling in the table and plotting points if the pavement is dry (the coefficient of friction is 0.7).

d	$v = 5.47\sqrt{\mu d}$
0	
50	
100	
150	
200	
250	
300	
400	
500	



Length of Skid Marks vs. Speed of Car

Graph the equation if the coefficient of friction on wet pavement is 0.4.

As the road conditions become more slippery, how does the graph change?

Every school—whether it's a high school, community college, college, or university—operates under its own set of rules and procedures. Understanding how your school works and where to go for help are essential parts of being successful in college. It's important to understand how your school works so that, for example, you know where and when to turn in your financial aid application and you know where to get help if you have questions about choosing the classes you need for graduation. Take this survey to learn how well you know your school. Check all boxes that apply.

- ☐ I know the address of my school's website.
- ☐ I can navigate the school's website to find most information that I need.
- ☐ I am aware of whether my school has a handbook containing useful information.
- ☐ I have signed up to receive emergency campus messages by email, text, or automated phone call.
- ☐ I am aware of important dates such as when to register for classes, when tuition is due, and when financial aid forms are due.
- ☐ I know where to register for classes on campus.
- ☐ On campus, I know where to ask questions about financial aid.
- ☐ I can locate the bookstore.
- ☐ I know the location of the library.
- ☐ I know the difference between an adviser and a counselor.
- ☐ I know the location of the campus health center.
- ☐ I know the location of student services offices that might be of interest to me. Some examples are veterans' support services, the office to help students with disabilities, and child care.
- ☐ I know the locations of all of my instructors' offices as well as their office hours.
- ☐ I know the location of the tutoring center/math lab, and I know their procedures for getting help when I need it.
- ☐ I can locate the Testing Center and know its rules and hours of operation.
- ☐ I am aware of clubs, organizations, and activities on campus, and I know where to go to become involved in those that interest me.
- ☐ I know the location of the office where I can go if I have questions about or want help finding a job.

Think about the items that you have, and have *not*, checked in this survey. Which apply to you and might contribute to your success in college? In the Study Strategies at the beginning of this chapter, you will learn how to get to know your school.

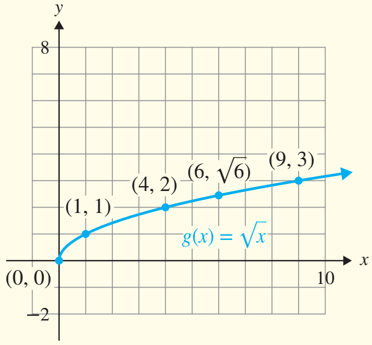
Chapter 9: Summary

Definition/Procedure	Example

9.1 Radical Expressions and Functions

Definition/Procedure	Example
If the radicand is a perfect square, then the square root is a *rational* number.	$\sqrt{49} = 7$ since $7^2 = 49$.
If the radicand is a negative number, then the square root is *not* a real number.	$\sqrt{-36}$ is not a real number.
If the radicand is positive and not a perfect square, then the square root is an *irrational* number.	$\sqrt{7}$ is irrational because 7 is not a perfect square.
The symbol $\sqrt[n]{a}$ is read as "the nth root of a." If $\sqrt[n]{a} = b$, then $b^n = a$. We call n the **index** of the radical.	$\sqrt[5]{32} = 2$ since $2^5 = 32$.
For any *positive* number a and any *even* index n, the **principal** ***n*th root** of a is $\sqrt[n]{a}$ and the **negative *n*th root** of a is $-\sqrt[n]{a}$.	$\sqrt[4]{16} = 2$ $-\sqrt[4]{16} = -2$
The *odd root* of a negative number is a negative number.	$\sqrt[3]{-125} = -5$ since $(-5)^3 = -125$.
The *even root* of a negative number is not a real number.	$\sqrt[4]{-16}$ is not a real number.
If n is a positive, *even* integer, then $\sqrt[n]{a^n} = \lvert a \rvert$. If n is a positive, *odd* integer, then $\sqrt[n]{a^n} = a$.	$\sqrt[4]{(-2)^4} = \lvert -2 \rvert = 2$ $\sqrt[3]{(-2)^3} = -2$
The **domain of a square root function** consists of all of the real numbers that can be substituted for the variable so that the radicand is nonnegative.	Determine the domain of the square root function. $f(x) = \sqrt{6x - 7}$ $6x - 7 \geq 0$ The value of the radicand must be ≥ 0. $6x \geq 7$ $x \geq \frac{7}{6}$ Solve. The domain of $f(x) = \sqrt{6x - 7}$ is $\left[\frac{7}{6}, \infty\right)$.
The **domain of a cube root function** is the set of all real numbers. We can write this in interval notation as $(-\infty, \infty)$.	The domain of $g(x) = \sqrt[3]{x}$ is $(-\infty, \infty)$.
To graph a square root function, find the domain, make a table of values, and graph.	Graph $g(x) = \sqrt{x}$.

x	$g(x)$
0	0
1	1
4	2
6	$\sqrt{6}$
9	3



9.2 Rational Exponents

Definition/Procedure	Example
If n is a positive integer greater than 1 and $\sqrt[n]{a}$ is a real number, then $a^{1/n} = \sqrt[n]{a}$.	$8^{1/3} = \sqrt[3]{8} = 2$
If m and n are positive integers and $\frac{m}{n}$ is in lowest terms, then $a^{m/n} = (a^{1/n})^m = (\sqrt[n]{a})^m$ provided that $a^{1/n}$ is a real number.	$16^{3/4} = (\sqrt[4]{16})^3 = 2^3 = 8$

Definition/Procedure	Example
If $a^{m/n}$ is a nonzero real number, then $$a^{-m/n} = \left(\frac{1}{a}\right)^{m/n} = \frac{1}{a^{m/n}}.$$	$25^{-3/2} = \left(\frac{1}{25}\right)^{3/2} = \left(\sqrt{\frac{1}{25}}\right)^3 = \left(\frac{1}{5}\right)^3 = \frac{1}{125}$ **BE CAREFUL** The negative exponent does not make the expression negative.

9.3 Simplifying Expressions Containing Square Roots

Definition/Procedure	Example
Product Rule for Square Roots Let a and b be nonnegative real numbers. Then, $\sqrt{a}\cdot\sqrt{b} = \sqrt{a\cdot b}$.	$\sqrt{5}\cdot\sqrt{7} = \sqrt{5\cdot 7} = \sqrt{35}$
An expression containing a square root is simplified when all of the following conditions are met: 1) The radicand does not contain any factors (other than 1) that are perfect squares. 2) The radicand does not contain any fractions. 3) There are no radicals in the denominator of a fraction. To *simplify square roots,* rewrite using the product rule as $\sqrt{a\cdot b} = \sqrt{a}\cdot\sqrt{b}$, where a or b is a perfect square. After simplifying a radical, look at the result and ask yourself, *"Is the radical in simplest form?"* If it is not, simplify again.	Simplify $\sqrt{24}$. $\sqrt{24} = \sqrt{4\cdot 6}$ 4 is a perfect square. $= \sqrt{4}\cdot\sqrt{6}$ Product rule $= 2\sqrt{6}$ $\sqrt{4} = 2$
Quotient Rule for Square Roots Let a and b be nonnegative real numbers such that $b \neq 0$. Then, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.	$\sqrt{\frac{72}{25}} = \frac{\sqrt{72}}{\sqrt{25}}$ Quotient rule $= \frac{\sqrt{36}\cdot\sqrt{2}}{5}$ Product rule; $\sqrt{25} = 5$ $= \frac{6\sqrt{2}}{5}$ $\sqrt{36} = 6$
If a is a nonnegative real number and m is an integer, then $\sqrt{a^m} = a^{m/2}$.	$\sqrt{k^{18}} = k^{18/2} = k^9$ (provided k represents a nonnegative real number)
Two Approaches to Simplifying Radical Expressions Containing Variables Let a represent a nonnegative real number. To simplify $\sqrt{a^n}$ where *n is odd and positive,* **i) Method 1:** Write a^n as the product of two factors so that the exponent of one of the factors is the *largest* number less than n that is divisible by 2 (the index of the radical). **ii) Method 2:** 1) Divide the exponent in the radicand by the index of the radical. 2) The exponent on the variable *outside* of the radical will be the *quotient* of the division problem. 3) The exponent on the variable *inside* of the radical will be the *remainder* of the division problem.	i) Simplify $\sqrt{x^9}$. $\sqrt{x^9} = \sqrt{x^8\cdot x^1}$ 8 is the largest number less than 9 that is divisible by 2. $= \sqrt{x^8}\cdot\sqrt{x}$ Product rule $= x^{8/2}\sqrt{x}$ $= x^4\sqrt{x}$ $8 \div 2 = 4$ ii) Simplify $\sqrt{p^{15}}$. $\sqrt{p^{15}} = p^7\sqrt{p^1}$ $15 \div 2$ gives a quotient of 7 and a remainder of 1. $= p^7\sqrt{p}$

Definition/Procedure	Example

9.4 Simplifying Expressions Containing Higher Roots

Definition/Procedure	Example
Product Rule for Higher Roots If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers such that the roots exist, then $\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$.	$\sqrt[3]{3} \cdot \sqrt[3]{5} = \sqrt[3]{15}$
Let P be an expression and let n be a positive integer greater than 1. Then $\sqrt[n]{P}$ **is completely simplified when all of the following conditions are met:** 1) The radicand does not contain any factors (other than 1) that are perfect nth powers. 2) The exponents in the radicand and the index of the radical do not have any common factors (other than 1). 3) The radicand does not contain any fractions. 4) There are no radicals in the denominator of a fraction. To *simplify radicals with any index,* reverse the process of multiplying radicals, where a or b is an nth power. $$\sqrt[n]{a \cdot b} = \sqrt[n]{a} \cdot \sqrt[n]{b}$$	Simplify $\sqrt[3]{40}$. **Method 1:** Think of two numbers that multiply to 40 so that one of them is a *perfect cube*. $40 = 8 \cdot 5$ 8 is a perfect cube. Then, $\sqrt[3]{40} = \sqrt[3]{8 \cdot 5}$ $= \sqrt[3]{8} \cdot \sqrt[3]{5}$ Product rule $= 2\sqrt[3]{5}$ $\sqrt[3]{8} = 2$ **Method 2:** Begin by using a factor tree to find the prime factorization of 40. $40 = 2^3 \cdot 5$ $\sqrt[3]{40} = \sqrt[3]{2^3 \cdot 5}$ $= \sqrt[3]{2^3} \cdot \sqrt[3]{5}$ Product rule $= 2\sqrt[3]{5}$ $\sqrt[3]{2^3} = 2$
Quotient Rule for Higher Roots If $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers, $b \neq 0$ and n is a natural number, then $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$.	$\sqrt[4]{\dfrac{32}{81}} = \dfrac{\sqrt[4]{32}}{\sqrt[4]{81}} = \dfrac{\sqrt[4]{16} \cdot \sqrt[4]{2}}{3} = \dfrac{2\sqrt[4]{2}}{3}$
Simplifying Higher Roots with Variables in the Radicand If a is a nonnegative number and m and n are integers such that $n > 1$, then $\sqrt[n]{a^m} = a^{m/n}$.	Simplify $\sqrt[4]{a^{12}}$. $\sqrt[4]{a^{12}} = a^{12/14} = a^3$
If the exponent does not divide evenly by the index, we can use two methods for simplifying the radical expression. If a is a nonnegative number and m and n are integers such that $n > 1$, then **i) Method 1:** Use the product rule. To simplify $\sqrt[n]{a^m}$, write a^m as the product of two factors so that the exponent of one of the factors is the *largest* number less than m that is divisible by n (the index). **ii) Method 2:** Use the quotient and remainder (presented in Section 9.3).	i) Simplify $\sqrt[5]{c^{17}}$. $\sqrt[5]{c^{17}} = \sqrt[5]{c^{15} \cdot c^2}$ 15 is the largest number less than 17 that is divisible by 5. $= \sqrt[5]{c^{15}} \cdot \sqrt[5]{c^2}$ Product rule $= c^{15/5} \cdot \sqrt[5]{c^2}$ $= c^3\sqrt[5]{c^2}$ $15 \div 5 = 3$ ii) Simplify $\sqrt[4]{m^{11}}$. $\sqrt[4]{m^{11}} = m^2\sqrt[4]{m^3}$ $11 \div 4$ gives a quotient of 2 and a remainder of 3.

9.5 Adding, Subtracting, and Multiplying Radicals

Definition/Procedure	Example
Like radicals have the same index and the same radicand. In order to add or subtract radicals, they must be like radicals. **Steps for Adding and Subtracting Radicals** 1) Write each radical expression in simplest form. 2) Combine like radicals.	Perform the operations, and simplify. a) $5\sqrt{2} + 9\sqrt{7} - 3\sqrt{2} + 4\sqrt{7}$ $= 2\sqrt{2} + 13\sqrt{7}$ b) $\sqrt{72} + \sqrt{18} - \sqrt{45}$ $= \sqrt{36} \cdot \sqrt{2} + \sqrt{9} \cdot \sqrt{2} - \sqrt{9} \cdot \sqrt{5}$ $= 6\sqrt{2} + 3\sqrt{2} - 3\sqrt{5}$ $= 9\sqrt{2} - 3\sqrt{5}$

Definition/Procedure	Example

Combining Multiplication, Addition, and Subtraction of Radicals

Multiply expressions containing radicals using the same techniques that are used for multiplying polynomials.

Multiply and simplify.

a) $\sqrt{m}(\sqrt{2m}+\sqrt{n})$

$$\begin{aligned} &= \sqrt{m}\cdot\sqrt{2m}+\sqrt{m}\cdot\sqrt{n} && \text{Distribute.}\\ &= \sqrt{2m^2}+\sqrt{mn} && \text{Multiply.}\\ &= m\sqrt{2}+\sqrt{mn} && \text{Simplify.}\end{aligned}$$

b) $(\sqrt{k}+\sqrt{6})(\sqrt{k}-\sqrt{2})$

Since we are multiplying two binomials, multiply using FOIL.

$$(\sqrt{k}+\sqrt{6})(\sqrt{k}-\sqrt{2})$$

$$\begin{aligned} &= \underbrace{\sqrt{k}\cdot\sqrt{k}}_{F}-\underbrace{\sqrt{2}\cdot\sqrt{k}}_{O}+\underbrace{\sqrt{6}\cdot\sqrt{k}}_{I}-\underbrace{\sqrt{6}\cdot\sqrt{2}}_{L}\\ &= k^2-\sqrt{2k}+\sqrt{6k}-\sqrt{12} && \text{Product rule}\\ &= k^2-\sqrt{2k}+\sqrt{6k}-2\sqrt{3} && \sqrt{12}=2\sqrt{13}\end{aligned}$$

Squaring a Radical Expression with Two Terms

To square a binomial we can either use FOIL or one of the special formulas from Chapter 6:

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$

$$\begin{aligned}(\sqrt{7}+5)^2 &= (\sqrt{7})^2+2(\sqrt{7})(5)+(5)^2\\ &= 7+10\sqrt{7}+25\\ &= 32+10\sqrt{7}\end{aligned}$$

Multiply $(a+b)(a-b)$

To multiply binomials of the form $(a+b)(a-b)$ use the formula $(a+b)(a-b)=a^2-b^2$.

$$\begin{aligned}(3+\sqrt{10})(3-\sqrt{10}) &= (3)^2-(\sqrt{10})^2\\ &= 9-10\\ &= -1\end{aligned}$$

9.6 Dividing Radicals

The process of eliminating radicals from the denominator of an expression is called **rationalizing the denominator.**

First, we give examples of rationalizing denominators containing one term.

Rationalize the denominator of each expression.

a) $\dfrac{9}{\sqrt{2}}=\dfrac{9}{\sqrt{2}}\cdot\dfrac{\sqrt{2}}{\sqrt{2}}=\dfrac{9\sqrt{2}}{2}$

b) $\dfrac{5}{\sqrt[3]{2}}=\dfrac{5}{\sqrt[3]{2}}\cdot\dfrac{\sqrt[3]{2^2}}{\sqrt[3]{2^2}}=\dfrac{5\sqrt[3]{2^2}}{\sqrt[3]{2^3}}=\dfrac{5\sqrt[3]{4}}{2}$

The **conjugate** of an expression of the form $a+b$ is $a-b$.

$\sqrt{11}-4$ conjugate: $\sqrt{11}+4$

$-8+\sqrt{5}$ conjugate: $-8-\sqrt{5}$

Rationalizing a Denominator with Two Terms

If the denominator of an expression contains two terms, including one or two square roots, then to rationalize the denominator, we multiply the numerator and denominator of the expression by the conjugate of the denominator.

Rationalize the denominator of $\dfrac{4}{\sqrt{2}-3}$.

$$\begin{aligned}\frac{4}{\sqrt{2}-3} &= \frac{4}{\sqrt{2}-3}\cdot\frac{\sqrt{2}+3}{\sqrt{2}+3} && \text{Multiply by the conjugate of the denominator.}\\ &= \frac{4(\sqrt{2}+3)}{(\sqrt{2})^2-(3)^2} && (a+b)(a-b)=a^2-b^2\\ &= \frac{4(\sqrt{2}+3)}{2-9} && \text{Square the terms.}\\ &= \frac{4(\sqrt{2}+3)}{-7}=-\frac{4\sqrt{2}+12}{7}\end{aligned}$$

Definition/Procedure	Example

9.7 Solving Radical Equations

Solving Radical Equations Containing Square Roots ***Step 1:*** Get a radical on a side by itself. ***Step 2:*** Square both sides of the equation to eliminate a radical. ***Step 3:*** Combine like terms on each side of the equation. ***Step 4:*** If the equation still contains a radical, repeat Steps 1–3. ***Step 5:*** Solve the equation. ***Step 6:*** Check the proposed solutions *in the original equation,* and discard extraneous solutions.	Solve $t = 2 + \sqrt{2t-1}$. $t - 2 = \sqrt{2t-1}$ Get the radical by itself. $(t-2)^2 = (\sqrt{2t-1})^2$ Square both sides. $t^2 - 4t + 4 = 2t - 1$ $t^2 - 6t + 5 = 0$ Get all terms on the same side. $(t-5)(t-1) = 0$ Factor. $t - 5 = 0$ or $t - 1 = 0$ $t = 5$ or $t = 1$ Check $t = 5$ and $t = 1$ in the *original* equation. $t = 5$: $t = 2 + \sqrt{2t-1}$ $5 \stackrel{?}{=} 2 + \sqrt{2(5)-1}$ $5 \stackrel{?}{=} 2 + \sqrt{9}$ $5 = 2 + 3$ True $t = 1$: $t = 2 + \sqrt{2t-1}$ $1 \stackrel{?}{=} 2 + \sqrt{2(1)-1}$ $1 \stackrel{?}{=} 2 + 1$ $1 = 3$ False $t = 5$ *is* a solution, but $t = 1$ is *not* because $t = 1$ does not satisfy the original equation. The solution set is $\{5\}$.

9.8 Complex Numbers

Definition of *i*: $i = \sqrt{-1}$ Therefore, $i^2 = -1$. A **complex number** is a number of the form $a + bi$, where a and b are real numbers. a is called the **real part** and b is called the **imaginary part.** The set of real numbers is a subset of the set of complex numbers.	*Examples of complex numbers:* $-2 + 7i$ 5 (since it can be written $5 + 0i$) $4i$ (since it can be written $0 + 4i$)
Simplifying Square Roots with Negative Radicands Use the product rule and $i = \sqrt{-1}$.	Simplify $\sqrt{-25}$. $\sqrt{-25} = \sqrt{-1} \cdot \sqrt{25}$ $= i \cdot 5$ $= 5i$
When multiplying or dividing square roots with negative radicands, write each radical in terms of i first.	Multiply $\sqrt{-12} \cdot \sqrt{-3}$. $\sqrt{-12} \cdot \sqrt{-3} = i\sqrt{12} \cdot i\sqrt{3} = i^2\sqrt{36}$ $= -1 \cdot 6 = -6$
Adding and Subtracting Complex Numbers To add and subtract complex numbers, combine the real parts and combine the imaginary parts.	Subtract $(10 + 7i) - (-2 + 4i)$. $(10 + 7i) - (-2 + 4i) = 10 + 7i + 2 - 4i$ $= 12 + 3i$
Multiply complex numbers like we multiply polynomials. Remember to replace i^2 with -1.	Multiply and simplify. a) $4(9 + 5i) = 36 + 20i$ F O I L b) $(-3 + i)(2 - 7i) = -6 + 21i + 2i - 7i^2$ $= -6 + 23i - 7(-1)$ $= -6 + 23i + 7$ $= 1 + 23i$

Definition/Procedure	Example
Complex Conjugates 1) The conjugate of $a + bi$ is $a - bi$. 2) The product of $a + bi$ and $a - bi$ is a real number. 3) Find the product $(a + bi)(a - bi)$ using FOIL or recall that $(a + bi)(a - bi) = a^2 + b^2$.	Multiply $-5 - 3i$ by its conjugate. The conjugate of $-5 - 3i$ is $-5 + 3i$. Use $(a + bi)(a - bi) = a^2 + b^2$. $(-5 - 3i)(-5 + 3i) = (-5)^2 + (3)^2 = 25 + 9 = 34$
Dividing Complex Numbers To divide complex numbers, multiply the numerator and denominator by the *conjugate of the denominator*. Write the quotient in the form $a + bi$.	Divide $\frac{6i}{2 + 5i}$. Write the result in the form $a + bi$. $\frac{6i}{2+5i} = \frac{6i}{2+5i} \cdot \frac{(2-5i)}{(2-5i)} = \frac{12i - 30i^2}{(2)^2 + (5)^2} = \frac{12i - 30(-1)}{29} = \frac{30}{29} + \frac{12}{29}i$
Simplify Powers of *i* We can simplify powers of i using $i^2 = -1$.	Simplify i^{14}. $i^{14} = (i^2)^7$ Power rule $= (-1)^7$ $i^2 = -1$ $= -1$ Simplify.

Chapter 9: Review Exercises

*Additional answers can be found in the Answers to Exercises appendix.

(9.1) Find each root, if possible.

1) $\sqrt{\frac{169}{4}}$ $\frac{13}{2}$

2) $\sqrt{-16}$ not real

3) $-\sqrt{81}$ -9

4) $\sqrt[5]{32}$ 2

5) $\sqrt[3]{-1}$ -1

6) $-\sqrt[4]{81}$ -3

7) $\sqrt[6]{-64}$ not real

8) $\sqrt{9 - 16}$ not real

Simplify. Use absolute values when necessary.

9) $\sqrt{(-13)^2}$ 13

10) $\sqrt[5]{(-8)^5}$ -8

11) $\sqrt{p^2}$ $|p|$

12) $\sqrt[6]{c^6}$ $|c|$

13) $\sqrt[3]{h^3}$ h

14) $\sqrt[4]{(y + 7)^4}$ $|y + 7|$

15) $f(x) = \sqrt{5x + 3}$

a) Find $f(4)$. $\sqrt{23}$

b) Find $f(p)$. $\sqrt{5p + 3}$

c) Find the domain of f. $\left[-\frac{3}{5}, \infty\right)$

16) $g(x) = \sqrt[3]{x - 12}$

a) Find $g(4)$. -2

b) Find $g(t + 7)$ $\sqrt[3]{t - 5}$

c) Find the domain of g. $(-\infty, \infty)$

17) Graph $k(x) = \sqrt{x + 4}$.

18) Graph $h(x) = -\sqrt[3]{x}$.

(9.2)

19) Explain how to write $8^{2/3}$ in radical form.

20) Explain how to eliminate the negative from the exponent in an expression like $9^{-1/2}$.
Take the reciprocal of the base. $9^{-1/2} = \left(\frac{1}{9}\right)^{1/2}$

Evaluate.

21) $36^{1/2}$ 6

22) $32^{1/5}$ 2

23) $\left(\frac{27}{125}\right)^{1/3}$ $\frac{3}{5}$

24) $-16^{1/4}$ -2

25) $32^{3/5}$ 8

26) $\left(\frac{64}{27}\right)^{2/3}$ $\frac{16}{9}$

27) $81^{-1/2}$ $\frac{1}{9}$

28) $\left(\frac{1}{27}\right)^{-1/3}$ 3

29) $81^{-3/4}$ $\frac{1}{27}$

30) $1000^{-2/3}$ $\frac{1}{100}$

31) $\left(\frac{27}{1000}\right)^{-2/3}$ $\frac{100}{9}$

32) $\left(\frac{25}{16}\right)^{-3/2}$ $\frac{64}{125}$

From this point forward, assume all variables represent positive real numbers.

Simplify completely. The answer should contain only positive exponents.

33) $3^{6/7} \cdot 3^{8/7}$ 9

34) $(169^4)^{1/8}$ 13

35) $(8^{1/5})^{10}$ 64

36) $\frac{8^2}{8^{11/3}}$ $\frac{1}{32}$

37) $\dfrac{7^2}{7^{5/3}\cdot 7^{1/3}}$ 1

38) $(2k^{-5/6})(3k^{1/2})$ $\dfrac{6}{k^{1/3}}$

39) $(64a^4b^{12})^{5/6}$ $32a^{10/3}b^{10}$

40) $\left(\dfrac{t^4u^3}{7t^7u^5}\right)^{-2}$ $49t^6u^4$

41) $\left(\dfrac{81c^{-5}d^9}{16c^{-1}d^2}\right)^{-1/4}$ $\dfrac{2c}{3d^{7/4}}$

Rewrite each radical in exponential form, then simplify. Write the answer in simplest (or radical) form.

42) $\sqrt[4]{36^2}$ 6

43) $\sqrt[12]{27^4}$ 3

44) $(\sqrt{17})^2$ 17

45) $\sqrt[3]{7^3}$ 7

46) $\sqrt[5]{t^{20}}$ t^4

47) $\sqrt[4]{k^{28}}$ k^7

48) $\sqrt{x^{10}}$ x^5

49) $\sqrt{w^6}$ w^3

(9.3) Simplify completely.

50) $\sqrt{28}$ $2\sqrt{7}$

51) $\sqrt{1000}$ $10\sqrt{10}$

52) $\dfrac{\sqrt{63}}{\sqrt{7}}$ 3

53) $\sqrt{\dfrac{18}{49}}$ $\dfrac{3\sqrt{2}}{7}$

54) $\dfrac{\sqrt{48}}{\sqrt{121}}$ $\dfrac{4\sqrt{3}}{11}$

55) $\sqrt{k^{12}}$ k^6

56) $\sqrt{\dfrac{40}{m^4}}$ $\dfrac{2\sqrt{10}}{m^2}$

57) $\sqrt{x^9}$ $x^4\sqrt{x}$

58) $\sqrt{y^5}$ $y^2\sqrt{y}$

59) $\sqrt{45t^2}$ $3t\sqrt{5}$

60) $\sqrt{80n^{21}}$ $4n^{10}\sqrt{5n}$

61) $\sqrt{72x^7y^{13}}$ $6x^3y^6\sqrt{2xy}$

62) $\sqrt{\dfrac{m^{11}}{36n^2}}$ $\dfrac{m^5\sqrt{m}}{6n}$

Perform the indicated operation, and simplify.

63) $\sqrt{5}\cdot\sqrt{3}$ $\sqrt{15}$

64) $\sqrt{6}\cdot\sqrt{15}$ $3\sqrt{10}$

65) $\sqrt{2}\cdot\sqrt{12}$ $2\sqrt{6}$

66) $\sqrt{b^7}\cdot\sqrt{b^3}$ b^5

67) $\sqrt{11x^5}\cdot\sqrt{11x^8}$ $11x^6\sqrt{x}$

68) $\sqrt{5a^2b}\cdot\sqrt{15a^6b^4}$ $5a^4b^2\sqrt{3b}$

69) $\dfrac{\sqrt{200k^{21}}}{\sqrt{2k^5}}$ $10k^8$

70) $\dfrac{\sqrt{63c^{17}}}{\sqrt{7c^9}}$ $3c^4$

(9.4) Simplify completely.

71) $\sqrt[3]{16}$ $2\sqrt[3]{2}$

72) $\sqrt[3]{250}$ $5\sqrt[3]{2}$

73) $\sqrt[4]{48}$ $2\sqrt[4]{3}$

74) $\sqrt[3]{\dfrac{81}{3}}$ 3

75) $\sqrt[4]{z^{24}}$ z^6

76) $\sqrt[5]{p^{40}}$ p^8

77) $\sqrt[3]{a^{20}}$ $a^6\sqrt[3]{a^2}$

78) $\sqrt[5]{x^{14}y^7}$ $x^2y\sqrt[5]{x^4y^2}$

79) $\sqrt[3]{16z^{15}}$ $2z^5\sqrt[3]{2}$

80) $\sqrt[3]{80m^{17}n^{10}}$ $2m^5n^3\sqrt[3]{10m^2n}$

81) $\sqrt[4]{\dfrac{h^{12}}{81}}$ $\dfrac{h^3}{3}$

82) $\sqrt[5]{\dfrac{c^{22}}{32d^{10}}}$ $\dfrac{c^4\sqrt[5]{c^2}}{2d^2}$

Perform the indicated operation, and simplify.

83) $\sqrt[3]{3}\cdot\sqrt[3]{7}$ $\sqrt[3]{21}$

84) $\sqrt[3]{25}\cdot\sqrt[3]{10}$ $5\sqrt[3]{2}$

85) $\sqrt[4]{4t^7}\cdot\sqrt[4]{8t^{10}}$ $2t^4\sqrt[4]{2t}$

86) $\sqrt[5]{\dfrac{x^{21}}{x^{16}}}$ x

87) $\sqrt[3]{n}\cdot\sqrt{n}$ $\sqrt[6]{n^5}$

88) $\dfrac{\sqrt[4]{a^3}}{\sqrt[3]{a}}$ $\sqrt[12]{a^5}$

(9.5) Perform the operations, and simplify.

89) $8\sqrt{5}+3\sqrt{5}$ $11\sqrt{5}$

90) $\sqrt{125}+\sqrt{80}$ $9\sqrt{5}$

91) $\sqrt{80}-\sqrt{48}+\sqrt{20}$ $6\sqrt{5}-4\sqrt{3}$

92) $9\sqrt[3]{72}-8\sqrt[3]{9}$ $10\sqrt[3]{9}$

93) $3p\sqrt{p}-7\sqrt{p^3}$ $-4p\sqrt{p}$

94) $9n\sqrt{n}-4\sqrt{n^3}$ $5n\sqrt{n}$

95) $10d^2\sqrt{8d}-32d\sqrt{2d^3}$ $-12d^2\sqrt{2d}$

96) $\sqrt{6}(\sqrt{7}-\sqrt{6})$ $\sqrt{42}-6$

97) $3\sqrt{k}(\sqrt{20k}+\sqrt{2})$ $6k\sqrt{5}+3\sqrt{2k}$

98) $(5-\sqrt{3})(2+\sqrt{3})$ $7+3\sqrt{3}$

99) $(\sqrt{6a}-9)(\sqrt{2a}-7)$ $2a\sqrt{3}-7\sqrt{6a}-9\sqrt{2a}+63$

100) $(8+3\sqrt{c})(8-3\sqrt{c})$ $64-9c$

101) $(\sqrt{2r}+5\sqrt{s})(3\sqrt{s}+4\sqrt{2r})$ $23\sqrt{2rs}+8r+15s$

102) $(2\sqrt{5}-4)^2$ $36-16\sqrt{5}$

103) $(1+\sqrt{y+1})^2$ $2+2\sqrt{y+1}+y$

104) $(\sqrt{6}-\sqrt{5})(\sqrt{6}+\sqrt{5})$ 1

(9.6) Rationalize the denominator of each expression.

105) $\dfrac{14}{\sqrt{3}}$ $\dfrac{14\sqrt{3}}{3}$

106) $\dfrac{20}{\sqrt{6}}$ $\dfrac{10\sqrt{6}}{3}$

107) $\dfrac{\sqrt{18k}}{\sqrt{n}}$ $\dfrac{3\sqrt{2kn}}{n}$

108) $\dfrac{\sqrt{45}}{\sqrt{m^5}}$ $\dfrac{3\sqrt{5m}}{m^3}$

109) $\dfrac{7}{\sqrt[3]{2}}$ $\dfrac{7\sqrt[3]{4}}{2}$

110) $-\dfrac{15}{\sqrt[3]{9}}$ $-5\sqrt[3]{3}$

111) $\dfrac{\sqrt[3]{x^2}}{\sqrt[3]{y}}$ $\dfrac{\sqrt[3]{x^2y^2}}{y}$

112) $\sqrt[4]{\dfrac{3}{4k^2}}$ $\dfrac{\sqrt[4]{12k^2}}{2k}$

113) $\dfrac{2}{3+\sqrt{3}}$ $\dfrac{3-\sqrt{3}}{3}$

114) $\dfrac{z-4}{\sqrt{z}+2}$ $\sqrt{z}-2$

Simplify completely.

115) $\dfrac{8-24\sqrt{2}}{8}$ $1-3\sqrt{2}$

116) $\dfrac{-\sqrt{48}-6}{10}$ $\dfrac{-2\sqrt{3}-3}{5}$

(9.7) Solve.

117) $\sqrt{x+8}=3$ $\{1\}$

118) $10-\sqrt{3r-5}=2$ $\{23\}$

119) $\sqrt{3j+4}=-\sqrt{4j-1}$ $\varnothing$

120) $\sqrt[3]{6d-14}=-2$ $\{1\}$

121) $a=\sqrt{a+8}-6$ $\{-4\}$

122) $1+\sqrt{6m+7}=2m$ $\{3\}$

123) $\sqrt{4a+1}-\sqrt{a-2}=3$ $\{2, 6\}$

124) $\sqrt{6x+9}-\sqrt{2x+1}=4$ $\{12\}$

125) Solve for V: $r=\sqrt{\dfrac{3V}{\pi h}}$ $V=\dfrac{1}{3}\pi r^2\text{h}$

126) The velocity of a wave in shallow water is given by $c=\sqrt{gH}$, where g is the acceleration due to gravity (32 ft/sec^2), and H is the depth of the water (in feet). Find the velocity of a wave in 10 ft of water.
$8\sqrt{5}$ ft/sec or about 17.9 ft/sec

(9.8) Simplify.

127) $\sqrt{-49}$ $7i$

128) $\sqrt{-8}$ $2i\sqrt{2}$

129) $\sqrt{-2}\cdot\sqrt{-8}$ -4

130) $\sqrt{-6}\cdot\sqrt{-3}$ $-3\sqrt{2}$

Perform the indicated operations.

131) $(2+i)+(10-4i)$ $12-3i$

132) $(4+3i)-(11-4i)$ $-7+7i$

133) $\left(\dfrac{4}{5}-\dfrac{1}{3}i\right)-\left(\dfrac{1}{2}+i\right)$ $\dfrac{3}{10}-\dfrac{4}{3}i$

134) $\left(-\dfrac{3}{8}-2i\right)+\left(\dfrac{5}{8}+\dfrac{3}{2}i\right)-\left(\dfrac{1}{4}-\dfrac{1}{2}i\right)$ 0

Multiply and simplify.

135) $5(-6+7i)$ $-30+35i$

136) $-8i(4+3i)$ $24-32i$

137) $3i(-7+12i)$ $-36-21i$

138) $(3-4i)(2+i)$ $10-5i$

139) $(4-6i)(3-6i)$ $-24-42i$

140) $\left(\frac{1}{5}-\frac{2}{3}i\right)\left(\frac{3}{2}-\frac{2}{3}i\right)$

Identify the conjugate of each complex number, then multiply the number and its conjugate.

141) $2-7i$ conjugate: $2+7i$; product: 53

142) $-2+3i$ conjugate: $-2-3i$; product: 13

Divide. Write the quotient in the form *a* + *bi*.

143) $\dfrac{6}{2+5i}$ $\frac{12}{29}-\frac{30}{29}i$

144) $\dfrac{-12}{4-3i}$ $-\frac{48}{25}-\frac{36}{25}i$

145) $\dfrac{8}{i}$ $-8i$

146) $\dfrac{4i}{1-3i}$ $-\frac{6}{5}+\frac{2}{5}i$

147) $\dfrac{9-4i}{6-i}$ $\frac{58}{37}-\frac{15}{37}i$

148) $\dfrac{5-i}{-2+6i}$ $-\frac{2}{5}-\frac{7}{10}i$

Simplify.

149) i^{10} -1

150) i^{51} $-i$

Chapter 9: Test

*Additional answers can be found in the Answers to Exercises appendix.

Fill in the blank with *always*, *sometimes*, or *never* to make the statement true.

1) The even root of a negative number is ___never___ a real number.

2) The product of a complex number and its conjugate is ___always___ a real number.

3) A negative number can ___sometimes___ be a solution to an equation containing a square root.

4) The domain of a square root function is ___sometimes___ $[0, \infty)$.

Find each real root, if possible.

5) $\sqrt{144}$ 12

6) $\sqrt[3]{-27}$ -3

7) $\sqrt{-16}$ not real

Simplify. Use absolute values when necessary.

8) $\sqrt[4]{w^4}$ $|w|$

9) $\sqrt[5]{(-19)^5}$ -19

10) Let $h(c)=\sqrt{3c+7}$.

a) Find $h(-2)$. 1

b) Find $h(a-4)$. $\sqrt{3a-5}$

c) Determine the domain of h. $\left[-\frac{7}{3}, \infty\right)$

11) Determine the domain of $f(x)=\sqrt{x-2}$, and graph the function.

Evaluate.

12) $16^{1/4}$ 2

13) $27^{4/3}$ 81

14) $(49)^{-1/2}$ $\frac{1}{7}$

15) $\left(\frac{8}{125}\right)^{-2/3}$ $\frac{25}{4}$

From this point forward, assume all variables represent positive real numbers.

16) Simplify completely. The answer should contain only positive exponents.

a) $m^{3/8}\cdot m^{1/4}$ $m^{5/8}$

b) $\dfrac{35a^{1/6}}{14a^{5/6}}$ $\frac{5}{2a^{2/3}}$

c) $(2x^{3/10}y^{-2/5})^{-5}$ $\frac{y^2}{32x^{3/2}}$

17) Simplify completely.

a) $\sqrt{75}$ $5\sqrt{3}$

b) $\sqrt[3]{48}$ $2\sqrt[3]{6}$

c) $\sqrt{\dfrac{24}{2}}$ $2\sqrt{3}$

Simplify completely.

18) $\sqrt{y^6}$ y^3

19) $\sqrt[4]{p^{24}}$ p^6

20) $\sqrt{t^9}$ $t^4\sqrt{t}$

21) $\sqrt{63m^5n^8}$ $3m^2n^4\sqrt{7m}$

22) $\sqrt[3]{c^{23}}$ $c^7\sqrt[3]{c^2}$

23) $\sqrt[3]{\dfrac{a^{14}b^7}{27}}$ $\frac{a^4b^2\sqrt[3]{a^2b}}{3}$

Perform the operations, and simplify.

24) $\sqrt{3}\cdot\sqrt{12}$ 6

25) $\sqrt[3]{z^4}\cdot\sqrt[3]{z^6}$ $z^3\sqrt[3]{z}$

26) $\dfrac{\sqrt{120w^{15}}}{\sqrt{2w^4}}$ $2w^5\sqrt{15w}$

27) $9\sqrt{7}-3\sqrt{7}$ $6\sqrt{7}$

28) $\sqrt{12}-\sqrt{108}+\sqrt{18}$ $3\sqrt{2}-4\sqrt{3}$

29) $2h^3\sqrt[4]{h}-16\sqrt[4]{h^{13}}$ $-14h^3\sqrt[4]{h}$

Multiply and simplify.

30) $\sqrt{6}(\sqrt{2}-5)$ $2\sqrt{3}-5\sqrt{6}$

31) $(3-2\sqrt{5})(\sqrt{2}+1)$ $3\sqrt{2}+3-2\sqrt{10}-2\sqrt{5}$

32) $(2\sqrt{x}+\sqrt{3})(2\sqrt{x}-\sqrt{3})$ $4x-3$

33) $(\sqrt{2p+1}+2)^2$ $2p+5+4\sqrt{2p+1}$

34) $2\sqrt{t}(\sqrt{t}-\sqrt{3u})$ $2t-2\sqrt{3tu}$

Rationalize the denominator of each expression.

35) $\dfrac{2}{\sqrt{5}}$ $\frac{2\sqrt{5}}{5}$

36) $\dfrac{8}{\sqrt{7}+3}$ $12-4\sqrt{7}$

37) $\dfrac{\sqrt{6}}{\sqrt{a}}$ $\frac{\sqrt{6a}}{a}$

38) $\dfrac{5}{\sqrt[3]{9}}$ $\frac{5\sqrt[3]{3}}{3}$

39) Simplify completely. $\dfrac{2-\sqrt{48}}{2}$ $1-2\sqrt{3}$

Solve.

40) $\sqrt{5h+4}=3$ $\{1\}$

41) $z=\sqrt{1-4z}-5$ $\{-2\}$

42) $\sqrt[3]{n-5}-\sqrt[3]{2n-18}=0$ $\{13\}$

43) $\sqrt{3k+1} - \sqrt{2k-1} = 1$ $\{1, 5\}$

44) In the formula $r = \sqrt{\dfrac{V}{\pi h}}$, V represents the volume of a cylinder, h represents the height of the cylinder, and r represents the radius.

a) A cylindrical container has a volume of 72π in^3. It is 8 in. high. What is the radius of the container? 3 in.

b) Solve the formula for V. $V = \pi r^2 h$

Simplify.

45) $\sqrt{-64}$ $8i$

46) $\sqrt{-45}$ $3i\sqrt{5}$

47) i^{19} $-i$

Perform the indicated operation, and simplify. Write the answer in the form $a + bi$.

48) $(-10 + 3i) - (6 + i)$ $-16 + 2i$

49) $(2 - 7i)(-1 + 3i)$ $19 + 13i$

50) $\dfrac{8+i}{2-3i}$ $1 + 2i$

Chapter 9: Cumulative Review for Chapters 1–9

*Additional answers can be found in the Answers to Exercises appendix.

1) Combine like terms.

$4x - 3y + 9 - \dfrac{2}{3}x + y - 1$ $\dfrac{10}{3}x - 2y + 8$

2) Write in scientific notation.

8,723,000 8.723×10^6

3) Solve $3(2c - 1) + 7 = 9c + 5(c + 2)$. $\left\{-\dfrac{3}{4}\right\}$

4) Graph $3x + 2y = 12$.

5) Write the equation of the line containing the points (5, 3) and (1, −2). Write the equation in slope-intercept form. $y = \dfrac{5}{4}x - \dfrac{13}{4}$

6) Solve by substitution.

$$2x + 7y = -12$$
$$x - 4y = -6$$
$(-6, 0)$

7) Multiply.

$(5p^2 - 2)(3p^2 - 4p - 1)$ $15p^4 - 20p^3 - 11p^2 + 8p + 2$

8) Divide.

$\dfrac{8n^3 - 1}{2n - 1}$ $4n^2 + 2n + 1$

Factor completely.

9) $4w^2 + 5w - 6$ $(4w - 3)(w + 2)$

10) $8 - 18t^2$ $2(2 + 3t)(2 - 3t)$

11) Solve $6y^2 - 4 = 5y$. $\left\{-\dfrac{1}{2}, \dfrac{4}{3}\right\}$

12) Solve $3(k^2 + 20) - 4k = 2k^2 + 11k + 6$. $\{6, 9\}$

13) *Write an equation, and solve.* The width of a rectangle is 5 in. less than its length. The area is 84 in^2. Find the dimensions of the rectangle. length = 12 in., width = 7 in.

Perform the operations, and simplify.

14) $\dfrac{5a^2 + 3}{a^2 + 4a} - \dfrac{3a - 2}{a + 4}$ $\dfrac{2a^2 + 2a + 3}{a(a + 4)}$

15) $\dfrac{10m^2}{9n} \cdot \dfrac{6n^2}{35m^5}$ $\dfrac{4n}{21m^3}$

16) Solve $\dfrac{3}{r^2 + 8r + 15} - \dfrac{4}{r + 3} = 1$. $\{-8, -4\}$

17) Solve $|6g + 1| \geq 11$. Write the answer in interval notation. $(-\infty, -2] \cup \left[\dfrac{5}{3}, \infty\right)$

18) Solve using Gaussian elimination.

$$x + 3y + z = 3$$
$$2x - y - 5z = -1$$
$$-x + 2y + 3z = 0$$
$(4, -1, 2)$

19) Simplify. Assume all variables represent nonnegative real numbers.

a) $\sqrt{500}$ $10\sqrt{5}$

b) $\sqrt[3]{56}$ $2\sqrt[3]{7}$

c) $\sqrt{p^{10}q^7}$ $p^5q^3\sqrt{q}$

d) $\sqrt[4]{32a^{15}}$ $2a^3\sqrt[4]{2a^3}$

20) Evaluate.

a) $81^{1/2}$ 9

b) $8^{4/3}$ 16

c) $(27)^{-1/3}$ $\dfrac{1}{3}$

21) Multiply and simplify $2\sqrt{3}(5 - \sqrt{3})$. $10\sqrt{3} - 6$

22) Rationalize the denominator. Assume the variables represent positive real numbers.

a) $\sqrt{\dfrac{20}{50}}$ $\dfrac{\sqrt{10}}{5}$

b) $\dfrac{6}{\sqrt[3]{2}}$ $3\sqrt[3]{4}$

c) $\dfrac{x}{\sqrt[3]{y^2}}$ $\dfrac{x\sqrt[3]{y}}{y}$

d) $\dfrac{\sqrt{a} - 2}{1 - \sqrt{a}}$ $\dfrac{a - 2 - \sqrt{a}}{1 - a}$

23) Solve.

a) $\sqrt{2b - 1} + 7 = 6$ $\varnothing$

b) $\sqrt{3z + 10} = 2 - \sqrt{z + 4}$ $\{-3\}$

24) Simplify.

a) $\sqrt{-49}$ $7i$

b) $\sqrt{-56}$ $2i\sqrt{14}$

c) i^8 1

25) Perform the indicated operation, and simplify. Write the answer in the form $a + bi$.

a) $(-3 + 4i) + (5 + 3i)$ $2 + 7i$

b) $(3 + 6i)(-2 + 7i)$ $-48 + 9i$

c) $\dfrac{2 - i}{-4 + 3i}$ $-\dfrac{11}{25} - \dfrac{2}{25}i$